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LOCAL CHARACTERIZATIONS OF OKA MANIFOLDS

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Abstract. We prove that strong dominability, the local approximation property (LAP), and the Oka property are equivalent for complex manifolds. The proof turns approximation across an attached disc in one complex variable, uniformly in holomorphic parameters, into convex approximation in arbitrary dimension. We also characterize Oka manifolds by the local elliptic property (LEP), formulated in terms of centered holomorphic families of discs.

In dimension $n \geq 1$, LEP propagates across sets of zero $(2n-1)$-dimensional Hausdorff measure. The Oka property is local in the Euclidean topology, and it descends under surjective holomorphic maps between connected complex manifolds.

Deformations of very free rational curves give the Oka property for every smooth connected complex projective rationally connected manifold X, and for $X \setminus A$ whenever $A \subset X$ is a closed algebraic subset of codimension at least two. Compact Kähler manifolds admitting a Kähler metric with nonnegative holomorphic sectional curvature, or having nef tangent bundle, are also Oka. For $n \geq 1$, the complement of a smooth hypersurface $D \subset \mathbb{P}^n$ of degree d is Oka if and only if $d \leq n+1$.

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1. Introduction

The complex structure of a manifold governs the holomorphic maps into it, giving rise to questions of rigidity and flexibility. If a complex manifold Y is Kobayashi hyperbolic [50], it admits no nonconstant entire curve $\mathbb{C} \to Y$; for compact Y, Brody's lemma [8] gives the converse. At the opposite extreme, Oka manifolds exhibit holomorphic flexibility: every continuous map from a Stein manifold into an Oka manifold is homotopic to a holomorphic map.

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This homotopy principle also has approximation and interpolation forms. Oka manifolds thus embody the principle that, for Stein sources, topological obstructions are the only obstructions to holomorphic realization.

Oka theory has its origins in Oka's solution of the second Cousin problem [56] and Grauert's theorem for holomorphic fiber bundles with homogeneous fibers over Stein spaces [42]. Gromov placed the subject within the framework of the h-principle and introduced the dominating sprays that define the holomorphically elliptic manifolds [43]. Forstnerič introduced subellipticity and proved the Oka principle for subelliptic manifolds [23].

Forstnerič also established approximation, interpolation, and finite-order jet interpolation [25], showed that convex approximation is equivalent to the Oka property with approximation [26], and later proved that convex approximation implies the full parametric Oka property [27]. His monograph [28] gives a systematic account of the theory; see also the survey of Forstnerič and Lárusson [31] and his recent exposition [29].

Definition 1.1 ([27])**.** Let Y be a complex manifold with a distance d_Y inducing its topology. For a fixed integer $d \geq 1$, the manifold Y satisfies CAP_d if, for every compact convex set $K \subset \mathbb{C}^d$, holomorphic map $f : U \to Y$ on an open neighborhood U of K, and $\epsilon > 0$, there is a holomorphic map $F : \mathbb{C}^d \to Y$ such that

$$\sup_{z \in K} d_Y(F(z), f(z)) < \epsilon.$$

The manifold Y has the *convex approximation property* (CAP) if it satisfies CAP_d for every $d \geq 1$. A manifold with CAP is called an *Oka manifold*.

By Forstnerič's theorem [27, Theorem 1.1], this is equivalent to the parametric Oka property, with approximation and interpolation, for maps from reduced Stein spaces.

Definition 1.2. A pair $D \subset D' \subset \mathbb{C}$ is a *special planar pair* [2, p. 22, before Definition 7.11] if D and D' are closed Jordan discs with piecewise smooth boundaries and there is a closed Jordan disc $E \subset \mathbb{C}$ with piecewise smooth boundary such that $D' = D \cup E$ and $D \cap E$ is homeomorphic to $[0, 1]$ and is a proper subset of both ∂D and ∂E (Figure 1).

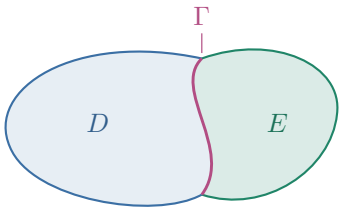


FIGURE 1. A special planar pair $D \subset D' = D \cup E$, with $\Gamma = D \cap E$.

Definition 1.3 ([55, Definition 2.1])**.** Let Y be a complex manifold. We say that Y satisfies the *local approximation property* (LAP) at $x \in Y$ if there is an open neighborhood $V \ni x$ such that, for every special planar pair $D \subset D'$, every integer $q \geq 0$, every closed polydisc $P \subset \mathbb{C}^q$, and every holomorphic map f from a neighborhood of $D \times P$ into V, the restriction $f|_{D \times P}$ is uniformly approximable by maps holomorphic near $D' \times P$ with values in Y. We call V a witness neighborhood. The manifold Y satisfies LAP if it does so at every point.

For $r > 0$, write $\Delta_r = \{t \in \mathbb{C} : |t| < r\}$ for the open disc of radius r centered at 0, and $\overline{\Delta}_r$ for its closure. Write $\Delta = \Delta_1$. A closed polydisc $P \subset \mathbb{C}^q$ is a product $P = \prod_{j=1}^{q}(a_j + \overline{\Delta}_{r_j})$, where $a_j \in \mathbb{C}$ and $r_j > 0$. For $q = 0$, the passive factor P is omitted.

Alarcón and Forstnerič introduced Oka-1 manifolds, for which the Oka principle with approximation and jet interpolation holds for maps from open Riemann surfaces [2, Definition 1.1].

They also introduced the local spray approximation property (LSAP) [2, Definition 7.11] and proved that a positive-dimensional complex manifold Y is Oka-1 if it satisfies LSAP outside a closed set of zero $(2\dim Y - 1)$-dimensional Hausdorff measure [2, Proposition 7.13(b)]. They obtained LSAP at a point from dominability by a spanning tube of complex lines [2, p. 23, after Remark 7.12]; spanning tubes themselves have LSAP [2, Proposition 7.13(a)]. Forstnerič and Lárusson developed further properties of LSAP [33, Section 5]. Both LSAP and LAP concern approximation across an attached planar disc for holomorphic families initially taking values in a fixed target neighborhood. LSAP allows a family-dependent reduction of the parameter radius, whereas LAP retains the whole prescribed compact parameter polydisc. For complex surfaces, LAP is equivalent to property $\mathcal{L}$ of OpenAI [55, Definition 2.1]; Definition 1.3 extends it to arbitrary target dimension.

Definition 1.4. Let Y be a complex manifold of dimension n and let $y \in Y$. The manifold Y is *dominable at* y if there is a holomorphic map $H : \mathbb{C}^n \to Y$ such that $H(0) = y$ and dH_0 is invertible. It is *strongly dominable* if it is dominable at every point.

Theorem 1.5. *Every complex manifold satisfying* LAP *is Oka. Consequently, for a complex manifold Y, the following conditions are equivalent:*

(i) *Y is strongly dominable;*
(ii) *Y satisfies* LAP*;*
(iii) *Y is Oka.*

The implication from the Oka property to strong dominability is classical [28, Theorem 5.5.1(e)]. Strong dominability implies LAP directly: a local inverse of a dominating entire map reduces the problem to approximation by entire maps into $\mathbb{C}^{\dim Y}$ (Proposition 2.1). Adapting OpenAI's approach [55], we prove LAP $\Rightarrow$ CAP, hence the Oka property, in Sections 3–5.

Theorem 1.5 thereby answers the strong-dominability question of Forstnerič and Lárusson [33], emphasized later in [29].

The pointwise case is already visible in Kusakabe's characterization. He characterized Oka manifolds by Gromov's condition Ell_1, which can be tested near compact convex sets [52]: for every m, every compact convex set $K \subset \mathbb{C}^m$, and every holomorphic map $f : W \to Y$ from an open neighborhood W of K, there are an open neighborhood U of K with $U \subseteq W$ and a holomorphic map $F : U \times \mathbb{C}^N \to Y$ for some N, with $F(s, 0) = f(s)$ and $F(s, \cdot)$ a submersion at 0 for every $s \in U$.

For $m = 0$, this requires, for every $y \in Y$, an entire map $H : \mathbb{C}^N \to Y$ with $H(0) = y$ that is a submersion at 0. This is equivalent to strong dominability: restricting H to a suitable n-dimensional linear subspace of $\mathbb{C}^N$, where $n = \dim Y$, gives a map $\mathbb{C}^n \to Y$ with invertible differential at 0; conversely, take $N = n$. By Theorem 1.5, this case alone already implies the Oka property.

For the geometric applications, we introduce the local elliptic property (LEP) in terms of centered holomorphic families and prove its pointwise equivalence with LAP. Very free curves will provide these families for rationally connected manifolds.

Definition 1.6. Let Y be a complex manifold of dimension $n \geq 1$. We say that Y satisfies the *local elliptic property* (LEP) at $x \in Y$ if there are an open neighborhood $V \ni x$, a convergence radius $\tau > 0$, radii $R_\nu \to \infty$, and holomorphic maps

$$h_{j,\nu} : V \times \Delta_{R_\nu} \longrightarrow Y, \qquad h_{j,\nu}(y, 0) = y, \qquad 1 \leq j \leq n,$$

converging locally uniformly on $V \times \Delta_\tau$ to maps h_j, such that

$$\partial_t h_1(x, 0), \ldots, \partial_t h_n(x, 0)$$

form a basis of $T_x Y$. The manifold Y satisfies LEP if it satisfies LEP at every point. In dimension zero, LEP holds at every point.

Convergence in Definition 1.6 is required only on $V \times \Delta_\tau$. A finite spanning collection of initial tangent vectors at x may be replaced by a basis chosen among them.

Theorem 1.7. *Let Y be a complex manifold and $x \in Y$. Then Y satisfies* LEP *at x if and only if it satisfies* LAP *at x. In particular, Y satisfies* LEP *if and only if it is Oka.*

Write $G_Y \subset Y$ for the points at which Y satisfies LEP. By continuity, the initial tangent vectors remain linearly independent throughout a smaller V, so the same families establish LEP there. Thus G_Y is open. Theorem 1.7 is proved in Section 6, where we also show that dominability at a point implies LEP there.

The next three results concern the source dimension, finite stratifications, and open covers; here CAP_d is convex approximation in source dimension d (Definition 1.1).

Theorem 1.8. *Let X be an n-dimensional complex manifold, where $n \geq 1$. Then X is Oka if and only if it satisfies* CAP_n.

Forstnerič and Lárusson introduced a *finite Oka stratification*: a finite decreasing filtration by closed complex subvarieties, ending in the empty set, whose successive differences are manifolds with Oka connected components [32, Definition 1]. They proved that such a stratification implies strong dominability [32, Theorem 2].

Theorem 1.9. *Every complex manifold with a finite Oka stratification is Oka.*

Theorem 1.10. *If every point of a complex manifold has an open Oka neighborhood, then the manifold is Oka.*

Theorem 1.8 answers Forstnerič's stabilization question at the proposed threshold $\dim X$ [26, Problem 6.2]. Theorem 1.9 settles the question of Forstnerič and Lárusson [32, p. 3716]: a finite Oka stratification yields an Oka manifold. Kusakabe proved localization for Zariski-open covers and asked whether Euclidean-open covers suffice [52]; Theorem 1.10 shows that they do. The three proofs are given in Subsection 6.3.

For $s \geq 0$, let $\mathcal{H}^s$ denote s-dimensional Hausdorff outer measure with respect to a smooth Riemannian metric. Its null sets are independent of this choice, since smooth metrics are locally bi-Lipschitz equivalent.

Theorem 1.11. *Let Y be a complex manifold of dimension $n \geq 1$ and $E \subset Y$ a set satisfying $\mathcal{H}^{2n-1}(E) = 0$. If Y satisfies* LEP *at every point of $Y \setminus E$, then it satisfies* LEP *everywhere and is Oka, without assuming that E is closed.*

Dominability outside such a set therefore implies the full Oka property. With E closed, Alarcón and Forstnerič obtained the Oka-1 conclusion from LSAP at every point of $Y \setminus E$ [2, Proposition 7.13(b)]. For analytic exceptional sets, Theorem 1.11 has the following direct consequence:

Theorem 1.12. *Let X be a connected complex manifold and $A \subset X$ a proper closed complex analytic subset. If $X \setminus A$ is Oka, then X is Oka.*

Theorem 1.12 settles Forstnerič's union problem [39, Problem 3.11], and with it the first part of Question D of Forstnerič and Lárusson [31]: a connected complex manifold is Oka as soon as it contains a nonempty Zariski-open Oka subset.

Theorem 1.11 also gives descent along arbitrary surjective holomorphic maps from Oka manifolds (Theorem 7.3). Together with Forstnerič's surjectivity theorem [35], this descent result characterizes positive-dimensional connected Oka manifolds by surjective entire maps (Corollary 7.5), which answers Forstnerič's Problem 1.4(a).

Geometric applications. A smooth complex projective manifold X is *rationally connected* if a general pair of its points lies on the image of a map $\mathbb{P}^1 \to X$, and is *Fano* if K_X^{-1} is ample. Deformations of very free rational curves give:

Theorem 1.13. *Every smooth connected complex projective rationally connected manifold is Oka. In particular, every smooth complex Fano manifold is Oka.*

Theorem 1.13 covers the rationally connected case of Forstnerič's Problem 11.3 [39], and in the Fano range it answers the hypersurface part of Problem E of Forstnerič and Lárusson [31]: every smooth hypersurface of degree $d \le n$ in $\mathbb{P}^n$ is Oka.

For rationally connected projective manifolds, Campana and Winkelmann constructed dense entire curves whose images contain prescribed countable subsets [11]. The Oka-1 conjecture that Alarcón and Forstnerič formulated for these manifolds [2] was proved by Du, Guo, and Xie [20]; their analytic constructions of rational curves are given in [19]. Guo and Xie [41] established the Oka property for smooth quartic hypersurfaces of dimension at least three, and An and Xie [4] for all smooth Fano threefolds.

In Section 8, we prove the general rationally connected case by using first-jet evaluation of very free curves to construct centered entire families with independent initial tangent vectors. Choosing these curves to avoid a closed algebraic subset $A \subset X$ of codimension at least two gives the Oka property for $X \setminus A$ (Corollary 8.4).

Corollaries 8.6 and 8.7 show that compact Kähler manifolds with nonnegative holomorphic sectional curvature or nef tangent bundle are Oka. In a noncompact setting, Forstnerič and Kusakabe proved that, for $n \ge 2$ and $k \ge 1$, the unit disc bundle defined by a semipositive Hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(k)$ is Oka [30].

For complements in affine space, Kusakabe's theorem covers compact polynomially convex sets in $\mathbb{C}^n$, $n > 1$ [53]; Forstnerič and Wold treated broader classes of unbounded convex sets [37]. Further examples come from the periodic-transport constructions of Du, Guo, Wang, and Xie [18].

In the projective setting, let X be a connected complex projective manifold and D a reduced simple normal crossings divisor. Theorem 9.1 shows that $X \setminus D$ is Oka if there is a nonconstant map $f : \mathbb{P}^1 \to X$ with $f^{-1}(D) \subseteq \{\infty\}$ and ample $f^*T_X(-\log D)$. Here $T_X(-\log D)$ is the bundle of vector fields tangent to every component of D.

Chen and Zhu's theory of $\mathbb{A}^1$-curves on log smooth varieties [17], combined with Corollary 9.4, gives Oka complements of smooth irreducible ample rationally connected divisors in connected complex projective manifolds of dimension at least two. The same method also applies to complements of smooth hypersurface sections of smooth complete intersections when $-(K_X+D)$ is ample (Theorem 9.5 and Corollary 9.6).

A *holomorphic spray* on Y is a holomorphic map $s : E \to Y$ from a holomorphic vector bundle $E \to Y$ such that $s(0_y) = y$. It is *dominating* if $d(s|_{E_y})_{0_y}$ is surjective for every y; a manifold admitting such a spray is *holomorphically elliptic* (also called elliptic in Gromov's terminology). Gromov's theorem implies that every such manifold is Oka [43]. For smooth hypersurface complements we obtain:

Theorem 1.14. *Let $D \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \ge 1$, where $n \ge 1$. The following conditions are equivalent:*

(i) $\mathbb{P}^n \setminus D$ *is Oka;*
(ii) $\mathbb{P}^n \setminus D$ *is holomorphically elliptic;*
(iii) $d \le n + 1$.

For $n = 1$, smoothness means that D consists of d distinct points; the positive cases are $\mathbb{C}$ and $\mathbb{C}^*$.

For complements, Kusakabe treated the quadric case [52] and Xie the smooth cubic case [62]; the plane cubic case answered Problem B of Forstnerič and Lárusson [31]. Theorem 1.14 resolves Forstnerič's Problem 5.3(a) for smooth hypersurfaces [39] and covers the complement part of Problem E of Forstnerič and Lárusson [31] for every smooth hypersurface, with the exact threshold $d \le n+1$.

At the critical degree $d = n+1$ the logarithmic tangent bundle has trivial determinant, so the preceding positivity criterion cannot apply. For $n \ge 2$, we instead use generic projection and successive polar equations to construct families of lines meeting D in at most two distinct points. Entire parametrizations of the line complements give centered families with independent initial tangent vectors on a nonempty Zariski-open subset of the complement; Theorem 1.11 then gives the Oka property everywhere.

For $d \ge n+2$, the rank obstruction of Carlson and Griffiths [16] excludes the Oka property.

Relation to the work of OpenAI. Xie and Zhao [63] constructed the first known Oka K3 surfaces, including every smooth hypersurface of multidegree $(2,2,2)$ in $(\mathbb{P}^1)^3$, and proved density results for Oka periods and local deformation parameters. They also conjectured that every complex K3 surface is Oka. OpenAI [55] proved this conjecture, using the same period-theoretic framework with different geometric constructions.

OpenAI [55, Theorem 3.1] proved the implication LAP $\Rightarrow$ CAP for complex surfaces. Our argument for general complex manifolds adapts their method.

Our proof uses an explicit scalar collapse for simultaneous sector filling with fixed passive parameters, and normalized attracting-basin coordinates for the passage to polydisc approximation. In the convex step, polynomial approximation of a local fiber inverse replaces the real-interface iteration by gluing on a fixed complex overlap.

Organization. Sections 2–5 prove Theorem 1.5, passing from local gluing and disc enlargement through thin attracting basins to polydisc and convex approximation. Section 6 proves the pointwise equivalence of LEP and LAP and gives direct consequences. Section 7 treats removability, holomorphic images, and complements. The geometric applications to rationally connected projective manifolds and divisor complements occupy Sections 8 and 9. Section 10 presents three conjectures and their compact Kähler analogues.

2. Local approximation and gluing

2.1. From strong dominability to LAP.

Proposition 2.1. *Let Y be a complex manifold. If Y is dominable at $x \in Y$, then it satisfies* LAP *at x. In particular, every strongly dominable complex manifold satisfies* LAP.

Proof. For $n = \dim Y > 0$, choose a holomorphic map $H : \mathbb{C}^n \to Y$ dominating at x and a local inverse $\lambda : V \to \mathbb{C}^n$ near x. For a special planar pair $D \subset D'$, a closed polydisc $P \subset \mathbb{C}^q$, and f holomorphic near $D \times P$ with values in V, the product $D \times P$ is polynomially convex: D has connected complement, and P is a polydisc. The Oka–Weil theorem [28, Theorem 2.3.1] therefore gives entire maps $g : \mathbb{C}^{1+q} \to \mathbb{C}^n$ approximating $\lambda \circ f$ on $D \times P$. Uniform continuity of H on a compact neighborhood of $\lambda(f(D \times P))$ shows that $H \circ g$ approximates f uniformly there. These maps are entire, proving LAP at x with witness neighborhood V. □

2.2. Local coordinates and one-step gluing.

To glue maps defined on two overlapping source domains, we allow each map to vary in a small holomorphic family. The additional variable v parametrizes small changes of the original map; the value $v = 0$ gives the map itself. We then seek small holomorphic choices $v = a(z)$ and $v = b(z)$ on the two domains so that the resulting maps agree on the overlap. The lemmas below construct these coordinates, split a holomorphic error into contributions from the two sides, and solve for a and b by a contraction.

2.2.1. *Conventions and graph coordinates.* Throughout, Y is a complex manifold of dimension n. We fix a complete distance d_Y inducing its topology. Completeness is used only when taking an exhaustion limit. The notation $\mathcal{O}(\operatorname{Op} K, Y)$ means maps holomorphic on some neighborhood of K. A closed polydisc of dimension zero is a point.

For a holomorphic map $f : U \to Y$ on a complex manifold U and a fiber radius $b > 0$, a *chart spray* centered at f is a holomorphic family $F : U \times \Delta_b^n \to Y$ with $F(z, 0) = f(z)$ for which $(z, v) \mapsto (z, F(z, v))$ is biholomorphic onto its image. The following standard graph construction is used by OpenAI [55, Lemma 2.6].

Lemma 2.2. *Let S be Stein, $f : S \to Y$ holomorphic, and suppose f^*TY is holomorphically trivial. For every $K \Subset S$, there exist a neighborhood U of K, a radius $b > 0$, and a chart spray $F : U \times \Delta_b^n \to Y$ centered at f.*

Proof. The graph of f is a Stein submanifold of $S \times Y$ and has a Stein neighborhood by Siu's theorem [59, Main Theorem]; see also [28, Theorem 3.1.1]. A frame of its vertical tangent bundle extends, after shrinking that neighborhood, to holomorphic vertical vector fields: apply Cartan's theorem B [28, Theorem 2.6.1(B)] to the restriction sequence for the vertical tangent sheaf. Compose their local flows. Near the zero section this gives a fiber-preserving holomorphic map whose vertical differential is the chosen frame. Its graph map therefore has invertible differential at every point $(z, 0)$. The inverse function theorem gives, near each such point, a product neighborhood on which the graph map is biholomorphic onto an open image. Cover K by finitely many smaller base neighborhoods and choose a fiber radius common to these products. On their union the graph map is a local biholomorphism and is injective: points with different base coordinates have different images, while points over the same base coordinate lie in one of the injective products. This gives the required neighborhood U and chart spray. □

In the applications, the source is a contractible Stein domain or a planar domain times a polydisc. A planar domain has the homotopy type of a one-dimensional CW complex, so complex vector bundles on it are topologically trivial [28, Corollary 3.12.2 and Theorem 8.2.1(iii)]. The Oka–Grauert principle [28, Theorem 5.3.1(ii)], originally proved in [42, Satz I], therefore provides the triviality required in Lemma 2.2. When using the inverse of a chart spray F, we restrict the source to a compact subset $K_0 \Subset U$ and the fiber variable to $\overline{\Delta}_{b'}^n$ with $b' < b$. The image of $K_0 \times \overline{\Delta}_{b'}^n$ under $(z, v) \mapsto (z, F(z, v))$ is then a compact subset of the open chart image. Consequently, sufficiently small perturbations of these values still lie in that image, so the inverse remains defined uniformly for $z \in K_0$.

2.2.2. *Bounded additive splitting.* For an open set $U \subset \mathbb{C}^m$, write $H^\infty(U, \mathbb{C}^n)$ for the Banach space of bounded holomorphic maps $u : U \to \mathbb{C}^n$, with

$$\|u\|_U = \sup_{z \in U} \|u(z)\|_\infty,$$

where $\|v\|_\infty = \max_j |v_j|$ is the maximum norm. Derivatives are measured in its induced operator norm. We use pairs of bounded open sets $A, B \subset \mathbb{C}^m$ whose closed fringes $\overline{A \setminus B}$ and $\overline{B \setminus A}$ are disjoint. Set $C = A \cap B$.

Lemma 2.3. *Suppose $A \cup B$ is a smoothly bounded strongly pseudoconvex domain. There exist a splitting bound $M > 0$ and bounded linear operators*

$$R_A : H^\infty(C, \mathbb{C}^n) \to H^\infty(A, \mathbb{C}^n), \qquad R_B : H^\infty(C, \mathbb{C}^n) \to H^\infty(B, \mathbb{C}^n)$$

with

$$R_A u - R_B u = u, \qquad \|R_A u\|_A + \|R_B u\|_B \le M \|u\|_C.$$

For bounded open sets $A, B \subset \mathbb{C}$ with the same separation condition, the conclusion also holds on $A \times P, B \times P$ for every open polydisc P; the operators preserve holomorphic dependence on P, and their bound depends only on A, B.

Proof. The first assertion is the classical additive Cousin splitting in [24, Lemma 4.6]. For completeness, set $\Omega = A \cup B$ and choose a smooth function $\chi : \mathbb{C}^m \to [0,1]$ with compact support, equal to 0 near $\overline{A \setminus B}$ and to 1 near $\overline{B \setminus A}$. For $u \in H^\infty(C, \mathbb{C}^n)$, the functions χu and $(\chi - 1)u$ extend smoothly by zero to A and B, respectively. Their $\bar\partial$ derivatives agree with the bounded, $\bar\partial$-closed form $\eta = u\bar\partial\chi$, extended by zero from C to Ω. A bounded linear $\bar\partial$ solution operator on Ω gives v with $\bar\partial v = \eta$. Hence

$$R_A u = \chi u - v|_A, \qquad R_B u = (\chi - 1)u - v|_B$$

are holomorphic, differ by u on C, and satisfy the required bound; the products denote their zero extensions.

For the planar assertion, the corresponding splitting on compact Cartan pairs is given in [38, Lemma 3.2]; we give the bounded-open-set version needed here directly. Write $\bar\partial\chi = \chi_{\bar z}\, d\bar z$ and, for $p \in P$, extend

$$a(\zeta, p) = u(\zeta, p)\chi_{\bar z}(\zeta), \qquad \zeta \in C,$$

by zero to the plane. This is a bounded measurable function supported in a fixed bounded set $K \subset \mathbb{C}$, independently of p. Define, componentwise,

$$v(z,p) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{a(\zeta,p)}{z-\zeta}\, dA(\zeta).$$

The integral is absolutely convergent. Indeed, for any fixed $R > 0$,

$$\sup_{z\in\mathbb{C}} \int_K \frac{dA(\zeta)}{|z-\zeta|} \le 2\pi R + \frac{\operatorname{area}(K)}{R},$$

by splitting into $|z-\zeta| < R$ and its complement. Thus $\|v\|_\infty \le C_\chi \|u\|_{C\times P}$ with a constant depending only on the fixed planar data. The fundamental-solution identity $\bar\partial_z(1/(\pi z)) = \delta_0$ gives $\bar\partial_z v = a$ in distributions. For fixed z, the integrand is holomorphic in p and is dominated by an integrable function independent of p; Morera's theorem therefore gives holomorphic parameter dependence. The same formulas for $R_A u$ and $R_B u$ now have vanishing $\bar\partial$ in all variables and hence are holomorphic on $A \times P$ and $B \times P$, respectively. Their difference is u, and their norm sum is at most $(2 + 2C_\chi)\|u\|_{C\times P}$. Neither the integration nor this estimate involves the dimension of P or the number of components of u. □

2.2.3. *Only the centers need to be glued.* OpenAI's center-gluing argument [55, Lemma 2.8] finds small holomorphic parameter choices for which the two families agree. With the source domains and their splitting bound M fixed, a contraction finds these choices; the lemma makes the required error bounds explicit.

Lemma 2.4. *Suppose the operators of Lemma 2.3 are fixed. Let $F_A : A \times \Delta^n_{\rho_+} \to Y$ be a holomorphic chart spray and $F_B : B \times \Delta^n_\rho \to Y$ a holomorphic family, where $\rho_+ > \rho > 0$. Suppose their transition is defined on $C \times \Delta^n_\rho$ and has the form*

$$F_A(z,\cdot)^{-1}(F_B(z,v)) = v + c(z,v).$$

Set $e = \|c\|_\infty$ and $\kappa = \|D_v c\|_\infty$ there. If $Me < \rho/2$ and $M\kappa < 1/2$, there are holomorphic corrections $a : A \to \mathbb{C}^n$ and $b : B \to \mathbb{C}^n$ satisfying

$$F_A(z, a(z)) = F_B(z, b(z)) \quad (z \in C), \qquad \|a\|_A + \|b\|_B \le Me.$$

The same conclusion holds if the families also depend holomorphically on $p \in P$, provided the splitting operators act on functions of (z,p), preserve holomorphic dependence on p, and have the same bound M uniformly on P. Here the radii are independent of p, and e and κ are the supremum norms on $C \times P \times \Delta^n_\rho$, subject to the same inequalities. The corrections are then holomorphic functions $a(z,p)$ and $b(z,p)$, and the estimates are uniform in p. Lemma 2.3 supplies such operators for planar A, B.

Proof. On the closed ball $\|u\|_C \le e$ in $H^\infty(C,\mathbb{C}^n)$, set

$$\mathcal{T}(u)(z) = c(z, R_B u(z)).$$

The bound $\|R_B u\|_B \le Me < \rho/2$ keeps $R_B u$ in the prescribed fiber domain. We have $\|\mathcal{T}(u)\|_C \le e$ and $\|\mathcal{T}(u) - \mathcal{T}(\widetilde{u})\|_C \le M\kappa\|u - \widetilde{u}\|_C$. At the fixed point set $a = R_A u$, $b = R_B u$. Then $a - b = u = c(z, b)$, which is exactly the transition identity required for the centers to agree. The norm estimate follows from the linear splitting bound. □

Consequently the glued map changes the old center by an amount tending to zero with e, uniformly on every retained compact set. In our applications, the enlarged compact has a contractible Stein neighborhood inside the domain of the glued map. The pullback of TY to this neighborhood is trivial, so Lemma 2.2 provides a new chart spray for the next step. Thus no spray needs to be carried across the overlap, and inverse fiber charts are required only for the old family. If the transition is uniformly small on a larger fiber polydisc, its derivative is small on a smaller one by Cauchy estimates; this is how the hypothesis on κ will always be obtained.

3. Disc enlargement with a fixed parameter domain

We enlarge a scalar disc while retaining a fixed passive polydisc P. An explicit scalar collapse keeps the images along the attaching corridors in witness neighborhoods, and the intervening sectors are filled and glued simultaneously.

Throughout this section, W denotes a witness neighborhood for LAP as in Definition 1.3. Spray parameters are treated as passive variables.

3.1. **The scalar collapse.**

The function below stays close to the identity on the old disc and sends each attaching ray into a small neighborhood of its boundary endpoint. Composing the original family with this function keeps the corridor images in witness neighborhoods while changing the map only slightly on the old disc.

Lemma 3.1. *For a radius $r > 0$ and an integer $N \ge 1$, set*

$$p_N(s) = \int_0^s \exp\bigl(-(t/r)^N\bigr)\,dt, \qquad \zeta_j = re^{2\pi ij/N} \quad (0 \le j \le N).$$

Then

$$\sup_{|s|\le r} |p_N(s) - s| \le \frac{r(e-1)}{N+1}, \qquad \sup_{t\ge r} |p_N(te^{2\pi ij/N}) - \zeta_j| \le \frac{r}{N+1}.$$

Proof. Termwise integration of the exponential series gives the first estimate. On the positive real ray,

$$0 \le r - p_N(r) \le \frac{r}{N+1}, \qquad \int_r^\infty e^{-(t/r)^N}\,dt \le \frac{r}{eN} \le \frac{r}{N+1}.$$

For the integral estimate substitute $u = (t/r)^N$ and use $u^{1/N-1} \le 1$ for $u \ge 1$. For $t \ge r$, write

$$p_N(t) - r = \int_r^t e^{-(u/r)^N}\,du - (r - p_N(r)).$$

Both terms on the right lie in $[0, r/(N+1)]$, so their difference has absolute value at most $r/(N+1)$. The identity $p_N(\omega s) = \omega p_N(s)$ for $\omega^N = 1$, which follows from the power series, gives the remaining rays. □

3.2. **Disc enlargement.** Alarcón and Forstnerič [2, proof of Lemma 5.1] use joining arcs to divide an annulus into discs, which they fill by approximation and spray gluing. We adapt the spokes-and-sectors construction of OpenAI [55, Proposition 2.9]. The scalar collapse of Lemma 3.1 constructs all radial corridors at once while keeping their images in the prescribed witness neighborhoods. This allows the intervening sectors to be filled and glued simultaneously.

Lemma 3.2. *For a disc radius $r > 0$ and a closed polydisc P, let f be a holomorphic map from a neighborhood of $\overline{\Delta}_r \times P$ to Y. Suppose that, for every $|\zeta| = r$, the compact fiber $f(\{\zeta\} \times P)$ is contained in one of the witness neighborhoods W. For every outer radius $S > r$, f is uniformly approximable on $\overline{\Delta}_r \times P$ by holomorphic maps from a neighborhood of $\overline{\Delta}_S \times P$ to Y.*

Proof. Fix an approximation tolerance $\epsilon > 0$. By Lemma 2.2, choose a chart spray $f^\sharp$ centered at f. For each boundary point ζ, compactness of $f(\{\zeta\} \times P) \subset W$ and continuity of the spray keep its values in this same witness neighborhood W on a small scalar neighborhood, a slightly enlarged passive polydisc, and a small spray polydisc. Choose finitely many smaller boundary arcs whose closures lie in these scalar neighborhoods and still cover the circle. Taking common parameter margins gives a closed polydisc P^+ with $P \Subset \operatorname{int} P^+$ and a radius $\beta > 0$ valid for all the arcs. The corresponding closed products have compact images in their respective witness neighborhoods W, hence positive margins. Fix fiber radii $0 < \rho < \rho_+ < \beta$.

Choose N large enough that each uniform subdivision arc lies in one of the chosen scalar neighborhoods and both estimates for p_N in Lemma 3.1 are smaller than the corresponding margins. Require also that

$$\sup_{\overline{\Delta}_r \times P} d_Y\big(f(p_N(s), p), f(s, p)\big) < \epsilon/2.$$

The near-identity estimate makes $f^\sharp(p_N(s), p, v)$ available on the old disc. With N fixed, truncate the rays at a common radius larger than S. The collapsed-ray estimate places their images in the witness neighborhoods with positive margins. Continuity on these finitely many compact ray products gives thin radial corridors on which the same containments hold uniformly in (p, v). Thus the values near each boundary arc and the two adjacent corridors lie in the corresponding witness neighborhood W, uniformly for $(p, v) \in P^+ \times \overline{\Delta}^n_\beta$; see Figure 2(a).

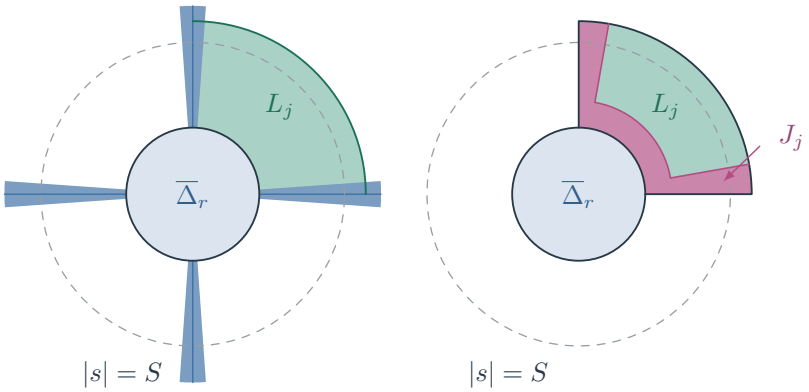
For each intervening sector, take a U-shaped collar J_j as in Figure 2(b). Its bottom follows the boundary arc and its sides follow the two corridors, so the whole parameter family on J_j takes values in a witness neighborhood W. Filling the U gives a sector L_j whose outer edge lies beyond radius S; the added disc meets J_j along a proper boundary arc, hence $J_j \subset L_j$ is a special planar pair. Compactness allows all collars and buffers to be chosen uniformly in (p, v).

Choose a bounded planar neighborhood A of the old disc together with the corridors, and bounded, pairwise disjoint planar open sets B_j in the intervening sectors. With small overlaps inside the controlled neighborhoods and $B = \bigsqcup_j B_j$, they satisfy

$$\overline{\Delta}_S \subset A \cup B, \qquad \overline{A \setminus B} \cap \overline{B \setminus A} = \varnothing, \qquad A \cap B_j \subset J_j, \quad B_j \subset L_j.$$

Retain p in a fixed neighborhood of P whose closure lies in $\operatorname{int} P^+$. The splitting operators act only in the scalar variable, preserve holomorphic dependence on p, and have a fixed bound. At this point N, the source domains, the parameter margins, and the splitting bound are fixed. Only the sector-approximation errors will now be decreased.

On A use the chart spray $F_A(s, p, v) = f^\sharp(p_N(s), p, v)$. Applying LAP to the collar families in their respective witness neighborhoods, with (p, v) passive, gives maps G_j near $L_j \times P^+ \times \overline{\Delta}^n_\beta$ arbitrarily close to F_A on the respective collars. Set $F_B = G_j$ on each B_j; the disjointness of the sets B_j makes this a holomorphic family on the whole new side. Since there are finitely many sectors, their approximation errors can all satisfy one prescribed gluing tolerance. On the compact overlap and the fiber polydisc $\overline{\Delta}^n_{\rho_+}$, the old chart image has a positive margin inside



(a) Old side and corridors (b) Sector and collar

FIGURE 2. The planar construction in Lemma 3.2. (a) The old disc and radial corridors form the old side; the sectors are filled simultaneously. The sector L_j is shown in both panels. (b) Its collar J_j consists of bottom and side strips. The middle of the outer edge is not part of the collar. Widths are enlarged for visibility; small boundary buffers and the parameter factors are omitted.

the full chart image with radius β. Thus a sufficiently close F_B has a well-defined transition

$$F_A(s,p,\cdot)^{-1}(F_B(s,p,v)) = v + E(s,p,v)$$

there. Uniform continuity of this fixed inverse makes $\|E\|$ tend to zero with the sector approximation errors. Cauchy estimates across the fixed gap $\rho_+ - \rho$ then make $\|D_v E\|$ tend to zero on Δ^n_ρ; here $D_v E$ is the derivative with respect to the fiber variable v. Both smallness inequalities of the parameter version of Lemma 2.4 therefore hold once the sector errors are sufficiently small. Apply that lemma once on A, B. Its corrections tend uniformly to zero, so continuity of the old spray gives a map F holomorphic near $\overline{\Delta}_S \times P$ with

$$\sup_{\overline{\Delta}_r \times P} d_Y\big(F(s,p), f(p_N(s),p)\big) < \epsilon/2$$

when the sector approximations are sufficiently accurate. The choice of N and the triangle inequality prove the assertion. □

4. From LAP to polydisc approximation

The disc-enlargement lemma of Section 3 enlarges one variable while retaining the others as parameters. It requires the image of each whole boundary fiber to lie in one LAP witness neighborhood. The construction below makes the boundary fibers of an auxiliary scalar parametrization small and provides a global basin parametrization close to the identity on the old polydisc. This gives a finite extension, which can then be iterated to obtain approximation by entire maps.

4.1. Thin attracting basins.

We construct an attracting basin with thin transverse fibers. We begin with the following theorem of Rosay and Rudin.

Theorem (Rosay–Rudin [57, Theorem 9.1]). *Let F be a holomorphic automorphism of $\mathbb{C}^m$ fixing 0. Write $A = dF_0 : \mathbb{C}^m \to \mathbb{C}^m$ for its differential at the origin, with eigenvalues $\lambda_1, \dots, \lambda_m$. For $k \geq 0$, F^k denotes the k-fold iterate of F. If*

$$\max_j |\lambda_j| < 1, \qquad \big(\max_j |\lambda_j|\big)^2 < \min_j |\lambda_j|,$$

then its attracting basin $\Omega_F = \{z \in \mathbb{C}^m : F^k(z) \to 0\}$ *is biholomorphic to* $\mathbb{C}^m$*. More precisely, the limit*

$$g = \lim_{k\to\infty} A^{-k}F^k : \Omega_F \longrightarrow \mathbb{C}^m$$

exists uniformly on compact subsets of Ω_F *and defines a biholomorphism satisfying* $g\circ F = A\circ g$*.*

We adapt the polynomial automorphism and logarithmic transverse coordinates of OpenAI [55, Lemma 3.3 and the proof of Proposition 3.4] to obtain a global basin parametrization close to the identity on the old polydisc. The construction combines the normalized basin coordinates of Rosay–Rudin [57, Theorem 9.1] with the thin-basin viewpoint of Buzzard and Hubbard [9, Section 3]. In dimension two the polynomial automorphism is a scaled Hénon map; in higher dimensions it is a scaled composition of shift-like automorphisms in the setting of Bedford and Pambuccian [7].

A proper domain in $\mathbb{C}^m$ biholomorphic to $\mathbb{C}^m$ is called a *Fatou–Bieberbach domain*.

Lemma 4.1. *Fix* $m \geq 2$ *and radii* $1 < \alpha < c < b$*. Given tolerances* $\eta, \delta > 0$ *for approximation to the identity and for boundary-fiber diameters, respectively, there exist a Fatou–Bieberbach domain* $\Omega \subset \mathbb{C}^m$*, a biholomorphism* $h : \mathbb{C}^m \to \Omega$*, an integer* $D \geq 1$*, and an entire map* $\Phi : \mathbb{C}\times\mathbb{C}^{m-1} \to \mathbb{C}^m$ *with the following properties.*

(1) The old polydisc is almost fixed:

$$\|h - \mathrm{id}\,\|_{\overline{\Delta}_1^m} < \eta, \qquad |h_1| < \alpha \quad \text{on } \overline{\Delta}_1^m.$$

(2) The part of Ω *with* $|x_1| \leq c$ *satisfies*

$$\Omega \cap \{|x_1| \leq c\} \subset \Delta_b^m.$$

(3) There are single-valued holomorphic coordinates

$$\kappa = (s, \zeta) : \Omega \cap \{|x_1| > \alpha\} \longrightarrow \mathbb{C}\times\Delta_1^{m-1}$$

such that $\Phi\circ\kappa = \mathrm{id}$ *and* $\Phi_1(s,\zeta) = s^D$*. Here* s *is the scalar coordinate, while* $\zeta = (\zeta_2, \dots, \zeta_m)$ *denotes the transverse coordinates.*

(4) For $r = c^{1/D}$*, the filled cylinder has small boundary fibers:*

$$\Phi(\overline{\Delta}_r \times \overline{\Delta}_2^{m-1}) \Subset \Delta_b^m, \qquad \sup_{|s|=r} \operatorname{diam}_\infty \Phi(\{s\}\times\overline{\Delta}_2^{m-1}) < \delta.$$

Here $\operatorname{diam}_\infty$ *denotes diameter in the maximum norm.*

The proof has two tasks. For the automorphism H_d constructed below, write $L = d(H_d)_0$. Near the old polydisc, the normalized iterates $L^{-k}H_d^k$ converge to basin coordinates g_d. Their inverse $h_d = g_d^{-1}$ is close to the identity for large d and will be chosen as h. On $\Omega\cap\{|x_1| > \alpha\}$, exclusion of an escaping region bounds the logarithms of the successive coordinate ratios. Division by powers of the degree d in the coordinate recursion then gives $O(d^{-1})$ variation of Φ on each fixed scalar boundary fiber; see Figure 3. The identity $x_1 = s^D$ identifies $|x_1| = c$ with $|s| = c^{1/D}$, while the radii keep the filled cylinder inside Δ_b^m. The band $\alpha < |x_1| < c$ provides the overlap in Proposition 4.2.

Proof. Choose nested radii and a contraction factor satisfying

$$1 < r_m < \cdots < r_2 < r_1 < \alpha, \qquad 0 < a < 1, \qquad r_j < a r_{j-1} \quad (2 \leq j \leq m).$$

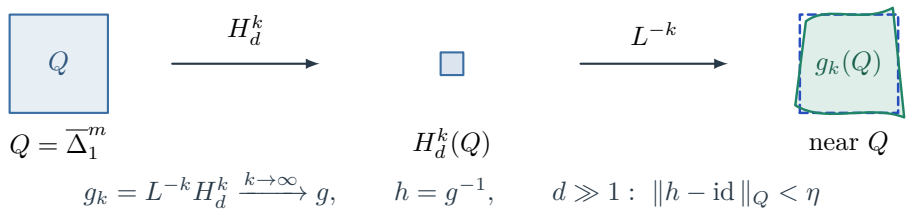
For a large integer $d \geq 2$, put

$$p_j(w) = (w/r_j)^d, \qquad H_d(x) = (ax_m, p_2(x_2) - ax_1, \dots, p_m(x_m) - ax_{m-1}).$$

Its differential $L = d(H_d)_0$ at the origin is

$$L(x) = (ax_m, -ax_1, \dots, -ax_{m-1}).$$

(a) Normalize the contraction near the old polydisc



(b) Thin boundary fibers allow local disc enlargement

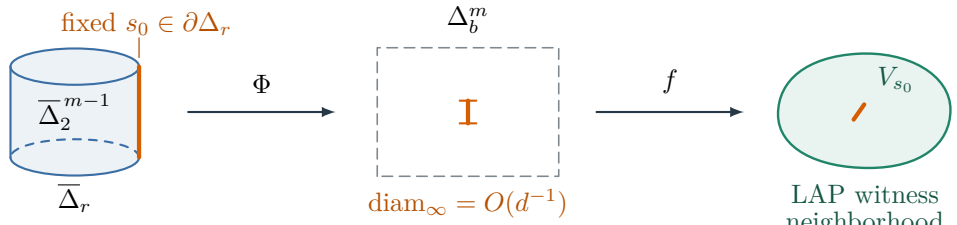


FIGURE 3. Normalized Hénon-type transport and thin boundary fibers (Lemma 4.1), schematically; d is the polynomial degree. (a) For fixed d, linear normalization compensates for contraction, and the limit as $k \to \infty$ gives the basin coordinates g. Taking d large makes $h = g^{-1}$ close to the identity on Q. The copies of Q are shown at the same scale. (b) The boundary fibers have Φ-images of diameter $O(d^{-1})$, uniformly in $s \in \partial\Delta_r$. For sufficiently large d, this diameter is less than δ, and the map f in Proposition 4.2 sends the image of each fixed fiber over $s_0 \in \partial\Delta_r$ into a LAP witness neighborhood V_{s_0}, which may depend on s_0. The cylinder is an auxiliary model for scalar disc enlargement. The relation $\Phi \circ \kappa = \mathrm{id}$ holds on the basin end $\Omega \cap \{|x_1| > \alpha\}$; the band $\alpha < |x_1| < c$ contains the overlap used for gluing.

For $y = H_d(x)$, the inverse is obtained successively from

$$x_m = y_1/a, \qquad x_{j-1} = (p_j(x_j) - y_j)/a \quad (j = m, m-1, \dots, 2).$$

Thus H_d is a polynomial automorphism. Let Ω_d be the basin of its attracting fixed point 0.

Basin coordinates. The map L is a times a signed cyclic permutation. Its eigenvalues have modulus a, so the recalled spectral condition reduces to $a^2 < a$, which holds since $0 < a < 1$. The theorem gives a biholomorphism

$$g_d = \lim_{k\to\infty} L^{-k}H_d^k : \Omega_d \longrightarrow \mathbb{C}^m, \qquad g_d \circ H_d = L \circ g_d.$$

Let $h_d = g_d^{-1}$. We show that $h_d \to \mathrm{id}$ uniformly on $\overline{\Delta}_1^m$. Fix an auxiliary radius $1 < u < r_m$ and $a < q_* < \sqrt{a}$. Writing $H_d = L + N_d$, set

$$\varepsilon_d = \frac{u^{d-2}}{r_m^d} \longrightarrow 0.$$

For sufficiently large d, the bounds

$$\|N_d(z)\|_\infty \le \varepsilon_d \|z\|_\infty^2, \qquad \|H_d(z)\|_\infty \le (a + u\varepsilon_d)\|z\|_\infty \le q_*\|z\|_\infty$$

hold on Δ_u^m. The contraction estimate shows that $\Delta_u^m \subset \Omega_d$. For $z \in \Delta_u^m$, the contraction estimate and the form of L give

$$\|H_d^k(z)\|_\infty \le q_*^k \|z\|_\infty, \qquad \|L^{-k-1}\| = a^{-k-1}.$$

Since $H_d = L + N_d$, successive normalized iterates satisfy

$$L^{-k-1}H_d^{k+1}(z) - L^{-k}H_d^k(z) = L^{-k-1}N_d(H_d^k(z)),$$

whose right-hand side has norm at most $(\varepsilon_d/a)(q_*^2/a)^k\|z\|_\infty^2$. As $q_*^2/a<1$, this series converges uniformly on Δ_u^m. Telescoping and taking the limit therefore yield

$$g_d(z)-z=\sum_{k\ge 0}L^{-k-1}N_d(H_d^k(z)),\qquad \|g_d(z)-z\|_\infty\le\frac{\varepsilon_d}{a(1-q_*^2/a)}\|z\|_\infty^2.$$

Thus $g_d\to\mathrm{id}$ locally uniformly on Δ_u^m. Fix an intermediate radius $1<v<u$ and set $e_d=\|g_d-\mathrm{id}\,\|_{\overline{\Delta}_v^m}\to 0$. For large d, $e_d<v-1$, so $\|g_d(x)\|_\infty>1$ on $\partial\Delta_v^m$. For $z\in\overline{\Delta}^m$, the path $h_d(tz)$, $0\le t\le 1$, starts at 0 and cannot leave Δ_v^m. A first exit would give $\|g_d(h_d(tz))\|_\infty>1$, contradicting $g_d(h_d(tz))=tz$ and $\|tz\|_\infty\le 1$. Hence $h_d(\overline{\Delta}^m)\subset\Delta_v^m$ and

$$\|h_d-\mathrm{id}\,\|_{\overline{\Delta}^m}\le\|g_d-\mathrm{id}\,\|_{\overline{\Delta}_v^m}=e_d\longrightarrow 0.$$

Taking d large enough that $e_d<\min\{\eta,\alpha-1\}$ gives $\|h_d-\mathrm{id}\,\|_{\overline{\Delta}^m}<\eta$ and

$$|h_{d,1}(z)|\le 1+e_d<\alpha\qquad(z\in\overline{\Delta}^m).$$

This proves (1).

Escape estimates and transverse bounds. Choose an escape threshold $S>\max(1,r_2)$. For large d, the region

$$\mathcal{E}=\{x:\max_{j\ge 2}|x_j|>\max(S,|x_1|)\}$$

is forward invariant, meaning $H_d(\mathcal{E})\subset\mathcal{E}$, and escaping: $\|H_d^n(x)\|_\infty\to\infty$ for every $x\in\mathcal{E}$ as $n\to\infty$. Choose d so that $(S/r_2)^d\ge(2+a)S$. If the transverse maximum is $M=|x_k|>\max(S,|x_1|)$, then

$$|H_d(x)_k|\ge(M/r_2)^d-aM\ge 2M,\qquad |H_d(x)_1|\le aM.$$

Thus the transverse maximum at least doubles at each iteration, while remaining larger than the first coordinate and S. Consequently Ω_d avoids $\mathcal{E}$. The nonempty escaping region also shows that $\Omega_d\neq\mathbb{C}^m$. For $x\in\Omega_d$, let $y=(y_1,\dots,y_m)$ denote its first forward iterate:

$$y=H_d(x),\qquad y_1=ax_m,\quad y_j=p_j(x_j)-ax_{j-1}\quad(2\le j\le m).$$

Set $M=\max(S,|x_1|)$. Since x and y avoid $\mathcal{E}$,

$$\|x\|_\infty\le M,\qquad |y_1|\le aM,\qquad \max_{j\ge 2}|y_j|\le\max(S,|y_1|)\le M.$$

Hence $|p_j(x_j)|=|ax_{j-1}+y_j|\le(1+a)M$. With $C=(1+a)S$ independent of d, this gives

$$|p_j(x_j)|\le C\max(1,|x_1|),\qquad 2\le j\le m. \tag{4.1}$$

In particular, $|x_1|\le c$ implies $|x_j|\le r_j(Cc)^{1/d}<b$ for large d, proving (2).

We claim that, for all sufficiently large d, every $x\in\Omega_d$ with $|x_1|\ge r_1$ has $x_1\cdots x_m\neq 0$ and satisfies

$$|y_j|\le qa|x_{j-1}|\qquad(2\le j\le m) \tag{4.2}$$

for some $q<1$ independent of d and x. There are two cases. If $|y_1|\le r_1$, we propagate lower bounds from x_1 to x_m. Estimate (4.1) at y gives $|y_j|\le r_j(Cr_1)^{1/d}$. Choose $0<\epsilon<r_m$ small enough that

$$q_0=\max_{2\le j\le m}\frac{r_j+\epsilon}{a(r_{j-1}-\epsilon)}<1.$$

For large d, $|y_j|\le r_j+\epsilon$. Starting from $|x_1|\ge r_1$, suppose inductively that $|x_{j-1}|\ge r_{j-1}-\epsilon$. Then

$$|y_j|\le r_j+\epsilon\le q_0a(r_{j-1}-\epsilon)\le q_0a|x_{j-1}|,$$

and $p_j(x_j)=ax_{j-1}+y_j$ yields

$$|x_j|\ge r_j\{(1-q_0)a(r_{j-1}-\epsilon)\}^{1/d}\ge r_j-\epsilon.$$

The last inequality holds for all j once d is large, since each quantity in braces is a fixed positive number. This closes the induction. If instead $w=|y_1|>r_1$, the equality $|x_m|=w/a$

allows backward propagation from x_m to x_1. Choose $r_2/r_1 < q_1 < 1$. Applying (4.1) at $y \in \Omega_d$ gives

$$|y_j| \le r_j (Cw)^{1/d} \qquad (2 \le j \le m).$$

The two estimates needed below are uniform for $w \ge r_1$, since

$$\frac{r_j (Cw)^{1/d}}{w} \le \frac{r_2 (Cr_1)^{1/d}}{r_1} \longrightarrow \frac{r_2}{r_1} < q_1, \qquad \frac{(w/(ar_j))^d}{w} \ge \frac{1}{r_1} \left(\frac{r_1}{ar_2} \right)^d \longrightarrow \infty.$$

Thus, for large d, $|y_j| \le q_1 w$ and $(w/(ar_j))^d \ge (1+q_1)w$. The backward induction step is

$$|x_{j-1}| \ge \frac{(|x_j|/r_j)^d - |y_j|}{a} \ge \frac{(1+q_1)w - q_1 w}{a} = \frac{w}{a} \quad \text{if } |x_j| \ge w/a.$$

Hence $|x_j| \ge w/a$ for every j, and therefore $|y_j| \le q_1 a |x_{j-1}|$. Both inductions give strictly positive lower bounds for every $|x_j|$. Taking $q = \max(q_0, q_1)$ proves (4.2).

Logarithmic tube coordinates. With $y = H_d(x)$, (4.2) places the ratios $p_j(x_j)/(ax_{j-1})$ in the closed disc centered at 1 with radius $q < 1$. Their principal logarithms

$$\xi_j = \log \frac{p_j(x_j)}{ax_{j-1}} = \log \left(1 + \frac{y_j}{ax_{j-1}} \right)$$

are single-valued on $\Omega_d \cap \{|x_1| > \alpha\}$ and satisfy $|\xi_j| \le B := -\log(1-q)$, where B is independent of d. The positive normalization factors ρ_j and holomorphic functions A_j are determined by $\rho_1 = 1$, $A_1 = 0$, and

$$\rho_j = r_j (a\rho_{j-1})^{1/d}, \qquad A_j = (A_{j-1} + \xi_j)/d \quad (2 \le j \le m).$$

Set $z_j = (x_j/\rho_j) e^{-A_j}$. The recurrences give

$$z_j^d = \frac{x_j^d}{a r_j^d \rho_{j-1}} e^{-A_{j-1} - \xi_j} = \frac{x_{j-1}}{\rho_{j-1}} e^{-A_{j-1}} = z_{j-1}.$$

Set $s = z_m = (x_m/\rho_m) e^{-A_m}$ using the last coordinate; this gives a single-valued holomorphic function without choosing a root of x_1. The displayed identities give $z_j = s^{d^{m-j}}$, hence

$$x_j = \rho_j s^{d^{m-j}} e^{A_j} \quad (1 \le j \le m). \tag{4.3}$$

In particular, $x_1 = s^D$ with $D = d^{m-1}$. Set $\zeta_j = \xi_j/(B+1)$ for $2 \le j \le m$. The corresponding parametrization is

$$\Phi_{d,j}(s,\zeta) = \rho_j s^{d^{m-j}} \exp \left(\sum_{\ell=2}^{j} \frac{(B+1)\zeta_\ell}{d^{j-\ell+1}} \right), \qquad 1 \le j \le m,$$

where the sum is empty for $j = 1$. This map is entire and inverts $\kappa = (s, \zeta)$ on $\Omega_d \cap \{|x_1| > \alpha\}$; in particular, $\Phi_d \circ \kappa = \mathrm{id}$ there. Since $|\zeta_j| \le B/(B+1) < 1$, property (3) follows.

Write $A_j(\zeta)$ for the exponent in the formula for $\Phi_{d,j}$. For $\zeta, \zeta' \in \overline{\Delta}_2^{m-1}$, the geometric sum gives

$$|A_j(\zeta)| \le K_d := \frac{2(B+1)}{d-1},$$
$$|A_j(\zeta) - A_j(\zeta')| \le \frac{4(B+1)}{d-1}.$$

Also $\rho_j \to r_j$ for $j \ge 2$ by its defining recurrence. For $|s| \le c^{1/D}$ and $j \ge 2$, this yields

$$|\Phi_{d,j}(s,\zeta)| \le \rho_j c^{1/d^{j-1}} e^{K_d} \longrightarrow r_j < b,$$

uniformly on the filled cylinder, while $|\Phi_{d,1}| \le c < b$. To estimate a fiber, integrate the derivative of the exponential along the segment between $A_j(\zeta)$ and $A_j(\zeta')$:

$$|\Phi_{d,j}(s,\zeta) - \Phi_{d,j}(s,\zeta')| \le \rho_j c^{1/d^{j-1}} e^{K_d} \frac{4(B+1)}{d-1} \longrightarrow 0.$$

Since the first coordinate is independent of ζ, these bounds give

$$\sup_{|s|=c^{1/D}} \operatorname{diam}_\infty \Phi_d(\{s\} \times \overline{\Delta}_2^{m-1}) = O(d^{-1}).$$

Taking d sufficiently large makes this diameter less than δ. Together with the filled-cylinder inclusion, this proves (4). Choose d large enough for all four properties, and take $\Omega = \Omega_d$, $h = h_d$, $\Phi = \Phi_d$. □

4.2. **Finite extension and exhaustion.**

Proposition 4.2. *Let Y be a complex manifold satisfying* LAP. *For an integer $m \geq 1$, let f be a holomorphic map from a neighborhood of $Q = \overline{\Delta}^m$ to Y. For every $R > 1$ and $\epsilon > 0$, there is a holomorphic map $\widetilde{f}$ from a neighborhood of $\overline{\Delta}_R^m$ to Y such that*

$$\sup_Q d_Y(\widetilde{f}, f) < \epsilon.$$

Proof. When $m = 1$, choose a scalar radius $r > 1$ inside the domain of f. Each boundary value belongs to a witness neighborhood W, so Lemma 3.2, with no passive variables, gives the required finite extension.

For $m \geq 2$, let h denote the basin biholomorphism from Lemma 4.1, to be chosen below close to the identity on Q. Write z for the original source point and $x = h(z)$ for its image in the basin. We retain $f \circ h$ on the inner region, enlarge the outer part in the scalar basin coordinate, and glue the two maps on an overlap of the form $c_0 < |h_1| < c_1$, with thresholds $c_0 < c_1$ to be chosen below.

Choose polydisc radii $1 < b_0 < b' < b$ such that f is holomorphic near $\overline{\Delta}_b^m$, and a chart spray $f^\sharp(x, v)$ centered at f near $\overline{\Delta}_{b'}^m \times \{0\}$. Take a smoothly bounded strongly convex domain $\overline{\Delta}_R^m \Subset D_0 \Subset \mathbb{C}^m$, and fix the scalar thresholds

$$1 < \alpha < c_0 < c_1 < c < b_0.$$

Compactness gives $\lambda > 0$ such that, for every $y \in f(\overline{\Delta}_{b_0}^m)$, the ambient d_Y-ball $B_{d_Y}(y, 3\lambda)$ lies in a witness neighborhood W. Choose a common spray radius $\beta > 0$ so small that

$$d_Y(f^\sharp(x, v), f(x)) < \lambda \quad (x \in \overline{\Delta}_{b_0}^m,\ v \in \overline{\Delta}_\beta^n).$$

By uniform continuity, choose $\delta > 0$ so that source points in $\overline{\Delta}_{b_0}^m$ at maximum-norm distance less than δ have f-images at distance less than λ. Choose $\eta > 0$ so small that $\|h - \mathrm{id}\,\|_Q < \eta$ implies $h(Q) \subset \Delta_{b_0}^m$ and

$$\sup_Q d_Y(f \circ h, f) < \epsilon/2.$$

Apply Lemma 4.1 with $b = b_0$ and these η, δ. Write $\Omega, h, D, \Phi, \kappa = (s, \zeta)$ for the resulting data, and set the scalar radius $r = c^{1/D}$. For every $|s| = r$, the points $\Phi(s, \zeta)$ and $\Phi(s, 0)$ lie in $\Delta_{b_0}^m$ and have maximum-norm distance less than δ, by the boundary-fiber estimate of Lemma 4.1. Hence their f-images have distance less than λ. Moving from $f(\Phi(s, \zeta))$ to $f^\sharp(\Phi(s, \zeta), v)$ costs less than a further λ. The triangle inequality therefore puts the whole compact fiber

$$f^\sharp(\Phi(s, \zeta), v), \qquad \zeta \in \overline{\Delta}_2^{m-1}, \quad v \in \overline{\Delta}_\beta^n,$$

in $B_{d_Y}(f(\Phi(s, 0)), 2\lambda)$. The ball of radius 3λ about the same center lies in a witness neighborhood W, so this fiber has the required containment with a positive margin.

If $|h_1| \leq c_0$ throughout D_0, Lemma 4.1(2) shows that $f \circ h$ is holomorphic on D_0, and we are done. Otherwise, the outer coordinates κ are defined near the compact set $h(\overline{D}_0) \cap \{|x_1| \geq c_0\}$, since $c_0 > \alpha$. Their transverse values lie in Δ_1^{m-1}. Choose an outer scalar radius $S > r$ such that $|s(x)| < S$ whenever $x \in h(\overline{D}_0)$ and $|x_1| \geq c_0$.

The map

$$k(s, \zeta, v) = f^\sharp(\Phi(s, \zeta), v)$$

is holomorphic near $\overline{\Delta}_r \times \overline{\Delta}_2^{m-1} \times \overline{\Delta}_\beta^n$. The larger transverse polydisc leaves a margin around the transverse values of κ, which lie in Δ_1^{m-1}. We retain the passive parameter domain $\overline{\Delta}_1^{m-1} \times \overline{\Delta}_\beta^n$. With this product fixed, Lemma 3.2 gives maps $G(s, \zeta, v)$ holomorphic near

$$\overline{\Delta}_S \times \overline{\Delta}_1^{m-1} \times \overline{\Delta}_\beta^n$$

and arbitrarily close to k on $\overline{\Delta}_r \times \overline{\Delta}_1^{m-1} \times \overline{\Delta}_\beta^n$.

Consider the fixed source pair

$$U_A = D_0 \cap \{|h_1| < c_1\}, \qquad U_B = D_0 \cap \{|h_1| > c_0\}.$$

It covers D_0, has separated closed fringes, and contains Q in $U_A \setminus \overline{U_B}$, because $|h_1| < \alpha < c_0$ on Q. Since $U_A \cup U_B = D_0$ is smoothly bounded and strongly convex, Lemma 2.3 supplies bounded splitting operators with a fixed constant M. On the two sides use

$$F_A(z, v) = f^\sharp(h(z), v), \qquad F_B(z, v) = G(\kappa(h(z)), v).$$

Since $c_1 < c$, Lemma 4.1(2) places $h(\overline{U_A})$ compactly in $\Delta_{b_0}^m$, so F_A is a chart spray near $\overline{U_A}$. The strict choice of S and the compact containment of the transverse values in Δ_1^{m-1} make F_B holomorphic near $\overline{U_B}$. On the overlap $U_A \cap U_B$,

$$\Phi(\kappa(h(z))) = h(z), \qquad |s(h(z))| \le c_1^{1/D} < r.$$

Consequently the pullback of k is exactly F_A, while that of G is F_B. The scalar approximation makes F_B arbitrarily close to F_A on the fixed overlap and the full spray polydisc.

Fix fiber radii $0 < \rho < \rho_+ < \beta$. The inverse-chart observation in Subsection 2.2 and Cauchy estimates across $\rho_+ - \rho$ show that, as the scalar approximation improves, the overlap transition converges uniformly to the identity on $\overline{\Delta}_{\rho_+}^n$ and its fiber derivative converges to I on Δ_ρ^n. The source pair and the splitting bound M are fixed, so Lemma 2.4 gives corrections tending uniformly to zero. On Q, the glued map is the old spray evaluated at its small correction; uniform continuity of that spray consequently gives a holomorphic map $\widetilde{f} : D_0 \to Y$ with

$$\sup_Q d_Y(\widetilde{f}, f \circ h) < \epsilon/2$$

when the scalar approximation is sufficiently accurate. Combining this with the choice of η gives

$$\sup_Q d_Y(\widetilde{f}, f) \le \sup_Q d_Y(\widetilde{f}, f \circ h) + \sup_Q d_Y(f \circ h, f) < \epsilon.$$

□

Iterating Proposition 4.2 with summable errors gives approximation by entire maps.

Theorem 4.3. *Let Y be a complex manifold satisfying* LAP. *Every holomorphic map from a neighborhood of a closed polydisc in $\mathbb{C}^m$ to Y can be approximated uniformly there by entire maps $\mathbb{C}^m \to Y$.*

Proof. Let f be the given map. Normalize its original polydisc to $Q = \overline{\Delta}^m$, fix $\epsilon > 0$, and set $f_0 = f$. Choose exhaustion radii $1 = R_0 < R_1 < \cdots \to \infty$ and apply Proposition 4.2 successively after rescaling. We obtain maps f_j holomorphic near $\overline{\Delta}_{R_j}^m$ such that

$$\sup_{\overline{\Delta}_{R_{j-1}}^m} d_Y(f_j, f_{j-1}) < \epsilon 2^{-j-1} \qquad (j \ge 1).$$

The summable bounds and completeness of d_Y give a locally uniform limit $F : \mathbb{C}^m \to Y$. In holomorphic coordinate charts on Y, local uniform convergence shows that F is holomorphic; cf. [40, §2]. Finally,

$$\sup_Q d_Y(F, f) \le \sum_{j \ge 1} \epsilon 2^{-j-1} = \epsilon/2 < \epsilon.$$

□

5. From LAP to the Oka property

Assume that Y satisfies LAP. We prove convex approximation by induction on the number r of variables in the general convex factor. The polydisc factor P is retained at every step. The property $(\mathcal{P}_r)$ below records this induction hypothesis.

Definition 5.1. A complex manifold Y satisfies property $(\mathcal{P}_r)$, where r is a nonnegative integer, if, for every compact convex set $K \subset \mathbb{C}^r$, every integer $s \geq 0$, every closed polydisc $P \subset \mathbb{C}^s$, and every map f holomorphic near $K \times P$ with values in Y, there are entire maps $\mathbb{C}^{r+s} \to Y$ approximating f uniformly on $K \times P$.

A factor of dimension zero is omitted. By Theorem 4.3, Y satisfies $(\mathcal{P}_0)$.

5.1. **Approximation with a real parameter.** The following argument uses the Stein-neighborhood and holomorphic retraction results in [28, Corollary 3.6.6 and Theorem 3.3.3], polynomial approximation in the real parameter, and the assumption that Y satisfies $(\mathcal{P}_{r-1})$.

Lemma 5.2. *Let $r \geq 1$ be an integer and Y a complex manifold satisfying $(\mathcal{P}_{r-1})$. Let $K = C \times P \subset \mathbb{C}^m$, where $C \subset \mathbb{C}^{r-1}$ is compact convex, $P \subset \mathbb{C}^q$ is a closed polydisc, and $m = r - 1 + q$. Let $I \subset \mathbb{R}$ be a nondegenerate compact interval and let $W \subset \mathbb{C}^m$ be an open neighborhood of K. If $H : I \times W \to Y$ is continuous and $H(t, \cdot)$ is holomorphic on W for every $t \in I$, then, for every tolerance $\epsilon > 0$, there is an entire map $F : \mathbb{C} \times \mathbb{C}^m \to Y$ such that*

$$\sup_{(t,w) \in I \times K} d_Y\big(F(t, w), H(t, w)\big) < \epsilon.$$

Proof. The proof has two steps. We first approximate H on $I \times K$ by a map holomorphic in a neighborhood of this product, using a graph embedding, Bernstein approximation in the real parameter, and a holomorphic retraction. We then use a Riemann map to turn the real parameter into a passive disc variable for $(\mathcal{P}_{r-1})$. A polynomial approximation to this change of variable gives an entire map in the original variables.

An affine change of the first variable reduces the proof to $I = [0, 1]$. Slightly enlarge C by a closed Euclidean ball and increase the radii of P to obtain a compact convex product K^+ with $K \subset \operatorname{int} K^+ \subset K^+ \subset W$. Such an enlargement remains in W by compactness of K. Choose also an open convex neighborhood X of K^+ with $X \subset W$. The domain X is Stein, and K^+ is $\mathcal{O}(X)$-convex because it is polynomially convex. Apply the theorem of Forstnerič and Wold [28, Corollary 3.6.6] (see also [36, Corollary 2.2]) with the Stein base X, the holomorphic submersion $X \times Y \to X$, the compact set K^+, and the continuous family of holomorphic sections $w \mapsto (w, H(t, w))$, parametrized by $t \in I$. It gives a Stein open neighborhood $\Theta \subset \mathbb{C} \times X \times Y$ of the compact total graph

$$\Sigma = \{(t, w, H(t, w)) : t \in I,\ w \in K^+\}.$$

By the Remmert–Bishop–Narasimhan embedding theorem for Stein manifolds [28, Theorem 2.4.1(a)], choose a closed holomorphic embedding $e : \Theta \hookrightarrow \mathbb{C}^N$. Siu's holomorphic neighborhood retraction theorem [59, Corollary 1] (see also [28, Theorem 3.3.3]), applied to the Stein submanifold $e(\Theta) \subset \mathbb{C}^N$, gives an open neighborhood R of $e(\Theta)$ and a holomorphic retraction $\rho : R \to e(\Theta)$.

By compactness, there is a common open neighborhood $W_0 \subset X$ of K^+ on which

$$g(t, w) = e(t, w, H(t, w)), \qquad (t, w) \in I \times W_0,$$

is defined. The map g is continuous and holomorphic in w. For $\nu \geq 1$ form the Bernstein polynomial

$$B_\nu g(t, w) = \sum_{j=0}^{\nu} \binom{\nu}{j} t^j (1 - t)^{\nu - j} g(j/\nu, w).$$

This is holomorphic on $\mathbb{C} \times W_0$. The real and imaginary parts of the coordinate functions $t \mapsto g(t, w)$, indexed by $w \in K^+$ and the finitely many coordinates, form a bounded equicontinuous family in $C([0, 1])$: boundedness and equicontinuity both follow from continuity on the compact product $I \times K^+$. The uniform Bernstein approximation theorem [46, Proposition 3.1] thus gives

$$\sup_{(t,w)\in I\times K^+} \|B_\nu g(t, w) - g(t, w)\| \longrightarrow 0.$$

Let $Q = \pi_Y \circ e^{-1} \circ \rho : R \to Y$, where π_Y is the projection onto Y. Since $Q \circ g = H$ and $g(I \times K^+)$ is compact in R, uniform convergence and continuity give a degree ν such that $B_\nu g(I \times K^+) \subset R$ and

$$\sup_{I\times K^+} d_Y(Q \circ B_\nu g, H) < \epsilon/3.$$

For this fixed ν, the map $H_1 = Q \circ B_\nu g$ is holomorphic on

$$\Omega = \{(t, w) \in \mathbb{C} \times W_0 : B_\nu g(t, w) \in R\},$$

an open neighborhood of $I \times K^+$.

Choose a thin rounded rectangle $V \supset I$ and an open neighborhood $W_1 \supset K$ with $V \times W_1 \subset \Omega$. Let $\varphi : V \to \Delta$ be a Riemann map, and choose radii $0 < \varrho < \varrho' < 1$ with $\varphi(I) \subset \Delta_\varrho$. The map

$$\widetilde{H}(\zeta, w) = H_1(\varphi^{-1}(\zeta), w)$$

is holomorphic on $\Delta \times W_1$. Write $w = (y, u) \in \mathbb{C}^{r-1} \times \mathbb{C}^q$. After permuting the variables from (ζ, y, u) to (y, ζ, u), the compact approximation set is $C \times (\overline{\Delta}_{\varrho'} \times P)$. Apply $(\mathcal{P}_{r-1})$, with ζ as one additional passive polydisc variable, to obtain an entire map $E : \mathbb{C} \times \mathbb{C}^m \to Y$ with

$$\sup_{(\zeta,w)\in\overline{\Delta}_{\varrho'}\times K} d_Y\big(E(\zeta, w), \widetilde{H}(\zeta, w)\big) < \epsilon/3.$$

With E fixed, uniform continuity on $\overline{\Delta}_{\varrho'} \times K$ gives a tolerance $0 < \eta < \varrho' - \varrho$ such that

$$d_Y\big(E(\zeta, w), E(\zeta', w)\big) < \epsilon/3 \quad \text{if } \zeta, \zeta' \in \overline{\Delta}_{\varrho'}, \quad |\zeta - \zeta'| < \eta, \ w \in K.$$

By polynomial approximation on a real interval [46, Proposition 3.1], applied to the real and imaginary parts of $\varphi|_I$, choose a complex polynomial p such that $\sup_I |p - \varphi| < \eta$. In particular, $p(I) \subset \Delta_{\varrho'}$. The map $F(t, w) = E(p(t), w)$ is entire in all $1 + m$ variables, and for $(t, w) \in I \times K$ the triangle inequality gives

$$\begin{aligned} d_Y(F(t, w), H(t, w)) &\le d_Y(E(p(t), w), E(\varphi(t), w)) \\ &\quad + d_Y(E(\varphi(t), w), H_1(t, w)) + d_Y(H_1(t, w), H(t, w)) < \epsilon. \end{aligned}$$ □

5.2. **Approximation on real faces.**

Lemma 5.3. *Let $r \ge 1$ be an integer and Y a complex manifold satisfying $(\mathcal{P}_{r-1})$. If $S \subset \mathbb{R} \times \mathbb{C}^{r-1}$ is compact convex and $P \subset \mathbb{C}^q$ is an arbitrary closed polydisc, then every map holomorphic near $S \times P$ with values in Y admits uniform approximation on $S \times P$ by entire maps $\mathbb{C}^{r+q} \to Y$.*

We adapt OpenAI's slicing and rescaling [55, Lemma 4.3] to arbitrary target dimension, using Lemma 5.2 in place of its real-parameter arguments [55, Lemmas 4.1–4.2]. The main point is to join neighboring entire approximants on a common polydisc in the y-variables while preserving their endpoints. Expansion occurs only where the interpolating path already equals an entire endpoint map. Approximation and contraction then give paths on the common polydisc; after endpoint correction, these paths patch along the real variable and Lemma 5.2 applies once more.

Proof. Fix a tolerance $\epsilon > 0$ and write the given map as $h(t, y, u)$, where $y \in \mathbb{C}^{r-1}$ and u is passive. For $r = 1$, choose a nondegenerate compact interval $I \supseteq S$ and an open neighborhood $W \supset P$ with $I \times W \subset \operatorname{dom} h$, and apply Lemma 5.2 with $K = P$. Assume henceforth that $r \geq 2$.

Local entire approximants and interpolation. Choose small compact convex thickenings $S^+ \subset \mathbb{R} \times \mathbb{C}^{r-1}$ of S and P^+ of P, with P^+ a closed polydisc, such that h is holomorphic on an open neighborhood U of $S^+ \times P^+$. Let I be the projection of S^+ onto $\mathbb{R}$ and put

$$F_t = \{y : (t, y) \in S^+\} \qquad (t \in I).$$

Since S^+ is compact and convex, its projection I is a compact interval and each F_t is nonempty, compact, and convex. Compactness also gives upper semicontinuity: for every $a \in I$ and every open set $V \subset \mathbb{C}^{r-1}$ containing F_a, we have $F_t \subset V$ for all $t \in I$ sufficiently close to a. Otherwise, there would be $t_\nu \in I$ tending to a and $y_\nu \in F_{t_\nu} \setminus V$. Passing to a subsequence, we obtain $(t_\nu, y_\nu) \to (a, y) \in S^+$ with $y \notin V$, contradicting $F_a \subset V$.

Compactness and upper semicontinuity give, for each $a \in I$, a small closed ball thickening C_a of F_a and an open complex disc E_a centered at a such that

$$E_a \times C_a \times P^+ \subset U, \qquad F_t \subset \operatorname{int} C_a \quad (t \in E_a \cap I).$$

Take a finite subdivision $I = I_1 \cup \cdots \cup I_N$ with $N \geq 2$ and each $I_i \Subset E_a$ for an assigned a. For each I_i, slightly enlarge its diameter disc to a closed disc $D_i \Subset E_a$, so that $I_i \Subset \operatorname{int} D_i$, and put $C_i = C_a$. At each division point b_i, choose compact convex neighborhoods C_i^-, C_i^0 of F_{b_i} and pairwise disjoint closed joining intervals J_i about b_i, with $F_t \subset C_i^-$ for $t \in J_i$, such that

$$C_i^- \Subset \operatorname{int} C_i^0 \Subset \operatorname{int}(C_i \cap C_{i+1}),$$
$$J_i \Subset \operatorname{int} D_i^0, \qquad D_i^0 \Subset \operatorname{int}(D_i \cap D_{i+1})$$

for closed complex discs D_i^0. Using Lemma 2.2 and the Oka–Grauert principle on convex Stein neighborhoods in U, fix chart sprays $h_i^\sharp$ centered at h over $D_i^0 \times C_i^0 \times P$ and strictly smaller fiber polydiscs whose images are within $\epsilon/3$ of h. Choose a common threshold less than $\epsilon/3$ keeping inverse coordinates in these polydiscs, then apply the property $(\mathcal{P}_{r-1})$ satisfied by Y, with (t, u) passive, to obtain entire maps g_i within this threshold of h on $D_i \times C_i \times P^+$.

Figure 4 shows the two factors of a common joining neighborhood.

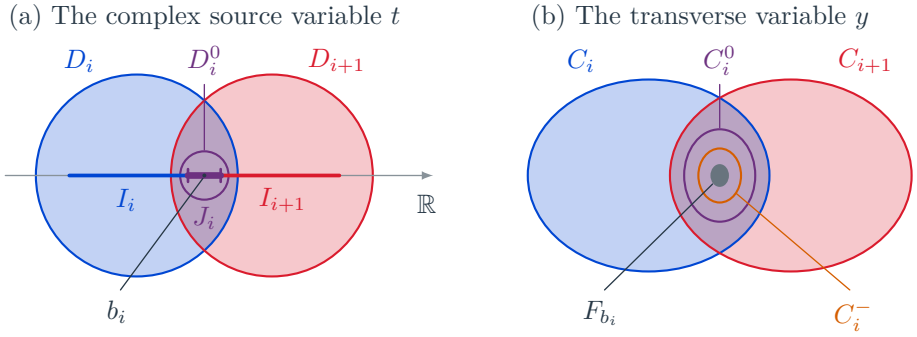


FIGURE 4. Joining neighborhoods in Lemma 5.3. Overlapping red and blue regions appear purple. The interval J_i extends to both sides of the common endpoint b_i. For real $t \in J_i$, the slices F_t lie in C_i^-; a common chart joins g_i and g_{i+1} on $D_i^0 \times C_i^0 \times P$. The passive factor P is omitted; the convex sets in the y-variables are shown schematically.

Fix a join and write $D = D_i^0$, $C^- = C_i^-$, $C = C_i^0$, $h^\sharp = h_i^\sharp$, $g_0 = g_i$, and $g_1 = g_{i+1}$. Here $x = (t, y, u)$ denotes a source point, while the real parameter $\tau \in [0, 1]$ indexes the path of

holomorphic maps. Near $D \times C \times P$, the holomorphic chart coordinates a_j of g_j lie in the chosen fiber polydisc. For a smooth $\chi : [0,1] \to [0,1]$ equal to 0 near 0 and 1 near 1, put

$$g_j(x) = h^\sharp(x, a_j(x)) \quad (j = 0, 1),$$
$$h_\tau(x) = h^\sharp(x, (1 - \chi(\tau))a_0(x) + \chi(\tau)a_1(x)).$$

The coordinate segments remain in that polydisc, giving a smooth path of holomorphic maps on a common neighborhood of $D \times C \times P$, equal to g_j near $\tau = j$, with

$$\sup_{\tau\in[0,1]} \sup_{x\in D\times C\times P} d_Y\big(h_\tau(x), h(x)\big) < \epsilon/3. \tag{5.1}$$

Expansion, approximation, and contraction. The interpolation is initially defined near C, but its endpoints must be corrected on one polydisc B common to all joins. Expanding the y-variable near the entire endpoint maps will make approximation on C control those endpoints on all of B after contraction. Fix a common closed polydisc $B \subset \mathbb{C}^{r-1}$ containing every C_i in its interior. Choose any $d \in \operatorname{int} C$ and then an expansion factor $L > 1$ sufficiently large that

$$s_L(B) := d + L^{-1}(B - d) \Subset \operatorname{int} C.$$

On disjoint relatively open endpoint intervals U_j whose closures lie where $h_\tau = g_j$, extend h_τ to all source points as g_j. Choose smooth $\lambda : [0,1] \to [1, L]$ with

$$\operatorname{supp}_{[0,1]}(\lambda - 1) \subset U_0 \cup U_1, \qquad \lambda = L \text{ near } 0, 1,$$

and expand the transverse variable:

$$\widetilde{h}_\tau(t, y, u) = h_\tau(t, d + \lambda(\tau)(y - d), u).$$

Expansion occurs only where h_τ is entire; near the inner endpoints of U_j the extension agrees with h_τ and $\lambda = 1$. Thus $\widetilde{h}_\tau$ is smooth in τ and holomorphic in (t, y, u) on a common neighborhood of $D \times C \times P$.

Before approximating, fix closed endpoint intervals V_j, W_j with $W_j \subset \operatorname{int}_{[0,1]} V_j$ and V_j in the relative interior of the region where $\lambda = L$ and $h_\tau = g_j$, and smooth cutoffs $\eta_j : W_j \to [0,1]$ equal to 0 near j and 1 near the inner endpoint. Figure 5(a) shows these nested endpoint intervals. Using Lemma 2.2 again, fix endpoint chart sprays $g_j^\sharp$ near $K_B = D \times B \times P$ and strictly smaller fiber polydiscs whose images are within $\epsilon/3$ of g_j. Then choose a tolerance $0 < \delta < \epsilon/3$ so that $d_Y(q, g_j(x)) < \delta$, $x \in K_B$, ensures that the chart inverse at (x, q) is defined and its fiber coordinate lies in the smaller polydisc, for both j.

Apply Lemma 5.2, with convex factor C and (t, u) passive, to obtain an entire map $E(\tau, t, y, u)$ within δ of $\widetilde{h}_\tau$ on $[0,1] \times D \times C \times P$. Undo the expansion by putting

$$s_\tau(y) = d + \lambda(\tau)^{-1}(y - d), \qquad A_\tau(t, y, u) = E(\tau, t, s_\tau(y), u).$$

These are entire source maps depending smoothly on real τ. Convexity gives

$$\bigcup_{\tau\in[0,1]} s_\tau(C^-) \subset \operatorname{conv}(C^- \cup \{d\}) \Subset \operatorname{int} C.$$

Since the affine rescalings cancel on U_j and $\lambda = 1$ elsewhere,

$$\widetilde{h}_\tau(t, s_\tau(y), u) = h_\tau(t, y, u) \qquad (\tau \in [0,1],\ (t, y, u) \in D \times C^- \times P).$$

Hence

$$d_Y(A_\tau(x), h_\tau(x)) < \delta \quad (\tau \in [0,1],\ x \in D \times C^- \times P). \tag{5.2}$$

For $\tau \in V_j$, $s_\tau = s_L$ sends B into C and $\widetilde{h}_\tau(t, s_L(y), u) = g_j(t, y, u)$ on K_B, giving

$$d_Y(A_\tau(x), g_j(x)) < \delta \quad (\tau \in V_j,\ x \in K_B). \tag{5.3}$$

Figure 5 shows how cancellation of the rescalings gives endpoint control on all of B, allowing exact correction in the endpoint charts.

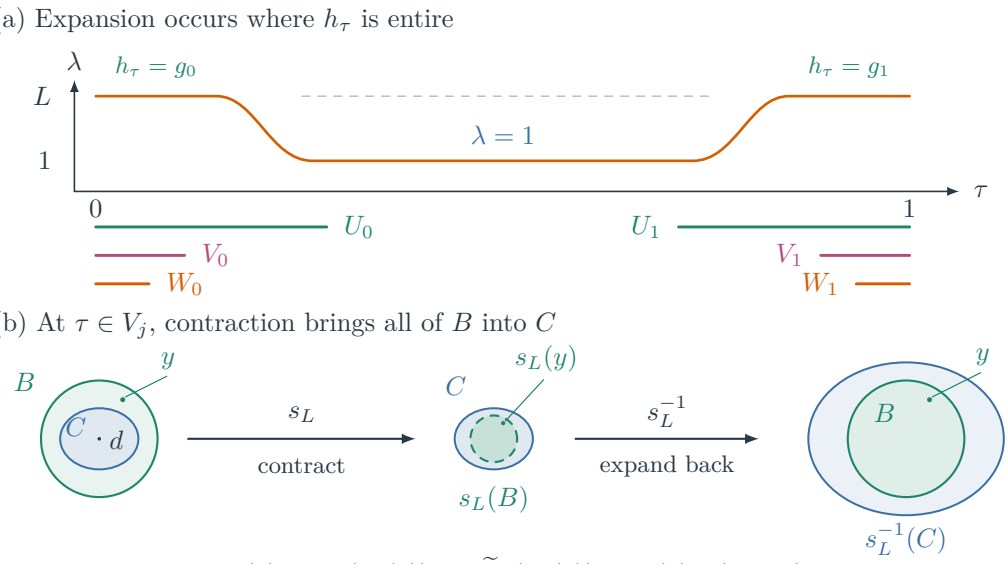


FIGURE 5. Endpoint rescaling in Lemma 5.3. (a) Expansion occurs only where h_τ is entire; V_j lies where $\lambda = L$, and correction takes place on W_j. (b) For $\tau \in V_j$, evaluating the approximation on C at $s_L(y)$ controls every $y \in B$, permitting exact correction near $\tau = j$. Here $E_\tau(y) = E(\tau, t, y, u)$, with (t, u) fixed. Regions and interval lengths are schematic.

Exact endpoints and patching. The fiber inverse coordinates $\beta_{j,\tau}(x) = (g_j^\sharp)_x^{-1}(A_\tau(x))$ are defined for $\tau \in V_j$ on K_B by (5.3). For this fixed E, compactness of $V_j \times K_B$ gives one common neighborhood of K_B on which they remain in the smaller fiber polydiscs, smoothly in τ and holomorphically in x. Set

$$\widehat{A}_\tau(x) = \begin{cases} g_j^\sharp(x, \eta_j(\tau)\beta_{j,\tau}(x)), & \tau \in W_j,\ j = 0, 1, \\ A_\tau(x), & \tau \notin W_0 \cup W_1. \end{cases}$$

Since $\eta_j = 1$ near the inner endpoints and 0 near j, this is a smooth family of holomorphic maps on a common neighborhood of K_B, exactly equal to g_j near $\tau = j$. On W_j the cutoff coordinates stay in the smaller fiber polydisc, giving error less than $\epsilon/3$ from $g_j = h_\tau$; off $W_0 \cup W_1$, (5.2) applies. Thus

$$\sup_{\tau \in [0,1]} \sup_{x \in D \times C^- \times P} d_Y(\widehat{A}_\tau(x), h_\tau(x)) < \epsilon/3. \tag{5.4}$$

For each join, take a smooth $\rho_i : J_i \to [0, 1]$ equal to 0 near the left endpoint and 1 near the right endpoint, and set

$$H(t, y, u) = \widehat{A}^i_{\rho_i(t)}(t, y, u) \quad (t \in J_i),$$

using the corresponding g_i between joins. Each corrected path equals its endpoint maps near the endpoints of its parameter interval, so these formulas agree on neighborhoods of the endpoints of J_i. Each join has a common neighborhood of $D_i^0 \times B \times P$; compactness of the J_i and a finite intersection give a common neighborhood of $B \times P$ on which H is smooth in real t and holomorphic in (y, u).

For $(t, y) \in S$, $t \in J_i$, we have $y \in C_i^-$, so (5.1) and (5.4) give error less than $2\epsilon/3$ for all $u \in P$. Off the joins the error is less than $\epsilon/3$. Apply Lemma 5.2 to H on $I \times B \times P$, with convex factor B and passive polydisc P, with error less than $\epsilon/3$. The resulting entire map is within ϵ of h on $S \times P$. □

5.3. **Convex approximation.** The strategy is that of Kusakabe's proof of the elliptic characterization [52], which uses local holomorphic right inverses, Oka–Weil approximation, and spray gluing. Near the cutting face, we first approximate the old chart spray by an entire family. We then solve for the fiber parameter that makes this family agree with the old spray. Approximating that parameter map by polynomials gives an entire family whose transition to the old spray is arbitrarily close to the identity on a fixed overlap. Fiberwise Cauchy estimates then allow the center-gluing lemma to apply.

Lemma 5.4. *Let $r \geq 1$ be an integer and Y a complex manifold satisfying $(\mathcal{P}_{r-1})$. Let $Q \subset \mathbb{C}^r$ be compact convex, let $P \subset \mathbb{C}^q$ be a closed polydisc, and suppose that $Q_- = Q \cap \{\Im z_1 \leq 0\}$ has nonempty interior in $\mathbb{C}^r$. If f is holomorphic on a neighborhood of $Q_- \times P$ with values in Y, then, for every tolerance $\epsilon > 0$, there is a map $\widetilde{f}$ holomorphic on a neighborhood of $Q \times P$ with values in Y such that*

$$\sup_{Q_- \times P} d_Y(\widetilde{f}, f) < \epsilon.$$

Proof. If $Q = Q_-$, take $\widetilde{f} = f$. Otherwise set $T = Q \cap \{\Im z_1 = 0\}$ and write $Z = (z, u) \in \mathbb{C}^{r+q}$. Choose a bounded convex open neighborhood S of $Q_- \times P$ with closure in the domain of f. It is contractible and Stein. The Oka–Grauert principle [28, Theorem 5.3.1(ii)] makes f^*TY holomorphically trivial on S, so Lemma 2.2 gives a chart spray $F(Z, v)$ centered at f, with fiber radius R and domain $N \times \Delta^n_R$, where $Q_- \times P \subset N \Subset S$ and $n = \dim Y$. For $n > 0$, choose a fiber radius $b > 0$ with $6b < R$.

A common fiber inverse. First we construct a single holomorphic fiber inverse near the whole cutting face (for a related implicit function theorem for sprays, see [51]). In the real hyperplane $\{\Im z_1 = 0\} \cong \mathbb{R} \times \mathbb{C}^{r-1}$ choose a compact convex neighborhood T^+ of T, and choose a slightly larger closed polydisc P^+ with $P \subset \operatorname{int} P^+$, so that $T^+ \times P^+ \subset N$. Apply Lemma 5.3 to F, regarding (u, v) as passive variables. We obtain an entire map $G(Z, w)$ approximating $F(Z, w)$ as closely as desired on

$$T^+ \times P^+ \times \overline{\Delta}^n_{5b}.$$

The graph map $(Z, w) \mapsto (Z, F(Z, w))$ is biholomorphic onto its open image. Its image of $T^+ \times P^+ \times \overline{\Delta}^n_{4b}$ is compact in that image. Therefore sufficiently close approximation ensures that

$$H(Z, w) = F(Z, \cdot)^{-1}(G(Z, w))$$

is defined on a neighborhood of $T^+ \times P^+ \times \overline{\Delta}^n_{4b}$ and is uniformly close to w there. By first making this difference sufficiently small on $\overline{\Delta}^n_{4b}$ and then using Cauchy estimates *only in the fiber variables*, we can arrange

$$\|H(Z, w) - w\| < b/4, \qquad \|D_w H(Z, w) - I\| < 1/4 \quad \text{on } T^+ \times P^+ \times \overline{\Delta}^n_{3b}.$$

We use the maximum norm on fibers and its induced operator norm on fiber derivatives.

Fix this choice of G. Compactness, openness of the chart image, and the strict inequalities just obtained give a complex open neighborhood U of $T \times P$, with $U \Subset N$, on which H is defined for w near $\overline{\Delta}^n_{3b}$ and

$$\|H(Z, w) - w\| < b/2, \qquad \|D_w H(Z, w) - I\| < 1/2.$$

For $Z \in U$ and $v \in \Delta^n_{2b}$, the map

$$w \longmapsto v - \big(H(Z, w) - w\big)$$

sends $\overline{\Delta}^n_{3b}$ into $\Delta^n_{5b/2}$ and is a contraction with constant at most $1/2$. Its unique fixed point, denoted by $W(Z, v)$, depends holomorphically on (Z, v): the successive iterates starting at

$w = v$ are holomorphic and converge locally uniformly, with a uniform geometric estimate. Thus

$$W : U \times \Delta_{2b}^n \longrightarrow \Delta_{3b}^n,$$
$$H(Z, W(Z, v)) = v, \qquad G(Z, W(Z, v)) = F(Z, v).$$

Fixing the source geometry. With G, W, and U fixed, set $K_0 = Q \times P$ and $h(Z) = \Im z_1$. Compactness first gives a half-width $\delta > 0$ for which $K_0 \cap \{h \le 3\delta\} \subset N$ and $K_0 \cap \{|h| \le 3\delta\} \subset U$. A sufficiently small bounded open neighborhood V of K_0 then satisfies

$$\overline{V} \cap \{h \le 2\delta\} \subset N, \qquad \overline{V} \cap \{|h| \le 2\delta\} \subset U.$$

Take a smooth convex approximation of the maximum of the affine defining functions of a bounded polyhedral neighborhood of K_0 whose closure lies in V, and add a small positive quadratic term. A suitable regular sublevel set then gives a smoothly bounded strongly convex domain $K_0 \Subset D \Subset V$, so

$$\overline{D} \cap \{h \le 2\delta\} \subset N, \qquad L := \overline{D} \cap \{|h| \le 2\delta\} \Subset U.$$

Set

$$A = D \cap \{h < \delta\}, \qquad B = D \cap \{h > -\delta\}, \qquad C = A \cap B.$$

Then $A \cup B = D$, the closed fringes are separated by the strip $-\delta < h < \delta$, and $\overline{C} \subset L$; see Figure 6. Since $A \cup B = D$ is smoothly bounded and strongly convex, the first assertion of Lemma 2.3, applied in all source variables $Z = (z, u)$, gives fixed splitting operators with bound M.

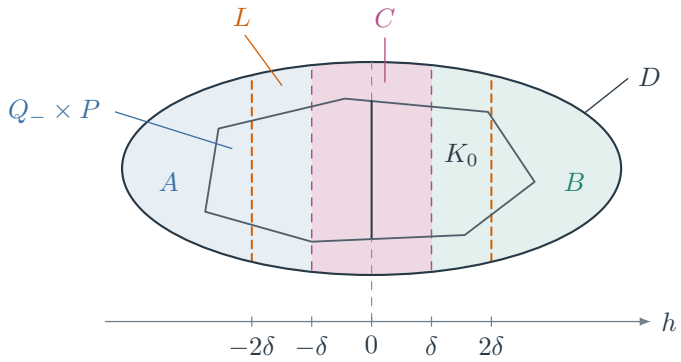


FIGURE 6. The source domains in Lemma 5.4, with $K_0 = Q \times P$ and $h = \Im z_1$, shown in a schematic real section. The old compact set $Q_- \times P$ lies in A. Approximation of the fiber inverse W over the wider compact collar L controls the transition over $C = A \cap B$, where the corrected centers agree.

Polynomial approximation and gluing. To make the new family entire, we will approximate W by polynomial maps q_j and use $G(Z, q_j(Z, v))$. The identity $H(Z, W(Z, v)) = v$ will control its transition to F. The compact L is convex. Hence $L \times \overline{\Delta}_b^n$ is polynomially convex, and W is holomorphic on a neighborhood of this product. The Oka–Weil theorem [28, Theorem 2.3.1], applied to its components, gives polynomial maps q_j with

$$\eta_j := \|q_j - W\|_{L \times \overline{\Delta}_b^n} \longrightarrow 0.$$

Choose once and for all fiber radii $0 < \rho < b' < b$, and take $\rho_+ \in (\rho, b')$. On the compact graph of W over $L \times \overline{\Delta}_{b'}^n$, the already constructed H is defined and satisfies $H(Z, W(Z, v)) = v$. This graph has a neighborhood compactly contained in the domain of H, on which its fiber

differential is bounded. For all large j, the segments from $W(Z,v)$ to $q_j(Z,v)$ remain in that neighborhood. It follows that the transition

$$H(Z,q_j(Z,v)) = v + c_j(Z,v)$$

is defined on a neighborhood of $L \times \overline{\Delta}^n_{b'}$ and, for a bound C_0 on that fiber differential, independent of j,

$$\|c_j\|_{L\times\overline{\Delta}^n_{b'}} \le C_0\eta_j.$$

The preceding estimates ensure that $G(Z,q_j(Z,v))$ lies in the F-chart on the overlap. Fiberwise Cauchy estimates on the nested polydiscs give

$$e_j := \|c_j\|_{C\times\Delta^n_\rho} \le C_0\eta_j, \qquad \kappa_j := \|D_v c_j\|_{C\times\Delta^n_\rho} \le \frac{nC_0}{b'-\rho}\eta_j.$$

All domains, radii, and the splitting constant M have been fixed before the accuracy η_j is chosen. We can therefore take j so large that

$$C_0\eta_j < \rho_+ - \rho, \qquad Me_j < \rho/2, \qquad M\kappa_j < 1/2.$$

The first inequality ensures that $v + c_j(Z,v) \in \Delta^n_{\rho_+}$ for $(Z,v) \in C \times \Delta^n_\rho$, so the transition takes values in the smaller chart used below.

Use Lemma 2.4 with

$$F_A(Z,v) = F(Z,v), \qquad F_B(Z,v) = G(Z,q_j(Z,v)).$$

The first family is a chart spray on $A \times \Delta^n_{\rho_+}$; the second is entire in (Z,v). The lemma gives corrections a_j on A and β_j on B, with $\|a_j\|_A + \|\beta_j\|_B \le Me_j$, and a holomorphic map on D defined by

$$\widetilde{f}_j(Z) = \begin{cases} F(Z,a_j(Z)), & Z \in A, \\ G(Z,q_j(Z,\beta_j(Z))), & Z \in B. \end{cases}$$

The two expressions agree on C by the transition identity. Since $Q_- \times P \subset A$ and $a_j \to 0$ uniformly, uniform continuity of F on a compact neighborhood of $(Q_- \times P) \times \{0\}$ gives

$$\sup_{Q_-\times P} d_Y(\widetilde{f}_j, f) \longrightarrow 0.$$

A sufficiently large j proves the lemma, with domain D containing the whole enlarged product $Q \times P$. □

Proof that LAP *implies* CAP. We prove by induction on r that Y satisfies $(\mathcal{P}_r)$. The case $r = 0$ follows from Theorem 4.3. For $r \ge 1$, assume that Y satisfies $(\mathcal{P}_{r-1})$. Let $K \subset \mathbb{C}^r$ be nonempty compact convex, $P \subset \mathbb{C}^q$ a closed polydisc, and f a map holomorphic near $K \times P$ with values in Y. Fix a tolerance $\epsilon > 0$. By compactness, a small convex neighborhood of K, multiplied by the full compact P, still lies in the domain of f. Enclose K in the interior of a full-dimensional real polytope K' in this neighborhood. Thus $K' \times P \Subset \operatorname{dom} f$. Choose a closed polydisc Q containing K' in its interior and write

$$K' = \bigcap_{j=1}^N \{\ell_j \le c_j\}, \qquad Q_k = Q \cap \bigcap_{j=k+1}^N \{\ell_j \le c_j\} \quad (0 \le k \le N),$$

with nonconstant real affine forms ℓ_j. Then $Q_0 = K'$, $Q_N = Q$, and

$$Q_{k-1} = Q_k \cap \{\ell_k \le c_k\}.$$

At each step we remove one half-space constraint and use Lemma 5.4 to extend the map approximately from $Q_{k-1} \times P$ to $Q_k \times P$. Every Q_k contains K' and therefore has nonempty interior.

To apply Lemma 5.4 to such a cut, write $\ell_k(z) = \Re \sum_{j=1}^r a_j z_j + d$ and introduce

$$w_1 = i\Big(\sum_{j=1}^r a_j z_j + d - c_k\Big).$$

Since $(a_1, \dots, a_r) \neq 0$, this extends to an invertible complex affine coordinate system on $\mathbb{C}^r$, and $\Im w_1 = \ell_k - c_k$. Convexity and holomorphy are preserved by this change of coordinates, and the passive polydisc P is unchanged. Thus the lemma applies after the coordinate change and its conclusion pulls back to the original variables.

Choose positive error tolerances ϵ_k with $\sum_{k=1}^N \epsilon_k < \epsilon/2$. Starting with $f_0 = f$, apply the lemma successively to obtain f_k holomorphic on a neighborhood of $Q_k \times P$ and satisfying

$$\sup_{Q_{k-1}\times P} d_Y(f_k, f_{k-1}) < \epsilon_k \qquad (1 \le k \le N).$$

If a cut is redundant, take $f_k = f_{k-1}$ instead. At each stage f_k is holomorphic near the full product $Q_k \times P$, so the next step uses the same passive polydisc P and a new chart spray with its own auxiliary margins. Since $K \subset Q_{k-1}$ for every k, the triangle inequality now gives

$$\sup_{K\times P} d_Y(f_N, f) < \epsilon/2.$$

Finally $Q \times P$ is a closed polydisc and f_N is holomorphic near it. Theorem 4.3 gives an entire map $g : \mathbb{C}^{r+q} \to Y$ with $\sup_{Q\times P} d_Y(g, f_N) < \epsilon/2$. Hence $\sup_{K\times P} d_Y(g, f) < \epsilon$, proving that Y satisfies $(\mathcal{P}_r)$. Induction gives this property for every $r \ge 0$; taking $q = 0$ shows that Y has CAP. □

Completion of the proof of Theorem 1.5. Every Oka manifold is strongly dominable [28, Theorem 5.5.1(e)], and Proposition 2.1 gives LAP from strong dominability. The preceding induction proves that LAP implies CAP, hence the Oka property. These implications prove the theorem. □

6. Equivalence between LAP and LEP

6.1. **From LEP to LAP.** We will pass from the centered families in LEP to LAP by successively substituting holomorphic scalar functions into them. The lemma below shows that the substitutions used here preserve approximation from an enlarged disc, with the remaining source variables as holomorphic parameters. Its proof adapts the gluing idea in OpenAI's two-strip argument [55, Proposition 2.2] to a thin overlap. Special planar pairs are as in Definition 1.2, and source splittings are made on slightly larger neighborhoods.

The model to keep in mind is a centered family $h_\nu : V \times \Delta_{R_\nu} \to Y$, with $h_\nu(y, 0) = y$. Here s denotes the full source point, including the holomorphic parameters; w is a spray variable allowing a small change of the old map, and t is the scalar family parameter into which a holomorphic function of s is substituted. For a small chart spray $G(s, w)$ around f_0, with image in V, we use

$$T_\nu(s, w, t) = h_\nu(G(s, w), t).$$

The approximation class in the lemma consists of maps on the old product that can already be approximated from the enlarged product; substitution into the limiting family preserves this class.

Lemma 6.1. *Let $D \subset D'$ be a special planar pair and, for an integer $q \ge 0$ and a closed polydisc $P \subset \mathbb{C}^q$, set $K = D \times P$ and $L = D' \times P$. The class $\mathcal{A}$ consists of maps to Y holomorphic near K that are uniformly approximable on K by maps to Y holomorphic near L.*

Suppose $f_0 \in \mathcal{A}$, $\dim Y = n$, and

$$G : \Omega \times \Delta_r^n \longrightarrow Y, \qquad G(s, 0) = f_0(s),$$

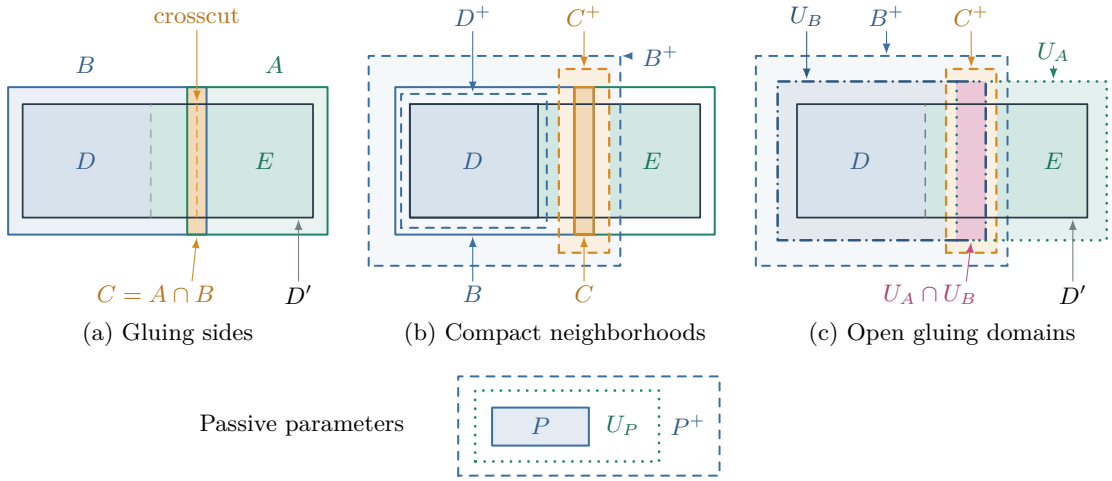
is a chart spray on a fixed neighborhood Ω of K; that is, $(s,w)\mapsto(s,G(s,w))$ is biholomorphic onto its image. Suppose the holomorphic maps

$$T_\nu : \Omega \times \Delta_r^n \times \Delta_{R_\nu} \longrightarrow Y, \qquad T_\nu(s,w,0) = G(s,w), \qquad R_\nu \longrightarrow \infty,$$

converge locally uniformly on $\Omega \times \Delta_r^n \times \Delta_\tau$ to T, for some $\tau > 0$. If φ is a scalar holomorphic function near K and $\|\varphi\|_K < \tau$, then $T(\cdot,0,\varphi) \in \mathcal{A}$.

The extra spray variable lets us recenter the construction at an approximant defined near the enlarged product. At scalar parameter zero, the recentered family agrees with the reference spray on a full fiber polydisc, including the nonzero parameters later used by the gluing corrections. For a related use of spray approximation and gluing, see the proof of [26, Proposition 3.1]. A scalar polynomial follows φ on the old product and is close to zero on the overlap (Figure 7). It therefore gives the desired substitution on the old side and a small transition to the reference spray where we glue. The domains and fiber margins are fixed first; the family index ν is chosen last.

In the following, by a crosscut we mean a simple arc with endpoints on the boundary of a disc and interior contained in the interior of the disc.



(a) Gluing sides (b) Compact neighborhoods (c) Open gluing domains

FIGURE 7. The planar choices in Lemma 6.1, with identical coordinates and scale in all three panels. (a) $D' = D \cup E$ is covered by the thickened sides A, B, and $C = A \cap B$ is disjoint from D. (b) The disjoint compact neighborhoods D^+, C^+ and the set B lie in $\operatorname{Int} B^+$. (c) The open domains satisfy $D' \subset U_A \cup U_B$, $U_A \cap U_B \Subset \operatorname{Int} C^+$, and $U_B \Subset \operatorname{Int} B^+$. The separate inset shows $P \Subset U_P \Subset \operatorname{Int} P^+$; dots mark U_A, U_P; dash-dots mark U_B.

Proof. Step 1: fix the old approximant, domains, and normalization. Fix an approximation tolerance $\epsilon > 0$ and an intermediate scalar radius $\|\varphi\|_K < t_1 < \tau$. Since $f_0 \in \mathcal{A}$, maps g holomorphic near L approximate f_0 uniformly on K. By the inverse-chart observation in Subsection 2.2,

$$\rho(s) = G(s,\cdot)^{-1}(g(s))$$

is holomorphic near K for sufficiently close g, and $\|\rho\|_K \to 0$ as $g \to f_0$ uniformly on K. Uniform continuity of T on a compact neighborhood of $\{(s,0,\varphi(s)) : s \in K\}$ therefore allows us to fix one such g with $\rho(K) \Subset \Delta_r^n$ and

$$\sup_K d_Y\big(T(s,\rho(s),\varphi(s)), T(s,0,\varphi(s))\big) < \epsilon/3.$$

A crosscut in the attached disc, close to but disjoint from D, gives a thin closed collar B containing D and a complementary closed region A, with slightly thickened boundaries,

such that $D' \subset \mathrm{Int}(A \cup B)$ and the overlap $C = A \cap B$ is a closed disc disjoint from D; see Figure 7(a). Take the collar sufficiently thin that the original G-chart and ρ are available near its product with P. Choose disjoint closed disc neighborhoods D^+, C^+ of D, C, a slightly larger closed neighborhood B^+ of B containing them, as in Figure 7(b), and fixed polydiscs in the remaining q source coordinates

$$P \Subset \mathrm{Int}\, P^+ \Subset \mathrm{Int}\, P^{++}.$$

Here Int denotes topological interior. These coordinates are *passive* only in the geometric construction: we enlarge the planar disc while keeping these polydiscs fixed. The maps may still depend holomorphically on all source coordinates. Make these choices inside the relevant holomorphic domains, with $B^+ \times P^{++} \Subset \Omega$, with g in the image of the G-chart there, and with φ holomorphic near $D^+ \times P^+$. Fix separated open gluing domains U_A, U_B and an open polydisc U_P satisfying

$$D' \subset U_A \cup U_B, \quad P \Subset U_P \Subset \mathrm{Int}\, P^+, \quad U_A \cap U_B \Subset \mathrm{Int}\, C^+, \quad U_B \Subset \mathrm{Int}\, B^+.$$

Figure 7(c) shows the planar choices. Their planar additive-splitting constant is now fixed.

Take a chart spray $F(s, v)$ centered at g over a neighborhood of the two sides. Lemma 2.2 applies on a sufficiently small contractible Stein neighborhood of the disc product. Choose a fixed fiber radius $b > 0$ so that F is available for $v \in \Delta^n_{4b}$ and

$$W(s, v) = G(s, \cdot)^{-1}(F(s, v))$$

is holomorphic near $B^+ \times P^{++} \times \overline{\Delta}^n_{3b}$, with image compactly inside Δ^n_r. Here $W(s, 0) = \rho(s)$. Set

$$Q_\nu(s, v, t) = T_\nu(s, W(s, v), t), \qquad Q(s, v, t) = T(s, W(s, v), t).$$

These families converge locally uniformly for $|t| < \tau$, and, for every allowed fiber value,

$$Q_\nu(s, v, 0) = Q(s, v, 0) = F(s, v). \tag{6.1}$$

Normalization on the whole fiber, rather than only at its center, makes the overlap transition for the limiting family tend to the identity as the scalar parameter tends to zero. The fixed fiber margin then gives derivative control by Cauchy estimates.

Step 2: choose the accuracies, then the scalar polynomial. By uniform continuity of Q, choose a correction bound $0 < \alpha < b$ and a scalar tolerance $\xi > 0$ so that, for $s \in K$, $\|v\| \le \alpha$, and $|u - \varphi(s)| < \xi$, one has $|u| < t_1$ and

$$d_Y\big(Q(s, v, u), Q(s, 0, \varphi(s))\big) < \epsilon/3.$$

Let M be the fixed splitting bound and choose a transition tolerance $e > 0$ with $Me < \min\{\alpha, b/2\}$ and $Mne/b < 1/2$. For a transition $v + c(s, v)$ with $\|c\|_\infty < e$ on Δ^n_{2b}, the Cauchy estimate in each fiber coordinate gives $\|D_v c\|_\infty \le ne/b$ on Δ^n_b, in the operator norm induced by the maximum norm. Thus both smallness hypotheses of Lemma 2.4 hold with $\rho = b$, and the sum of the correction norms is less than α.

Write $s = (z, p)$, $\Psi_s = F(s, \cdot)^{-1}$, and

$$K_* = (C^+ \times P^+) \times \overline{\Delta}^n_{2b}.$$

Since $Q(s, v, 0) = F(s, v)$ by (6.1) and F is available on the larger fiber polydisc Δ^n_{4b}, compactness and continuity give a scalar radius $0 < \eta < \tau$ such that $\Psi_s(Q(s, v, t))$ is defined and

$$\|\Psi_s(Q(s, v, t)) - v\| < e/2 \quad \big((s, v) \in K_*,\ |t| \le \eta\big).$$

The graph points $(s, Q(s, v, t))$ over this compact product form a compact subset of the open chart image. Local uniform convergence $Q_\nu \to Q$ and uniform continuity of the inverse therefore make $\Psi_s(Q_\nu(s, v, t))$ defined on the same product and within $e/2$ of $\Psi_s(Q(s, v, t))$ for all sufficiently large ν. The transition error is thus less than e.

The disjoint closed Jordan discs D^+ and C^+ have a union with connected complement in $\mathbb{C}$, so that union is polynomially convex. The closed polydisc P^+ is polynomially convex as well, and the product of these two compact sets is polynomially convex. Define a holomorphic function to equal φ near $D^+ \times P^+$ and zero near $C^+ \times P^+$; the neighborhoods can be chosen disjoint. Oka–Weil approximation [28, Theorem 2.3.1], with error smaller than $\min\{\xi, \eta\}$, therefore supplies a polynomial $\psi : \mathbb{C}^{1+q} \to \mathbb{C}$ such that

$$\|\psi - \varphi\|_{D^+ \times P^+} < \xi, \qquad \|\psi\|_{C^+ \times P^+} < \eta. \tag{6.2}$$

Step 3: choose the family index last, then glue. Fix ψ and choose ν so large that the preceding overlap estimates hold,

$$R_\nu > 1 + \|\psi\|_{B^+ \times P^+}, \tag{6.3}$$

and

$$\sup_{K \times \overline{\Delta}_b^n \times \overline{\Delta}_{t_1}} d_Y(Q_\nu, Q) < \epsilon/3.$$

The convergence estimate uses a compact product fixed before ψ is chosen. The polynomial ψ may be large elsewhere on $B^+ \times P^+$; those values enter only the radius requirement (6.3).

Use the reference spray on the attached side U_A and the substituted family on the old side $U_B \supset D$:

$$F_A(s, v) = F(s, v), \qquad F_B(s, v) = Q_\nu(s, v, \psi(s)).$$

The second is defined throughout the fixed B side by (6.3). On the overlap its transition into the F_A-chart is within e of the identity by (6.2). Lemma 2.4 provides holomorphic corrections a and b_*, both of norm less than α, with

$$F_A(s, a(s)) = F_B(s, b_*(s)).$$

Thus the two expressions define $\widetilde{g}$ holomorphic on $(U_A \cup U_B) \times U_P$, a neighborhood of L. On K,

$$\widetilde{g}(s) = Q_\nu(s, b_*(s), \psi(s)).$$

The three error terms below are each less than $\epsilon/3$ by the choices above:

$$\begin{aligned} &d_Y\big(\widetilde{g}(s), T(s, 0, \varphi(s))\big) \\ &\quad \le d_Y\big(Q_\nu(s, b_*(s), \psi(s)), Q(s, b_*(s), \psi(s))\big) \\ &\qquad + d_Y\big(Q(s, b_*(s), \psi(s)), Q(s, 0, \varphi(s))\big) \\ &\qquad + d_Y\big(T(s, \rho(s), \varphi(s)), T(s, 0, \varphi(s))\big) < \epsilon. \end{aligned}$$

□

Remark 6.2. For an actual centered family $h : V \times \mathbb{C} \to Y$, take $T_\nu(s, w, t) = h(G(s, w), t)$ on increasing scalar discs. The lemma then permits every scalar function holomorphic near K: choose $\tau > \|\varphi\|_K$. The family F_B need neither dominate nor remain in the old fiber chart away from the overlap.

If Y satisfies LEP at x, the basis condition in Definition 1.6 makes compositions of the centered families into local coordinates near x. A map into this coordinate neighborhood can then be built from the constant map x by applying one family at a time. Lemma 6.1 preserves approximability at each application, giving the following implication.

Lemma 6.3. *If Y satisfies* LEP *at x, then there is a witness neighborhood $W_x \ni x$ for* LAP.

Proof. For $a = (a_1, \dots, a_n)$ in a sufficiently small polydisc $Q \subset \mathbb{C}^n$ about 0, use the families of Definition 1.6 to form the exact partial compositions

$$\theta_0(a) = x, \qquad \theta_j(a) = h_j(\theta_{j-1}(a), a_j), \qquad \Theta(a) = \theta_n(a).$$

Shrink Q so that its closure lies in their domains, every partial image is relatively compact in V, and $|a_j| < \tau$ on $\overline{Q}$. Since $h_j(y, 0) = y$, one has $d_y h_j(y, 0) = \mathrm{id}$. The columns of $d\Theta_0$ are

therefore $\partial_t h_j(x,0)$. The inverse function theorem gives a neighborhood W_x whose closure lies in an inverse chart for Θ, with $\Theta^{-1}(\overline{W}_x) \Subset Q$.

For a special planar pair $D \subset D'$, an arbitrary closed polydisc P, and k holomorphic near $K = D \times P$ with values in W_x, let $a = \Theta^{-1} \circ k$. The successive maps are

$$k_0 = x, \qquad k_j = h_j(k_{j-1}, a_j), \qquad k_n = k.$$

Let $\mathcal{A}$ be the approximation class for $K \subset D' \times P$ in Lemma 6.1. The constant map k_0 belongs to $\mathcal{A}$. Suppose $k_{j-1} \in \mathcal{A}$. On a contractible Stein neighborhood of K, its pulled-back tangent bundle is holomorphically trivial by the Oka–Grauert principle [28, Theorem 5.3.1(ii)]. Lemma 2.2 supplies a chart spray H_j centered at k_{j-1}. Since $k_{j-1}(K) \Subset V$, shrink the base and fiber once so that H_j takes values in V, independently of ν. The families

$$T_{j,\nu}(s,w,t) = h_{j,\nu}(H_j(s,w),t), \qquad |t| < R_\nu,$$

satisfy $T_{j,\nu}(s,w,0) = H_j(s,w)$ and converge locally uniformly for $|t| < \tau$ to $h_j(H_j(s,w),t)$: the fixed substitution sends compact subsets into compact subsets of $V \times \Delta_\tau$. Also $a(K) \Subset Q$ gives $\|a_j\|_K < \tau$. Lemma 6.1 therefore yields $k_j = h_j(k_{j-1}, a_j) \in \mathcal{A}$. Induction proves the result. □

6.2. **From LAP to LEP.** To center the families from Lemma 3.2 exactly, we first use local inverses to find a source point near the origin mapping to each nearby target point. Translation in the scalar coordinate, with the transverse coordinates of this source point held fixed, then gives centered disc families on a common target neighborhood.

Lemma 6.4. *For $r > 0$ and closed polydiscs $P_j \subset \mathbb{C}^{n-1}$ with $0 \in \operatorname{Int} P_j$, $1 \le j \le n$, suppose that ψ_j is a biholomorphic coordinate parametrization near $\overline{\Delta}_r \times P_j$, $\psi_j(0,0) = x$, and the vectors $\partial_z \psi_j(0,0)$ form a basis of $T_x Y$. Also suppose that for some $S_\nu \to \infty$ there are holomorphic maps*

$$\sigma_{j,\nu} : \operatorname{Op}(\overline{\Delta}_{S_\nu} \times P_j) \longrightarrow Y$$

converging uniformly to ψ_j on $\overline{\Delta}_r \times P_j$. Then $x \in G_Y$.

Proof. Fix small source polydiscs Q_j around $(0,0)$, with closures in $\Delta_r \times \operatorname{Int} P_j$. On slightly larger fixed polydiscs, uniform convergence places the images of $\sigma_{j,\nu}$ in the ψ_j-chart for all sufficiently large ν. Cauchy estimates give

$$\psi_j^{-1} \circ \sigma_{j,\nu} = \mathrm{id} + e_{j,\nu}, \qquad e_{j,\nu} \longrightarrow 0 \quad \text{in } C^1(\overline{Q}_j).$$

For example, if $Q_j = \Delta^n_{\rho_j}$ after separate coordinate rescalings, arrange $\|e_{j,\nu}\| < \rho_j/4$ and $\|De_{j,\nu}\| < 1/2$ there. For a point in $\Delta^n_{\rho_j/2}$, the equation $z + e_{j,\nu}(z) = w$ has a unique solution by contraction on the closed $3\rho_j/4$ polydisc. The solutions depend holomorphically on w and converge uniformly on smaller products to w.

Intersect the resulting finitely many target neighborhoods and shrink once to a relatively compact neighborhood $V \ni x$. After discarding a common finite initial segment, this gives holomorphic branches

$$\ell_{j,\nu} = (s_{j,\nu}, p_{j,\nu}) : V \longrightarrow \Delta_r \times \operatorname{Int} P_j, \qquad \sigma_{j,\nu} \circ \ell_{j,\nu} = \mathrm{id}_V,$$

converging to ψ_j^{-1}, with images in fixed compact subproducts. In particular, $|s_{j,\nu}(y)| < C$ for one constant C, independent of j, ν, y.

The scalar shifts $s_{j,\nu}(y)$ are uniformly bounded by C. Reserve this amount of radius by setting $R_\nu = S_\nu - C$; these radii still tend to infinity. Discard further initial terms so that $R_\nu > 0$, and set

$$h_{j,\nu}(y,t) = \sigma_{j,\nu}(s_{j,\nu}(y) + t, p_{j,\nu}(y)).$$

These maps are holomorphic on the common products $V \times \Delta_{R_\nu}$ and satisfy $h_{j,\nu}(y,0) = y$ exactly. The images of $\ell_{j,\nu}$ lie in fixed compact subproducts, whose scalar coordinates have a common positive distance from $\partial\Delta_r$. Choose a common scalar radius $0 < \tau$ smaller than this

distance. On compact subsets of $V \times \Delta_\tau$, replacing $\sigma_{j,\nu}$ by ψ_j gives an error tending to zero, while $\ell_{j,\nu} \to \psi_j^{-1}$ and uniform continuity of ψ_j control the change in its argument. Hence $h_{j,\nu}$ converge locally uniformly to

$$h_j(y,t) = \psi_j(\psi_j^{-1}(y) + (t,0)).$$

The limiting families h_j have initial tangent vectors $\partial_t h_j(x,0) = \partial_z \psi_j(0,0)$, as required by Definition 1.6. □

Lemma 6.5. *If Y satisfies* LAP *at x, then it satisfies* LEP *at x.*

Proof. For $n = \dim Y \geq 1$, choose a witness neighborhood V for LAP at x and a coordinate parametrization ψ near $0 \in \mathbb{C}^n$, with $\psi(0) = x$ and image in V. Using each coordinate in turn as the scalar variable gives biholomorphic parametrizations ψ_j near fixed products $\overline{\Delta}_r \times P_j$, where $0 \in \operatorname{Int} P_j$ and the vectors $\partial_z \psi_j(0,0)$ form a basis of $T_x Y$. Every boundary fiber lies in the same witness neighborhood V. For a common sequence $S_\nu \to \infty$, Lemma 3.2 supplies maps $\sigma_{j,\nu}$ holomorphic near $\overline{\Delta}_{S_\nu} \times P_j$ and converging uniformly to ψ_j on $\overline{\Delta}_r \times P_j$, with each P_j fixed. Lemma 6.4 gives LEP at x. □

This completes the proof of Theorem 1.7.

6.3. **Direct consequences.**

Proof of Theorem 1.8. Only sufficiency needs proof. For $s > 0$, let B_s denote the Euclidean ball of radius s in $\mathbb{C}^n$. Fix $x \in X$, choose a chart radius $r > 0$, and take a coordinate parametrization $\phi : B_{2r} \to X$ with $\phi(0) = x$. By CAP_n, an entire map $G : \mathbb{C}^n \to X$ approximates ϕ arbitrarily closely on $\overline{B}_{3r/2}$. Choose the approximation so close that its image on this ball lies in $\phi(B_{2r})$ and, for $g = \phi^{-1} \circ G$,

$$\sup_{\overline{B}_r} |g(z) - z| < r/2, \qquad \sup_{\overline{B}_r} \|dg_z - I\| < 1/2.$$

The second estimate follows from Cauchy estimates on the larger ball. The map $z \mapsto z - g(z)$ is therefore a contraction of $\overline{B}_r$ into itself. Its fixed point a satisfies $g(a) = 0$, and dg_a is invertible. Thus $z \mapsto G(z + a)$ dominates X at x. Apply Theorem 1.5.

The same argument applies to the special convex sets in Forstnerič's original definition [26, Definition 1.1 and p. 692]: take two nested balls in the interior of such a set and rescale the coordinate parametrization to be defined near the whole set. □

Proof of Theorem 1.9. Combine the strong dominability theorem of Forstnerič and Lárusson [32, Theorem 2] with Theorem 1.5, componentwise. □

Proof of Theorem 1.10. By Theorem 1.5, it suffices to observe that a dominating map into an open subset also dominates the ambient manifold. □

7. Removability and holomorphic images

By the LEP characterization of Theorem 1.7, the Oka property can be read off from centered families constructed away from a small exceptional set. We first propagate LEP across the exceptional set, then apply the resulting criterion to holomorphic images and complements.

7.1. **Removability of small exceptional sets.**

Recall that G_Y is the open set of points of Y at which LEP holds. For $n = \dim Y \geq 1$, the exponent $2n - 1$ reflects the parameters of a coordinate circle centered at a fixed point: $2n - 2$ real parameters for its complex line and one for its radius. A set of zero $(2n-1)$-dimensional Hausdorff measure cannot meet every such circle.

The following proposition adapts the boundary-disc argument of [55, Corollary 2.10], using n independent directions and Lemma 6.4 to produce exactly centered families.

Proposition 7.1. *Let ϑ be a holomorphic coordinate chart on Y near $x \in Y$. If the closed coordinate disc $\vartheta(x) + \overline{\Delta}_r v$, $v \neq 0$, lies in the chart image and its boundary lies in $\vartheta(G_Y)$, then $x \in G_Y$.*

Proof. By compactness, small changes of v keep the closed disc in the chart image and its boundary in $\vartheta(G_Y)$. Choose $n = \dim Y$ linearly independent vectors v_j sufficiently close to v, and complete each to a basis $v_j, e_{j,2}, \dots, e_{j,n}$ of $\mathbb{C}^n$. The corresponding coordinate maps to Y are

$$\psi_j(z,p) = \vartheta^{-1}\left(\vartheta(x) + zv_j + \sum_{k=2}^{n} p_{k-1} e_{j,k}\right).$$

For each j, cover $|z| = r$ by finitely many closed arcs whose $\psi_j(\cdot, 0)$-images lie compactly in witness neighborhoods from Lemma 6.3. By continuity, fix a small closed polydisc $P_j \subset \mathbb{C}^{n-1}$ centered at 0 such that ψ_j is a coordinate parametrization near $\overline{\Delta}_r \times P_j$ and maps each closed arc product into a compact subset of its chosen witness neighborhood.

For a common sequence $S_\nu > r$ tending to infinity, Lemma 3.2 gives holomorphic maps $\sigma_{j,\nu}$ to Y near $\overline{\Delta}_{S_\nu} \times P_j$, converging uniformly to ψ_j on $\overline{\Delta}_r \times P_j$. Since $\psi_j(0,0) = x$ and the vectors $\partial_z \psi_j(0,0) = (d\vartheta_x)^{-1} v_j$ form a basis of $T_x Y$, Lemma 6.4 gives $x \in G_Y$. □

Proof of Theorem 1.11. By hypothesis, $Y \setminus E \subset G_Y$. Fix $x \in Y$ and take coordinates centered at x. Choose radii $0 < a < b$ so that the closed ball of radius b lies in the chart. On the annulus $a \leq |z| \leq b$, the smooth map

$$q(z) = ([z], |z|) \in \mathbb{P}^{n-1} \times [a, b]$$

is Lipschitz. The image under q of the coordinate representative of E in this annulus has zero $(2n-1)$-dimensional Hausdorff outer measure. Since $\mathbb{P}^{n-1} \times (a, b)$ has positive measure in that dimension, choose $([v], r)$ in this manifold outside the image, with $|v| = 1$. The corresponding circle $\{\zeta v : |\zeta| = r\}$ therefore avoids the coordinate representative of E. Its inverse image under the chart lies in G_Y, and the filled disc lies in the chart image. Proposition 7.1 gives $x \in G_Y$. This also covers $n = 1$, when $\mathbb{P}^0$ is a point. Thus $G_Y = Y$, and Theorem 1.7 makes Y Oka. □

By Proposition 2.1 and the pointwise assertion of Theorem 1.7, dominability of Y at every point outside E gives LEP for the target Y at those points. Hence Theorem 1.11 also applies in this case. In particular, if such an exceptional set E is closed and $Y \setminus E$ is Oka, then Y is Oka.

Proof of Theorem 1.12. For $n = \dim X \geq 1$, the proper analytic subset A has $\mathcal{H}^{2n-1}(A) = 0$; see [28, §1.3]. Strong dominability of $X \setminus A$, by Theorem 1.5, gives dominability of X at every point outside A after composition with the inclusion. The dominability consequence of Theorem 1.11 just established shows that X is Oka. □

The following result was proved by Wang, Xie, and Zhao [61, Corollary 1.3]; it also follows from OpenAI's theorem that every complex K3 surface is Oka [55, Theorem 1.1].

Corollary 7.2. *Every complex Kummer surface is Oka.*

Proof. For a Kummer surface X, the complement of its sixteen exceptional curves is Oka by Forstnerič and Lárusson [32, Lemma 7], so Theorem 1.12 applies. □

7.2. **Surjective holomorphic images and complements.** The proof for a surjective holomorphic map into Y uses the fact that its critical values have zero $(2 \dim Y - 1)$-dimensional Hausdorff measure, even when the map is not proper.

7.2.1. *Surjective holomorphic images.*

Theorem 7.3. *Let $p : X \to Y$ be a surjective holomorphic map of connected complex manifolds. If X is Oka, then Y is Oka.*

Proof. For $n = \dim Y \geq 1$, let E be the set of critical values of p. Stratify the critical locus so that p has constant rank on each stratum [58, Chapter 2, Proposition 3]. By the holomorphic constant-rank theorem, each stratum has a countable cover by coordinate neighborhoods whose images lie in complex submanifolds of dimension at most $n-1$. The resulting countable collection of images covers E, so $\mathcal{H}^{2n-1}(E) = 0$.

For each $y \in Y \setminus E$, strong dominability of X gives an entire map $F : \mathbb{C}^{\dim X} \to X$ dominating X at some $x \in p^{-1}(y)$. Since dp_x is surjective, the restriction of $p \circ F$ to a suitable n-dimensional linear subspace dominates Y at y. Theorem 1.11 therefore shows that Y is Oka. □

Corollary 7.4. *Let $\pi : \widetilde{X} \to X$ be a proper bimeromorphic holomorphic map of connected complex manifolds. If $\widetilde{X}$ is Oka, then X is Oka. In particular, this holds when $\widetilde{X} = \mathrm{Bl}_Z X$ for a smooth closed complex submanifold $Z \subsetneq X$.*

Proof. This follows directly from Theorem 7.3, since a proper bimeromorphic map is surjective. □

This answers the blow-down question of Forstnerič and Lárusson [32, Section 6].

Following Forstnerič [35, Definition 1.2], a connected complex manifold Y has the *basic Oka property with surjectivity* (BOPS) if every continuous map $S \to Y$ from a Stein manifold with $\dim S \geq \dim Y$ is homotopic to a surjective holomorphic map.

Corollary 7.5. *For a connected complex manifold Y of dimension $n \geq 1$, the following conditions are equivalent:*

(i) *Y is Oka;*
(ii) *there is a surjective holomorphic map $\mathbb{C}^n \to Y$;*
(iii) *there is a surjective holomorphic map $\mathbb{C}^N \to Y$ for some finite integer $N \geq 1$;*
(iv) *Y has BOPS.*

Proof. Forstnerič's surjectivity theorem gives both (ii) and (iv) from (i) [35, Theorem 1.1]. Clearly (ii) implies (iii), while Theorem 7.3 gives (iii)⇒(i). Applying BOPS to a constant map $\mathbb{C}^n \to Y$ gives (iv)⇒(ii). □

This answers Forstnerič's question whether BOPS implies the Oka property [35, Problem 1.4(a)].

7.2.2. *Product complements and blowups.*

Theorem 7.6. *Let X and Y be connected positive-dimensional Oka manifolds, with $A \subsetneq X$ and $B \subsetneq Y$ closed complex analytic subsets. If $Z \subset X \times Y$ is a closed complex analytic subset contained in $A \times B$, then $(X \times Y) \setminus Z$ is Oka.*

Proof. Both Z and $A \times B$ are proper closed analytic subsets of the connected complex manifold $X \times Y$, so their complements are connected. First consider $Z = A \times B$ and fix (p, q) in its complement. After exchanging the factors, assume $p \notin A$. Set $n = \dim X$ and $m = \dim Y$. Choose entire maps $f : \mathbb{C}^n \to X$ and $g : \mathbb{C}^m \to Y$, centered at p and q and nonsingular at zero. The image of g contains a neighborhood of q. Since B has empty interior, some $c \in \mathbb{C}^m$ satisfies $g(c) \notin B$.

The analytic set $f^{-1}(A)$ avoids zero. Cartan's theorem A [28, Theorem 2.6.1(A)], applied to its ideal sheaf, gives $h \in \mathcal{O}(\mathbb{C}^n)$ with $h|_{f^{-1}(A)} = 0$ and $h(0) = 1$. Define

$$\Phi(u, v) = \big(f(u), g(c + h(u)(v - c))\big).$$

Whenever $f(u) \in A$, the second component equals $g(c) \notin B$. Thus Φ takes values in the complement and $\Phi(0,0) = (p,q)$. Its differential at zero is block triangular with invertible diagonal blocks df_0 and dg_0. The complement is strongly dominable, hence Oka.

For general Z, write $W = (X \times Y) \setminus Z$ and $U = (X \times Y) \setminus (A \times B)$. The first part makes U Oka, and $W \setminus U = W \cap (A \times B)$ is a proper closed analytic subset of W. Theorem 1.12 therefore makes W Oka. □

Corollary 7.7. *Let X, Y, A, B be as in Theorem 7.6.*

(i) *If $Z \subset A \times B$ is a smooth closed complex submanifold of $X \times Y$, possibly disconnected, then $\mathrm{Bl}_Z(X \times Y)$ is Oka. In particular, this applies to $Z = A \times B$ when both A and B are smooth.*

(ii) *For every finite set $D \subset X \times Y$, the complement $(X \times Y) \setminus D$ and the blowup of $X \times Y$ at the distinct points of D are Oka.*

Proof. For a smooth center Z, the blowup is a connected complex manifold whose exceptional divisor has complement biholomorphic to $(X \times Y) \setminus Z$. The latter is Oka by Theorem 7.6, so Theorem 1.12 applies upstairs.

For a finite set D, its projections $A_D = \mathrm{pr}_X(D)$ and $B_D = \mathrm{pr}_Y(D)$ are proper closed analytic subsets because both factors have positive dimension. Since $D \subset A_D \times B_D$, the theorem gives the Oka complement, and (i) gives the Oka blowup. □

8. Rationally connected projective manifolds

8.1. **Very free curves and their consequences.** For a complex manifold X of dimension n, a rational curve $g : \mathbb{P}^1 \to X$ is *very free* if g^*TX is ample. By the Birkhoff–Grothendieck theorem [44, Theorem 2.1], this means

$$g^*TX \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i), \qquad a_i \geq 1.$$

Very free rational curves provide the centered entire families required by the analytic criterion. A submersive first-jet map lets us vary both the center and the initial tangent vector. A local section then gives finitely many centered families whose initial tangent vectors form a frame.

Fix distinct points $0, \infty \in \mathbb{P}^1$ and an affine coordinate t on $\mathbb{P}^1 \setminus \{\infty\}$ with $t(0) = 0$. For a parametrized curve g, write $g'(0) = dg_0(\partial/\partial t)$. With this coordinate fixed, the space of first jets at 0 of maps to a manifold Y is identified with TY via the value $g(0)$ and first derivative $g'(0)$.

Lemma 8.1. *Let Y be a complex n-manifold, S a complex manifold, and $F : S \times \mathbb{C} \to Y$ a holomorphic family. Suppose*

$$J : S \longrightarrow TY, \qquad J(s) = (F(s,0), \partial_t F(s,0))$$

is a submersion at some $s_0 \in S$. Then, on a neighborhood V of $x = F(s_0, 0)$, there are holomorphic maps $h_j : V \times \mathbb{C} \to Y$, $1 \leq j \leq n$, such that $h_j(y,0) = y$ and the vectors $\partial_t h_j(y,0)$ form a frame of $TY|_V$. If F extends to $S \times \mathbb{P}^1$, so do these families.

Proof. Choose a local holomorphic section $\sigma : W \to S$ of J, where $W \subset TY$ is a neighborhood of $J(s_0)$. Every nonempty open subset of a finite-dimensional vector space contains a basis, so choose $v_1, \ldots, v_n \in W \cap T_x Y$ forming a basis. Extend them to a local holomorphic frame $v_j(y)$ and shrink V until all its graphs lie in W. Set

$$h_j(y,t) = F(\sigma(y, v_j(y)), t).$$

The identity $J \circ \sigma = \mathrm{id}_W$ gives the asserted center values and initial tangent vectors. If F extends to $S \times \mathbb{P}^1$, the same formula defines h_j on $V \times \mathbb{P}^1$. □

We use the strong form of rational connectedness: a smooth projective rationally connected variety over $\mathbb{C}$ admits a very free curve through every prescribed point [48, IV.3.3.1 and Theorem IV.3.9].

Lemma 8.2. *Let $g : \mathbb{P}^1 \to X$ be very free, where X is a smooth projective complex manifold. In the scheme H of morphisms of its fixed projective degree, both H and the first-jet morphism*

$$J : H \longrightarrow TX, \qquad u \longmapsto (u(0), u'(0)),$$

are smooth at g.

Proof. Let $2[0]$ denote the length-two subscheme defined by $t^2 = 0$, and set $E = g^*TX$. In the splitting above, $a_i \geq 1$, so $E(-2[0]) = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i - 2)$ has summand degrees $a_i - 2 \geq -1$. Since $H^1(\mathbb{P}^1, \mathcal{O}(k)) = 0$ for $k \geq -1$ [60, Tag 01XT],

$$H^1(\mathbb{P}^1, E) = H^1(\mathbb{P}^1, E(-2[0])) = 0.$$

Fixed-source deformation theory identifies $T_g H$ with $H^0(\mathbb{P}^1, E)$ and makes H smooth at g [5, Theorem 5(1)–(2)]. Infinitesimal deformation of a first jet identifies $T_{J(g)}TX$ with $H^0(2[0], E|_{2[0]})$. The exact sequence

$$0 \longrightarrow E(-2[0]) \longrightarrow E \longrightarrow E|_{2[0]} \longrightarrow 0$$

and $H^1(\mathbb{P}^1, E(-2[0])) = 0$ make the restriction map $H^0(\mathbb{P}^1, E) \to H^0(2[0], E|_{2[0]})$ surjective. This map is dJ_g. Since H and TX are smooth at the relevant points, J is smooth at g. □

Proof of Theorem 1.13. For $\dim X > 0$, fix $x \in X$. By [48, IV.3.3.1 and Theorem IV.3.9], there is a very free curve $g : \mathbb{P}^1 \to X$ with $g(0) = x$. For a fixed projective embedding of X, let H be the morphism scheme of degree $\deg g^*\mathcal{O}_X(1)$. On a smooth neighborhood $S \subset H$ of g, the first-jet map of universal evaluation $S \times \mathbb{C} \to X$ is a submersion at g by Lemma 8.2. Lemma 8.1 therefore gives centered entire families on a common neighborhood of x whose initial tangent vectors span $T_x X$. Since x was arbitrary, Definition 1.6 gives LEP on X, and Theorem 1.7 shows that X is Oka. The Fano assertion follows from the rational connectedness theorem of Campana and Kollár–Miyaoka–Mori [14, 49]. □

The same argument survives deletion of an algebraic subset of codimension at least two. To obtain families at every remaining point, we first find an avoiding very free curve through that point.

Lemma 8.3. *Let X be as in Theorem 1.13 and $A \subset X$ a closed algebraic subset whose irreducible components have codimension at least two. For every $x \in X \setminus A$ there is a very free curve $g : \mathbb{P}^1 \to X$ with $g(0) = x$ and $g(\mathbb{P}^1) \cap A = \varnothing$.*

Proof. Choose a very free curve through x. Very freeness is open in its fixed-degree morphism scheme, by semicontinuity of $H^1(\mathbb{P}^1, g^*TX(-2[0]))$; see [5, p. 21]. By Lemma 8.2, evaluation at 0 is smooth on this locus: it is the composition of the smooth first-jet morphism with the smooth projection $TX \to X$. The tangent space to the evaluation fiber over x is the kernel of $H^0(\mathbb{P}^1, g^*TX) \to T_x X$, namely $H^0(\mathbb{P}^1, g^*TX(-[0]))$. Take a nonempty smooth irreducible open subset S of this fiber within the very-free locus. For every $g \in S$ and $t \neq 0$, very freeness gives

$$H^1(\mathbb{P}^1, g^*TX(-[0] - [t])) = 0,$$

since every summand of this bundle has degree at least -1. The restriction sequence at t for $g^*TX(-[0])$ then makes

$$H^0(\mathbb{P}^1, g^*TX(-[0])) \longrightarrow T_{g(t)}X$$

surjective: twisting by $-[0]$ does not change the fiber at t. These sections are the tangent vectors to S, so the differential of universal evaluation is already surjective in the S directions. Both source and target are smooth; hence $e : S \times (\mathbb{P}^1 \setminus \{0\}) \to X$ is smooth.

The incidence set $I = \{(g,t) : g(t) \in A\}$ is closed in the full product $S \times \mathbb{P}^1$ and misses $S \times \{0\}$. Since e is smooth, the inverse image of each component of A, if nonempty, has the same codimension in $S \times (\mathbb{P}^1 \setminus \{0\})$ as that component has in X. Consequently

$$\dim I \leq \dim S + 1 - 2 = \dim S - 1.$$

The projection $\mathrm{pr}_S : S \times \mathbb{P}^1 \to S$ is proper, so $\mathrm{pr}_S(I)$ is closed. Moreover,

$$\dim \mathrm{pr}_S(I) \leq \dim I < \dim S,$$

so this image is a proper subset. Any parameter outside it gives the required curve. □

Corollary 8.4. *Let X be a smooth connected complex projective rationally connected manifold and $A \subset X$ a closed algebraic subset whose irreducible components have codimension at least two. Then $X \setminus A$ is Oka.*

Proof. Fix $x \in X \setminus A$ and choose an avoiding very free curve through x. Avoidance of A on all of $\mathbb{P}^1$ is an open condition in the unrestricted morphism scheme: the complementary locus is the image of the closed universal incidence set under the proper projection to the morphism scheme, and is therefore closed. On this open locus, Lemma 8.2 still makes first-jet evaluation a submersion near the chosen curve. Lemma 8.1 now gives centered entire families into $X \setminus A$ whose initial tangent vectors form a basis. Thus Definition 1.6 gives LEP at every point of $X \setminus A$, and Theorem 1.7 proves the claim. □

Corollary 8.5. *Every smooth connected complex projective rationally connected manifold, in particular every complex projective Fano manifold, is holomorphically elliptic. Every smooth projective birational model of such a manifold is also Oka and holomorphically elliptic.*

Proof. Theorem 1.13 gives the Oka property. The theorem of Forstnerič–Lárusson that every projective Oka manifold is holomorphically elliptic [34] gives the first assertion. Rational connectedness is invariant under birational maps between smooth projective manifolds [5, p. 37], so the same argument applies to every indicated birational model. □

8.2. **Compact Kähler consequences.** The structure theorems below reduce these compact Kähler cases to rationally connected projective manifolds and complex tori. We then use the Oka property of products, holomorphic fiber bundles, and unramified coverings.

Corollary 8.6. *Every compact Kähler manifold admitting a Kähler metric with nonnegative holomorphic sectional curvature is Oka.*

Proof. Let X be such a manifold; we may assume that it is connected. Matsumura's structure theorem [54, Theorem 1.1 and Remark 1.2] supplies an integer $q \geq 0$ and a smooth projective rationally connected manifold F, possibly a point, such that the universal cover of X is biholomorphic to $\mathbb{C}^q \times F$. By Theorem 1.13, F is Oka. Products of Oka manifolds are Oka, and the Oka property descends through unramified holomorphic coverings [27, Corollary 1.3]. Hence X is Oka. □

On a compact complex manifold Z with a fixed Hermitian form ω, a holomorphic line bundle L is *nef* if, for every $\epsilon > 0$, it admits a smooth Hermitian metric whose normalized real Chern curvature is bounded below by $-\epsilon\omega$. For a holomorphic vector bundle $E \to Z$, let $\mathbb{P}(E)$ parametrize its one-dimensional quotients. The bundle E is *nef* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef [21, Definitions 1.2 and 1.9].

Corollary 8.7. *Every compact Kähler manifold with nef tangent bundle is Oka.*

Proof. Let X be such a manifold; we may assume that it is connected. The structure theorem of Demailly–Peternell–Schneider [21, Theorem 3.14 and Proposition 3.9] gives a connected finite étale cover $p : X' \to X$ whose Albanese map

$$a : X' \longrightarrow \mathrm{Alb}(X')$$

is a surjective holomorphic submersion with connected Fano fibers. If the fibers are points, a is an isomorphism, so X' is a complex torus and is Oka [28, Proposition 5.6.1].

Otherwise $TX' = p^*TX$ is nef, and hence so is $-K_{X'} = \det TX'$. The general fiber of a is Fano, so its anticanonical bundle is nef and big. Cao–Höring's local-triviality theorem [15] therefore makes a a holomorphic fiber bundle. Its fibers are Oka by Theorem 1.13, and its base is a complex torus. The Oka fiber bundle theorem [27, Corollary 1.3] shows that X' is Oka. In either case the Oka property descends along the unramified covering p. □

9. Divisor complements

9.1. **Logarithmic rational curves and divisor complements.** Let $D = \sum_{i=1}^{r} D_i$ be a reduced simple normal crossings divisor on a smooth projective manifold X of dimension n. Thus every D_i is smooth and locally their union is a union of coordinate hyperplanes. The bundle $T_X(-\log D)$ consists of vector fields tangent to all D_i; in coordinates where $D = \{x_1 \cdots x_k = 0\}$ it has frame

$$x_1 \frac{\partial}{\partial x_1}, \dots, x_k \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_n}.$$

We allow $D = 0$ and use the fixed source coordinate of Section 8.

For a divisor complement, deformation of an ordinary rational curve need not preserve avoidance of the boundary. The appropriate parameter space keeps the complete pullback divisor fixed at infinity. Its infinitesimal kernel is the logarithmic tangent bundle, and positivity of that bundle again gives a submersive first-jet map. Here the hypothesis supplies just one curve. Its deformations give families on a nonempty Zariski-open subset of the complement, after which analytic propagation reaches every point. We use the standard logarithmic deformation principle used in the proof of [17, Proposition 2.7]; here the contact conditions are realized as a fiber of an ordinary divisor map, and first-jet smoothness is checked directly.

Theorem 9.1. *Let X be a connected complex projective manifold and D a reduced simple normal crossings divisor. Suppose there is a nonconstant map $f : \mathbb{P}^1 \to X$ such that*

$$f^{-1}(D) \subseteq \{\infty\}, \qquad f^*T_X(-\log D) \text{ is ample.}$$

Then $U = X \setminus D$ is Oka.

The proof follows Lemma 9.2 below.

9.1.1. *Fixed contact and first jets.* Write $f^*D_i = m_i[\infty]$, where $m_i \geq 0$.

Fix a projective embedding of X. In the finite-type scheme of morphisms $\mathbb{P}^1 \to X$ of degree $\deg f^*\mathcal{O}_X(1)$ [22, §2.1], retain all connected components on which $\deg g^*\mathcal{O}_X(D_i) = m_i$ for every i. Let H be the Zariski-open subset of this union of components consisting of maps whose images are contained in no D_i.

For $m \geq 0$, the symmetric power $\mathrm{Sym}^m(\mathbb{P}^1)$ parametrizes effective divisors of degree m. There is an algebraic divisor map

$$c : H \longrightarrow B := \prod_{i=1}^{r} \mathrm{Sym}^{m_i}(\mathbb{P}^1), \qquad c(g) = (g^*D_i)_i.$$

To obtain c scheme-theoretically, pull back the defining section of each D_i along universal evaluation. It is not identically zero on any source fiber. Its zero scheme is therefore a relative effective Cartier divisor, flat over H, by the fiberwise Cartier criterion [60, Tag 062Y]. It is proper and fiberwise finite of constant degree m_i, so it determines the indicated morphism to the symmetric power of the smooth curve $\mathbb{P}^1$ [47, Remark 9.3.9].

Set $b_0 = (m_i[\infty])_i$ and take the scheme-theoretic fiber

$$M = c^{-1}(b_0).$$

Thus all contact points, including their multiplicities, are fixed, while the maps remain parametrized. Universal evaluation restricts to

$$\mathcal{F} : M \times \mathbb{A}^1 \longrightarrow U. \tag{9.1}$$

The fixed coordinate identifies first jets at 0 with TU, giving an algebraic morphism $J : M \to TU$, $J(g) = (g(0), g'(0))$.

Lemma 9.2. *If $g \in M$ and $E_g = g^*T_X(-\log D)$ is ample, then $J : M \to TU$ is smooth at g. Moreover M is smooth at g and $T_gM = H^0(\mathbb{P}^1, E_g)$.*

Proof. We first prove smoothness of the contact fiber and compute its tangent space. Set

$$Z_i = m_i[\infty], \qquad Q_g = \bigoplus_{i:m_i>0} g^*\mathcal{O}_X(D_i)|_{Z_i}.$$

The logarithmic inclusion gives an exact sequence

$$0 \longrightarrow E_g \longrightarrow g^*TX \longrightarrow Q_g \longrightarrow 0. \tag{9.2}$$

Indeed, near $g(\infty)$ take normal crossings coordinates and a source coordinate z vanishing at infinity. For the components met by g, one has $x_i \circ g = z^{m_i}u_i(z)$ with $u_i(0) \neq 0$. In the corresponding normal direction, the inclusion is multiplication by $z^{m_i}u_i(z)$, with quotient modulo z^{m_i}. All other directions are unchanged. Away from infinity the inclusion is an isomorphism. This proves both injectivity and the asserted torsion quotient.

The tangent space to the divisor target is

$$T_{Z_i}\operatorname{Sym}^{m_i}(\mathbb{P}^1) = H^0(Z_i, \mathcal{O}_{\mathbb{P}^1}(Z_i)|_{Z_i}) = H^0(Z_i, g^*\mathcal{O}_X(D_i)|_{Z_i}).$$

The final identification uses the defining section with zero divisor Z_i. With the corresponding convention for infinitesimal divisor displacement, dc_g is the quotient map to $H^0(Q_g)$: a variation of $x_i \circ g$ is taken modulo z^{m_i}. Thus Q_g records the infinitesimal motion of the contact divisors.

Ampleness gives

$$H^1(\mathbb{P}^1, E_g) = H^1(\mathbb{P}^1, E_g(-2[0])) = 0.$$

Since Q_g is supported at infinity, $H^1(\mathbb{P}^1, Q_g) = 0$. Sequence (9.2) therefore gives $H^1(\mathbb{P}^1, g^*TX) = 0$ and makes the quotient map $H^0(\mathbb{P}^1, g^*TX) \to H^0(Q_g)$ surjective. Fixed-source deformation theory [5, Theorem 5(1)–(2)] makes H smooth at g, with tangent space $H^0(\mathbb{P}^1, g^*TX)$. The quotient map is dc_g, so c is smooth at g, since B is smooth. Its fiber M is consequently smooth at g, with

$$T_gM = \ker dc_g = H^0(\mathbb{P}^1, E_g).$$

The supports of $2[0]$ and the contact divisors at infinity are disjoint, so $E_g|_{2[0]} = g^*TX|_{2[0]}$ under the natural inclusion. As in the proof of Lemma 8.2, dJ_g is the restriction map

$$H^0(\mathbb{P}^1, E_g) \longrightarrow H^0(2[0], E_g|_{2[0]}) \simeq T_{J(g)}TU.$$

It is surjective because $H^1(\mathbb{P}^1, E_g(-2[0])) = 0$. Since M and TU are smooth at the relevant points, J is smooth at g. □

Proof of Theorem 9.1. By Lemma 9.2, f belongs to the smoothness locus of the morphism $J : M \to TU$. Choose a nonempty Zariski-open neighborhood $S \subset M$ of f on which J is smooth. Then S is smooth. Let $\pi : TU \to U$ be the bundle projection. Evaluation at 0 is the smooth map $\mathrm{ev}_0 = \pi \circ J : S \to U$, and its image

$$U^+ = \mathrm{ev}_0(S)$$

is therefore nonempty and Zariski open.

For each $x \in U^+$ choose $g \in S$ over x. Apply Lemma 8.1 at g to the analytification of the universal family (9.1) restricted to $S \times \mathbb{A}^1$. This gives centered entire families near x whose

initial tangent vectors form a frame. Hence $U^+ \subset G_U$, where G_U is the set of points of U at which LEP holds. Since X is smooth and connected it is irreducible, and $U \setminus U^+$ is a proper closed algebraic, hence analytic, subset of U. Thus it has zero $(2\dim U - 1)$-dimensional Hausdorff measure, and Theorem 1.11 shows that U is Oka. □

Remark 9.3. The positivity hypothesis in Theorem 9.1 has a necessary numerical consequence. Write

$$f^*T_X(-\log D) = \bigoplus_{\nu=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_\nu), \qquad a_\nu \geq 1.$$

The pushforward cycle $f_*[\mathbb{P}^1]$ counts the image curve with the degree of f onto its image. Since $\det T_X(-\log D) = \mathcal{O}_X(-(K_X + D))$, we obtain

$$-(K_X + D) \cdot f_*[\mathbb{P}^1] = \deg f^*T_X(-\log D) = \sum_{\nu=1}^{n} a_\nu \geq n.$$

The inequality constrains the logarithmic-curve criterion, not the Oka property of the complement; the critical-degree case in Subsection 9.2 makes this distinction explicit.

9.1.2. *Ample boundary and the log-Fano application.* An algebraic variety U is $\mathbb{A}^1$-*connected* if there is an algebraic family $a : T \times \mathbb{A}^1 \to U$ of nonconstant proper maps whose two-point evaluation

$$T \times \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow U \times U, \qquad (s, t_1, t_2) \longmapsto (a(s, t_1), a(s, t_2))$$

is dominant [17, Definitions 1.1–1.2].

Corollary 9.4. *Let X be a connected complex projective manifold with a reduced simple normal crossings divisor D. If $X \setminus D$ is $\mathbb{A}^1$-connected, then it is Oka.*

Proof. In characteristic zero the dominant two-point evaluation is separable. Chen–Zhu's very-free-curve criterion [17, Proposition 1.13(2)] gives a nonconstant map $f : \mathbb{P}^1 \to X$ with $f^{-1}(D) = \{\infty\}$ and $f^*T_X(-\log D)$ ample. Theorem 9.1 applies. □

Theorem 9.5. *Let X be a connected complex projective manifold of dimension at least two and $D \subset X$ a smooth irreducible ample divisor. If D is rationally connected, then $X \setminus D$ is Oka and holomorphically elliptic. The same conclusion holds if $-(K_X + D)$ is ample.*

Proof. We apply Chen–Zhu's fully-free-center criterion [17, Theorem 1.3 and Remark 1.4] to the single boundary component D. Choose a very free curve $a : \mathbb{P}^1 \to D$; when $\dim D = 1$, take an isomorphism $\mathbb{P}^1 \simeq D$. The contact intersection matrix is the 1×1 matrix $(\deg a^*\mathcal{O}_D(D))$. Its entry is positive by ampleness, so it has rank one. Together with the very freeness of a, this verifies the hypotheses of the criterion, which gives $\mathbb{A}^1$-connectedness. Corollary 9.4 gives the Oka property.

Choose $k > 0$ with $\mathcal{O}_X(kD)$ very ample. In the associated projective embedding, the section defining kD cuts out a hyperplane. Consequently $X \setminus D$ is closed in the affine complement of that hyperplane. It is affine, hence Stein [28, §2.2, p. 49]. A Stein Oka manifold is holomorphically elliptic [28, Proposition 5.6.15], which proves the first assertion.

If $-(K_X + D)$ is ample, adjunction [45, Proposition II.8.20] gives $-K_D = -(K_X + D)|_D$. Thus D is Fano and is rationally connected [14, 49]; the first assertion applies. □

Corollary 9.6. *Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of multidegree $(d_1, \ldots, d_c)$ and dimension at least two, and D be a smooth hypersurface section of degree $e \geq 1$. If*

$$d_1 + \cdots + d_c + e \leq N,$$

then $X \setminus D$ is Oka and holomorphically elliptic.

Proof. Let H denote the hyperplane class. Adjunction [45, Proposition II.8.20] gives

$$-(K_X + D) = (N + 1 - d_1 - \cdots - d_c - e)H|_X,$$

which is ample under the stated inequality. The divisor D is also ample, since $D \sim eH|_X$.

The Koszul resolution and the cohomology of line bundles on projective space [45, Proposition III.7.10A and Theorem III.5.1] give $H^0(X, \mathcal{O}_X) = H^0(D, \mathcal{O}_D) = \mathbb{C}$. Hence X and D are connected, and smoothness makes D irreducible. Theorem 9.5 applies. □

9.2. **Smooth hypersurface complements.** To include the critical degree $d = n + 1$, we use line complements. The construction has three steps: generic projection makes successive polar equations transverse; these equations supply enough directions through a general point of the complement; and an exponential formula turns the directions into centered entire families. Analytic propagation then reaches the remaining points. The parametrizations vary holomorphically even when a line complement changes between $\mathbb{C}$ and $\mathbb{C}^*$.

9.2.1. *Successive polars and generic projection.* In the next two lemmas, N is the vector-space dimension and $X \subset \mathbb{P}^{N-1}$ denotes the hypersurface. In the application to $\mathbb{P}^n \setminus D$, these symbols mean $N = n + 1$ and $X = D$.

Successive polars encode high contact of lines with a hypersurface; see [6]. Use coordinates $v = (v_1, \dots, v_N)$ on $\mathbb{C}^N$. For a homogeneous polynomial F of degree d and a nonzero vector $a = (a_1, \dots, a_N)$, write

$$D_a = \sum_{i=1}^{N} a_i \frac{\partial}{\partial v_i}.$$

The equations $D_a^j F(v) = 0$ are bihomogeneous in (a, v), so their projective zero sets depend only on $[a]$ and $[v]$.

Suppose $d \geq 2$, $F(a) \neq 0$, and $v \neq 0$ satisfies $D_a^j F(v) = 0$ for $0 \leq j \leq d-2$. Write $dF_a(v)$ for the differential of F at a applied to v; by contrast, $D_a F(v)$ differentiates at v in the direction a. For $s \in \mathbb{C}$, homogeneity gives

$$F(a + sv) = \sum_{\ell=0}^{d} \frac{s^{d-\ell}}{\ell!} D_a^\ell F(v) = F(a) + s\, dF_a(v). \tag{9.3}$$

The first equality follows for $s \neq 0$ by expanding $s^d F(v + a/s)$ and hence holds identically as a polynomial in s. The polar equations kill precisely the coefficients of $s^2, \dots, s^d$. Thus the restriction of F to the line through $[a]$ and $[v]$ has degree at most one in the affine parameter s, and the line meets the hypersurface with multiplicity at least $d - 1$ at $[v]$.

For a general point $[a]$ of the complement, we therefore seek points $[v]$ satisfying these polar equations. Two properties are needed: transversality, so that the points $[v]$ can be continued holomorphically as $[a]$ moves, and spanning, so that the resulting line directions generate the tangent space at $[a]$.

Lemma 9.7. *Suppose $N \geq d \geq 3$ and a homogeneous polynomial F of degree d on $\mathbb{C}^N$ defines a smooth projective hypersurface $X = \{F = 0\} \subset \mathbb{P}^{N-1}$. There is a nonempty Zariski-open subset $\Omega \subset \mathbb{P}^{N-1} \setminus X$ such that, for every $[a] \in \Omega$ and $1 \leq k \leq d - 2$, the differentials of $D_a F, \dots, D_a^k F$ restricted to X are linearly independent at their common zeros. The differentials here are taken in local trivializations of the corresponding line bundles.*

Proof. Fix a hyperplane $L \simeq \mathbb{P}^{N-2}$. Projection from $[a] \notin X \cup L$ gives a finite map $p_a : X \to L$: a positive-dimensional fiber would contain its whole line through $[a]$, contradicting $[a] \notin X$. Its differential has kernel of dimension at most one. We apply Alzati–Ottaviani [1, Theorems 3.10 and 3.11] to the smooth projective hypersurface X and point centers $T = \{[a]\}$. In their notation, the embedding has ambient dimension $N - 1$ and codimension 1, and the center has dimension $t = 0$. The center space is $\mathcal{F} = \mathbb{P}^{N-1} \setminus (X \cup L)$, and $\dim L = N - 2 = \dim X$.

Here $J^k(X,L)$ is the bundle of k-jets of maps from X to L, and $j^k p_a$ records the value and derivatives of p_a through order k. Let $I_k = (\underbrace{1,\dots,1}_{k})$ be the finite symbol with no terminal zero: it prescribes only the first k entries, allowing further tangencies. The corresponding Thom–Boardman locus $\Sigma^{I_k} \subset J^k(X,L)$ is a smooth locally closed algebraic subvariety, by [1, Proposition 2.25 and Theorem 2.28], with the global construction in [1, Definition 2.29]. For a corank-one map between manifolds of equal dimension, Σ^{I_k} has codimension k; the local equations below identify its inverse image with the successive tangency locus.

The locus Σ^{I_k} is modular by [1, Theorem 3.10], so [1, Theorem 3.11] gives a nonempty Zariski-open subset $U_k \subset \mathcal{F}$ on which $j^k p_a$ is transverse to Σ^{I_k}. The finite intersection $\Omega = \bigcap_{k=1}^{d-2} U_k$ is nonempty and Zariski open, since $\mathcal{F}$ is irreducible.

We identify the equations supplied by this transversality. Near a common zero $[v]$ of the polars, choose a linear form ℓ with $\ell(a) = 0$ and $\ell(v) \neq 0$. In the chart $\ell = 1$, choose coordinates (z,u,t) with $z \in \mathbb{C}^{N-3}$ so that translation by a is translation in t and the projection forgets t. If P is the equation of X in this chart, then

$$\partial_t^j P = (D_a^j F)|_{\ell=1}.$$

At the point under consideration $P_t = 0$. Smoothness lets us choose u with $P_u \neq 0$ and write X locally as $u = g(z,t)$. Set

$$A = P_u(z,g(z,t),t), \qquad q_j = (\partial_t^j P)(z,g(z,t),t), \qquad h_j = \partial_t^j g.$$

After shrinking, A is a unit. Differentiating $P(z,g(z,t),t) = 0$ repeatedly gives

$$q_j + Ah_j \in (h_1,\dots,h_{j-1}), \qquad (q_1,\dots,q_j) = (h_1,\dots,h_j). \tag{9.4}$$

Indeed, the first identity starts with $q_1 + Ah_1 = 0$, and $\partial_t q_j = q_{j+1} + P_{ut^j}(z,g(z,t),t)h_1$ proves the induction. The equality of ideals follows because A is invertible.

In these coordinates $p_a(z,t) = (z,g(z,t))$. We show inductively that the inverse image of Σ^{I_j} is $S_j = \{h_1 = \cdots = h_j = 0\}$, where $I_j = (1,\dots,1)$ has length j. The critical locus is $S_1 = \{h_1 = 0\}$, and transversality gives $dh_1 \neq 0$ there. On S_1, the kernel of dp_a is spanned by ∂_t. Inductively, on $S_j = \{h_1 = \cdots = h_j = 0\}$ one has

$$\partial_t \in T_p S_j \quad \Longleftrightarrow \quad h_{j+1}(p) = 0, \qquad p \in S_j,$$

because $\partial_t h_i = h_{i+1}$. Thus the locus where the kernel of dp_a is tangent to S_j is S_{j+1}. This identification also holds for the defining ideals: the Jacobian of $(p_a, h_1, \dots, h_j)$ has an identity block in the z variables, so its maximal-minor ideal is $(h_1,\dots,h_{j+1})$. Modulo the preceding equations the new equation is h_{j+1}. Thus the successive Thom–Boardman equations pull back to these generators, and jet transversality gives their differential rank $j+1$. Induction therefore shows that $(h_1,\dots,h_k)$ has differential rank k at its common zeros for every $1 \le k \le d-2$; when $d = 3$ only the first step is needed. At a common zero, (9.4) relates the differentials of the q_j and h_j by an invertible triangular matrix, so the same rank statement holds for the polars. □

Lemma 9.8. *Under the hypotheses and with the open set Ω of Lemma 9.7, for each $[a] \in \Omega$ set*

$$Z_a = \{F = D_a F = \cdots = D_a^{d-2} F = 0\} \subset \mathbb{P}^{N-1}.$$

Then Z_a is a nonempty smooth complete intersection of multidegree $(d, d-1, \dots, 2)$, dimension $N-d$, and degree $d!$. Its points span $\mathbb{P}^{N-1}$. Moreover, the incidence variety

$$\mathcal{I} = \{([a],[v]) \in \Omega \times X : [v] \in Z_a\}$$

is smooth, and its projection $\pi : \mathcal{I} \to \Omega$ is a surjective holomorphic submersion with fibers Z_a.

Proof. Let $S = \mathbb{C}[v_1, \dots, v_N]$ and $J = (F, D_aF, \dots, D_a^{d-2}F)$. There are $d-1 < N$ homogeneous generators. If the projective zero set were empty, the radical of J would be the irrelevant maximal ideal of height N, contrary to Krull's height theorem [60, Tag 0BBZ]. Thus Z_a is nonempty. The preceding lemma shows that the equations, including F, have independent differentials in projective coordinates along Z_a. Hence it is smooth of dimension $N-d$, and J has height $d-1$. The polynomial ring is Cohen–Macaulay [60, Tag 00ND]. In its localization at the irrelevant maximal ideal, an ideal of height $d-1$ generated by these $d-1$ homogeneous elements is a complete intersection, and these generators form a regular sequence [60, Tag 02JN]. Since the elements are homogeneous, regularity can be checked at that maximal ideal, so the sequence is regular in S as well. The associated projective scheme is smooth by the differential-rank computation above and therefore has the reduced structure on Z_a. This proves the complete-intersection assertion, and Bézout's theorem [45, Theorem I.7.7] gives

$$\deg Z_a = d(d-1)\cdots 2 = d!.$$

To prove spanning, it remains to show that no nonzero linear form vanishes on Z_a, or equivalently that $H^0(\mathbb{P}^{N-1}, \mathcal{I}_{Z_a}(1)) = 0$. Twist the Koszul resolution [60, Tag 062F] of the ideal sheaf $\mathcal{I}_{Z_a}$ by $\mathcal{O}(1)$. Its terms $K_0, \dots, K_{d-2}$ are sums of negative line bundles. The standard cohomology formula [60, Tag 01XT] gives the vanishings

$$H^i(\mathbb{P}^{N-1}, \mathcal{O}(-k)) = 0 \qquad (k > 0,\ 0 \le i \le N-2)$$

and the successive short exact sequences of the resolution embed $H^0(\mathbb{P}^{N-1}, \mathcal{I}_{Z_a}(1))$ into $H^{d-2}(\mathbb{P}^{N-1}, K_{d-2}) = 0$, since $d-2 \le N-2$. As Z_a is reduced, a linear form vanishing at all its points must therefore be zero. Thus Z_a spans $\mathbb{P}^{N-1}$, including when $d = N$.

Finally, in local projective coordinates, the incidence equations have rank $d-2$ in the X variables. The holomorphic implicit function theorem gives local coordinates on $\mathcal{I}$ consisting of the center coordinates and $N-d$ free fiber coordinates. Thus $\mathcal{I}$ is smooth and $\pi : \mathcal{I} \to \Omega$ is a holomorphic submersion with fibers of dimension $N-d$. These fibers are the nonempty varieties Z_a, so π is surjective. □

9.2.2. *Entire parametrizations of the line complements.* The polar equations produce lines whose complements are $\mathbb{C}$ or $\mathbb{C}^*$. Their intersection divisors make the two cases explicit. Let $D = \{F = 0\}$. If $F(a) \neq 0$ and $[v] \in Z_a$, write $c = F(a)$ and $\lambda = dF_a(v)$. With homogeneous line coordinates $[r : s] \in \mathbb{P}^1$, homogenizing (9.3) gives

$$F(ra + sv) = r^{d-1}(rc + s\lambda).$$

The line $L_{a,v} = \{[ra+sv] : [r : s] \in \mathbb{P}^1\}$ is not contained in the hypersurface, and its intersection divisor is

$$D|_{L_{a,v}} = \begin{cases} (d-1)[v] + [\lambda a - cv], & \lambda \neq 0, \\ d[v], & \lambda = 0. \end{cases}$$

Consequently its complement is $\mathbb{C}^*$ or $\mathbb{C}$, respectively.

The entire function $E : \mathbb{C}^2 \to \mathbb{C}$ used by Buzzard–Lu in their parametrization of tangent-line complements of a smooth plane cubic [10, proof of Proposition 5.1, formulas (5.1)–(5.2)] has the following series expression, valid also at $\lambda = 0$:

$$E(\lambda, t) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{k+1}}{(k+1)!} = \begin{cases} (e^{\lambda t} - 1)/\lambda, & \lambda \neq 0, \\ t, & \lambda = 0. \end{cases}$$

The series converges locally uniformly on $\mathbb{C}^2$ and gives

$$1 + \lambda E(\lambda, t) = e^{\lambda t}, \qquad \partial_t E(\lambda, 0) = 1.$$

For $\lambda \neq 0$ its image is $\mathbb{C} \setminus \{-1/\lambda\}$; for $\lambda = 0$ its image is all of $\mathbb{C}$. The map $t \mapsto [a + cE(\lambda, t)v]$ parametrizes the entire line complement in either case and depends holomorphically on λ through zero.

Lemma 9.9. *Let F be a homogeneous polynomial on $\mathbb{C}^{n+1}$, $Y = \mathbb{P}^n \setminus \{F = 0\}$, and $V \subset Y$ an open set. Suppose there are nonzero holomorphic lifts $a, v_1, \dots, v_n : V \to \mathbb{C}^{n+1}$ such that $[a(y)] = y$, the classes of $v_1(y), \dots, v_n(y)$ form a basis modulo $\mathbb{C}a(y)$, and*

$$F(a(y) + sv_j(y)) = F(a(y)) + s\, dF_{a(y)}(v_j(y)) \quad (s \in \mathbb{C}). \tag{9.5}$$

Then every point of V admits centered entire families with initial tangent vectors spanning its tangent space.

Proof. Set $c(y) = F(a(y))$ and $\lambda_j(y) = dF_{a(y)}(v_j(y))$. The required families are

$$h_j(y,t) = [a(y) + c(y)E(\lambda_j(y), t)v_j(y)].$$

The expression inside brackets has F-value $c(y)e^{\lambda_j(y)t} \neq 0$, so it is nonzero and defines a holomorphic map $h_j : V \times \mathbb{C} \to Y$. It is centered at y. For projectivization $q : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$,

$$\partial_t h_j(y, 0) = dq_{a(y)}(c(y)v_j(y)).$$

Since $\ker dq_{a(y)} = \mathbb{C}a(y)$ and $c(y) \neq 0$, these initial tangent vectors form a basis of T_yY for every $y \in V$. □

9.2.3. *The degree threshold.*

Proposition 9.10. *If $D \subset \mathbb{P}^n$ is a smooth hypersurface of degree $1 \leq d \leq n+1$, where $n \geq 1$, then $\mathbb{P}^n \setminus D$ is Oka.*

Proof. Write $D = \{F = 0\}$ and $Y = \mathbb{P}^n \setminus D$. The manifold Y is connected: a line joining two of its points is not contained in D and loses only finitely many points on removing D.

For $d = 1$, $Y \simeq \mathbb{C}^n$. For $d = 2$, the quadratic form F is nondegenerate, so we may choose coordinates with $F(z) = z_0^2 + \cdots + z_n^2$. The vectors $e_0 \pm \mathrm{i}e_j$, $1 \leq j \leq n$, are isotropic. Their sums and differences span $\mathbb{C}^{n+1}$, so the smooth quadric spans $\mathbb{P}^n$. This also covers $n = 1$, when the quadric is two distinct points. At any $y_0 = [a_0] \in Y$, choose constant isotropic vectors $v_1, \dots, v_n$ whose classes form a basis modulo $\mathbb{C}a_0$, together with a local lift $a(y)$ near y_0, and shrink its domain V so that this basis property persists. The quadratic identity

$$F(a + sv_j) = F(a) + s\, dF_a(v_j) + s^2 F(v_j) = F(a) + s\, dF_a(v_j)$$

allows Lemma 9.9 to be applied at every point. The LEP criterion, Theorem 1.7, makes Y Oka.

Now suppose $3 \leq d \leq n+1$ and apply Lemma 9.8 with $N = n+1$ and $X = D$. For any $y_0 = [a_0] \in \Omega$, spanning by Z_{a_0} gives points $[v_1^0], \dots, [v_n^0] \in Z_{a_0}$ whose representatives form a basis modulo $\mathbb{C}a_0$. The submersion $\pi : \mathcal{I} \to \Omega$ has local holomorphic sections through $([a_0], [v_j^0])$. On a common neighborhood V of y_0, local trivializations of the tautological line bundle give nonzero holomorphic lifts $a, v_1, \dots, v_n : V \to \mathbb{C}^{n+1}$ with

$$[a(y)] = y, \qquad [v_j(y)] \in Z_{a(y)}.$$

The determinant $\det(a, v_1, \dots, v_n)$ is nonzero at y_0 and remains so after shrinking V. Hence the classes of $v_1(y), \dots, v_n(y)$ form a basis modulo $\mathbb{C}a(y)$ throughout V. Identity (9.3) supplies (9.5). Lemma 9.9 therefore gives centered entire families at every point of Ω.

Let $G_Y \subset Y$ be the set of points at which Y satisfies LEP. We have shown $\Omega \subset G_Y$. Since $Y \setminus \Omega$ is a proper closed algebraic, hence analytic, subset of the connected n-manifold Y, it has zero $(2n-1)$-dimensional Hausdorff measure. Theorem 1.11 shows that Y is Oka throughout $d \leq n+1$. □

Proof of Theorem 1.14. Proposition 9.10 proves the positive direction. Write $Y = \mathbb{P}^n \setminus D$. For $d \geq n+2$, the line bundle

$$\mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n} + D) = \mathcal{O}_{\mathbb{P}^n}(d - n - 1)$$

is ample. Carlson–Griffiths' degeneracy theorem [16, Corollary 4.17] for the complement of a simple normal crossings divisor with positive log canonical class applies to the smooth divisor D. It says that every holomorphic map $\mathbb{C}^n \to Y$ has rank less than n everywhere. An Oka manifold is strongly dominable by Theorem 1.5, so Y cannot be Oka. This also treats $n = 1$ and $d \geq 3$.

Finally, D is ample, so Y is affine and hence Stein, by the affine-complement argument in Subsection 9.1. A Stein Oka manifold is holomorphically elliptic, and a holomorphically elliptic manifold is Oka [28, Corollary 5.6.14 and Proposition 5.6.15]. Thus the two analytic properties are equivalent here, completing all three implications. □

Example 9.11. Smoothness is essential to the degree threshold. In homogeneous coordinates $[X : Y : Z]$ on $\mathbb{P}^2$, for every $m \geq 3$ the reduced plane divisor

$$D_m = \{Z(YZ^{m-1} - X^m) = 0\} \subset \mathbb{P}^2$$

has degree $m+1 > 3$ and is singular at $[0 : 1 : 0]$. Its complement lies in the chart $Z \neq 0$. In the affine coordinates $x = X/Z$ and $y = Y/Z$, the polynomial automorphism $(x, y) \mapsto (x, y - x^m)$ identifies it with $\mathbb{C} \times \mathbb{C}^*$. It is therefore Oka and holomorphically elliptic. This gives singular hypersurfaces with Oka complements in arbitrarily high degree.

10. Further questions

A compact Kähler manifold is *special* in Campana's sense if it admits no meromorphic fibration onto a positive-dimensional base of general type with its induced orbifold structure [12, Definition 2.1].

Conjecture 10.1. *Every connected smooth complex projective manifold that is special in Campana's sense is Oka-1.*

Conjecture 10.2. *Every connected smooth complex projective Oka-1 manifold is Oka.*

Conjecture 10.3. *Let $\pi : \mathcal{X} \to S$ be a smooth projective morphism of complex manifolds, with connected base S and connected fibers. If one fiber of π is Oka, then every fiber is Oka.*

The first two conjectures together would make every special smooth projective manifold Oka, answering this case of Campana–Winkelmann's question [13, Question 1.8]; see also Forstnerič [39, Problem 11.3]. Related Oka-1 and deformation questions appear in [2, Problem 2.11(c)] and [31, Problem F].

Every compact Kähler Oka manifold is special [12, Corollary 8.11]. Thus the first two conjectures would identify the special, Oka-1, and Oka classes in the projective category. Together with Campana's conjecture that specialness is invariant under smooth deformation [12, Conjecture 6.23], they would imply Conjecture 10.3.

The first two conjectures are consistent with the known cases: rationally connected manifolds are Oka (Theorem 1.13), hence Oka-1 and special; the same holds for all K3 surfaces [55, Theorem 1.1], the Schoen fiber products with disjoint singular-value sets, and sufficiently general Gross–Popescu Calabi–Yau threefolds [3, Theorems 8.7–8.8].

Analogues of all three conjectures can be posed for compact Kähler manifolds.

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