SCIENCE CHINA
Mathematics

• ARTICLES •

December 2025 Vol. 68 No. 12: 2847–2866
https://doi.org/10.1007/s11425-023-2402-0

Entire holomorphic curves into $\mathbb{P}^n(\mathbb{C})$ intersecting $n+1$ general hypersurfaces

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Received September 24, 2023; accepted March 14, 2025; published online August 20, 2025

Abstract Let $\{D_i\}_{i=1}^{n+1}$ be $n+1$ hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ with total degrees $\sum_{i=1}^{n+1}\deg D_i \geqslant n+2$, in general position and satisfying a generic geometric condition: every n hypersurfaces intersect only at smooth points, and their intersections are transversal. For every algebraically nondegenerate entire holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, we establish a Second Main Theorem:

$$\sum_{i=1}^{n+1} \delta_f(D_i) < n+1,$$

expressed as a defect inequality in Nevanlinna theory. This result provides the first example in the literature of a Second Main Theorem for $n+1$ general hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ with optimal total degrees.

Keywords Nevanlinna theory, Second Main Theorem, entire curves, parabolic Riemann surfaces, semi-abelian varieties

MSC(2020) 32H30, 32Q45, 14J99

Citation: Chen Z C, Huynh D T, Sun R R, et al. Entire holomorphic curves into $\mathbb{P}^n(\mathbb{C})$ intersecting $n+1$ general hypersurfaces. Sci China Math, 2025, 68: 2847–2866, https://doi.org/10.1007/s11425-023-2402-0

1 Introduction

Given a codimension one subvariety D in a complex manifold X such that the complement $X \setminus D$ satisfies certain complex hyperbolicity properties in the spirit of the Kobayashi conjecture [22] or the Green-Griffiths conjecture [17], one aims to achieve a quantitative strengthening in terms of a Second Main Theorem in Nevanlinna theory. Such a theorem bounds, in a proportional manner, the "growth rate" of an algebraically nondegenerate holomorphic map $f\colon S \to X$ from a source space S (typically $\mathbb{C}$) from above by the "intersection frequency" or "impact" of $f(S)$ with respect to D.

The classical example is Nevanlinna's celebrated work [24], which quantifies the following theorem.

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Theorem 1.1 (Little Picard theorem). *If p_1, p_2 and p_3 are three distinct points in $\mathbb{P}^1(\mathbb{C})$, then any meromorphic function $f\colon \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \setminus \{p_1, p_2, p_3\}$ is constant.*

For higher-dimensional target spaces X, various source spaces S and holomorphic maps $f\colon S \to X$, we refer the reader to [15, 25, 27, 34, 37] for subsequent developments. The central problem in this direction is the following conjecture.

Conjecture 1.2 (Fundamental conjecture of entire curves [18, 27]). Let D be a simple normal crossing divisor on the projective space $\mathbb{P}^n(\mathbb{C})$ of degree $d \geqslant n+2$. Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an entire holomorphic curve. If the image of f is not contained in any hypersurface, then the following Second Main Theorem type estimate holds:

$$(d-n-1)\,T_f(r) \leqslant N_f^{[k_0]}(r, D) + o(T_f(r)) \quad \|, \tag{1.1}$$

where $k_0 \in \mathbb{N}$ is a positive integer independent of f.

Here, $N_f^{[k_0]}(r, D)$ and $T_f(r)$ are standard notions in Nevanlinna theory, which will be introduced in the next paragraph. For non-negatively valued functions $\phi(r)$ and $\psi(r)$ defined for $r \geqslant r_0 \geqslant 0$, we write

$$\phi(r) \leqslant \psi(r) \quad \|$$

if the inequality holds for $r \geqslant r_0$ outside a Borel set of finite Lebesgue measure.

Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an entire holomorphic curve, and let $D \subset \mathbb{P}^n(\mathbb{C})$ be a hypersurface such that $f(\mathbb{C}) \not\subset D$. The order function

$$T_f(r) := \int_1^r \frac{dt}{t} \int_{\mathbb{D}_t} f^*\omega_{\mathsf{FS}}, \quad r > 1$$

is a geometric equivalent version of Nevanlinna's characteristic function, historically discovered independently by Shimizu and Ahlfors (see [27, pp. 11 and 12]). It measures the area growth of the image of the disc $\mathbb{D}_r$ centered at 0 with radius r with respect to the Fubini-Study metric ω_{FS}. For $k \in \mathbb{N} \cup \{\infty\}$, the level-$k$ truncated counting function

$$N_f^{[k]}(r, D) := \int_1^r \frac{dt}{t} \sum_{|z|<t} \min\{k, \mathrm{ord}_z f^*D\}$$

captures the intersection frequencies of $f(\mathbb{C}) \cap D$. The defect of f with respect to D is given by

$$\delta_f^{[k]}(D) := \liminf_{r\to\infty} \left(1 - \frac{N_f^{[k]}(r, D)}{\deg(D)\,T_f(r)}\right).$$

For brevity, when $k = \infty$, we write $N_f(r, D)$ and $\delta_f(D)$ instead of $N_f^{[\infty]}(r, D)$ and $\delta_f^{[\infty]}(D)$, respectively.

The *First Main Theorem* in Nevanlinna theory, which is a reformulation of the Lelong-Jensen formula, provides an upper bound for the counting function in terms of the order function:

$$N_f(r, D) \leqslant d\,T_f(r) + O(1),$$

which implies

$$0 \leqslant \delta_f(D) \leqslant 1.$$

The reverse direction, i.e., bounding the order function from above by the sum of counting functions of multiple divisors, is typically much more challenging. Results of this type are called *Second Main Theorems*. The problem of establishing a satisfactory estimate of the form (1.1) remains largely open in general. When D consists of $q \geqslant n+2$ hyperplanes $H_i \subset \mathbb{P}^n(\mathbb{C})$ in general position $(1 \leqslant i \leqslant q)$ and $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is linearly nondegenerate, such a Second Main Theorem with truncation at the level n was established by Cartan [6], yielding the following *defect relation*:

$$\sum_{i=1}^q \delta_f^{[n]}(H_i) \leqslant n+1.$$

When all components of D are hypersurfaces and the image of f is not contained in D, a Second Main Theorem without an effective truncation level was obtained by Eremenko and Sodin [14] using potential theoretic methods, which implies a defect relation bounded by $2n$. Under the additional assumption that f is algebraically nondegenerate, stronger estimates were established in [1, 33, 41]. Ru [33] proved that $\sum_{j=1}^{q} \delta_f(D_j) \leqslant n+1$. Yan and Chen [41] showed that for any $\epsilon > 0$, there exists some $M > 0$ such that for any algebraically nondegenerate f, the inequality $\sum_{j=1}^{q} \delta_f^{[M]}(D_j) \leqslant n+1+\epsilon$ holds. An and Phuong [1] provided an explicit estimate of M in terms of d_j, n and ϵ.

In [18], Griffiths conjectured (1.1) for $k_0 = \infty$ on the right-hand side. This conjecture quantifies as follows.

Conjecture 1.3 (Logarithmic Green-Griffiths conjecture [17])**.** If D is a simple normal crossing divisor on the projective space $\mathbb{P}^n(\mathbb{C})$ of degree $d \geqslant n+2$, then the image of any holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ omitting D lies in some proper algebraic subvariety of $\mathbb{P}^n(\mathbb{C})$.

When D has $q \leqslant n+1$ components, few Second Main Theorem type results toward (1.1) are known. In hindsight, the difficulty is closely related to establishing the (conjectured) hyperbolicity property of $X \setminus D$ using the jet differential technique introduced by Bloch [3]. Indeed, on the one hand, the logarithmic fundamental vanishing theorem of entire curves states that any negatively twisted logarithmic (along D) k-jet differential ω serves as an obstruction for the existence of entire curves $f : \mathbb{C} \to X \setminus D$, since f must satisfy the differential equation $f^*\omega \equiv 0$ (see, e.g., [34]). On the other hand, it is shown in [21, Theorem 3.1] that, for any negatively twisted logarithmic (along D) k-jet differential ω and for any entire curve $f : \mathbb{C} \to X$ not contained in D, if $f^*\omega \not\equiv 0$, then one can derive a Second Main Theorem (SMT) for the entire curve f with respect to the divisor D. Thus, to demonstrate the hyperbolicity of $X \setminus D$ or to obtain an SMT for f with respect to D, one aims to find sufficiently many negatively twisted logarithmic (along D) k-jet differentials $\{\omega_i\}_{i=1}^{M}$ with "tiny" common base loci that support no entire curve therein.

However, in practice, such an approach is highly challenging. For example, in the simplest case where $k = 1$ and $D = \emptyset$, there was a related conjecture by Debarre [11], which states that for general $c \geqslant n/2$ hypersurfaces $H_1, \ldots, H_c \subset \mathbb{P}^n(\mathbb{C})$ with sufficiently large degrees $\gg 1$, the intersection $X := H_1 \cap \cdots \cap H_c$ should have ample cotangent bundle T_X^*. The Debarre ampleness conjecture was first proved in [39] (see also arXiv:1510.06323), where the difficulty of controlling the base loci was addressed through the ad hoc symmetry of certain sophisticated deformed Fermat-type polynomial equations, utilizing explicit 1-jet differentials obtained in [4]. Another proof, appearing shortly thereafter in [5] (see also arXiv:1511.04709), employed more symmetric generalized Fermat-type polynomial equations.

In line with Siu's strategy [36] for the Kobayashi and Green-Griffiths conjectures, which involves the use of slanted vector fields [10, 23, 35] and certain Riemann-Roch calculations [9], a Second Main Theorem [21] was established for the case $q = 1$. This result applies to algebraically nondegenerate entire curves $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ with respect to a general hypersurface $D \subset \mathbb{P}^n(\mathbb{C})$ of large degree $d \geqslant 15\,(5\,n+1)\,n^n$. Thanks to the breakthrough by Riedl and Yang [31], the Zariski density assumption on $f(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ can now be removed.

From now on, a finite collection of hypersurfaces $D_1, \ldots, D_q$, where $q \geqslant n$, is said to be *intersecting transversally* if for any n distinct hypersurfaces $D_{j_1}, \ldots, D_{j_n}$ and for any point z in their intersection, the following conditions hold:

- z is a *smooth point* of each D_{j_k}, where $1 \leqslant k \leqslant n$;
- the *normal vectors* of the tangent spaces $T_z D_{j_k}$, $1 \leqslant k \leqslant n$ are *linearly independent*.

In this article, we focus on the case where

$$D = \bigcup_{i=1}^{q} D_i$$

consists of $q = n+1$ hypersurfaces $D_i \subset \mathbb{P}^n(\mathbb{C})$ (not all of which are hyperplanes) that are in general position and intersect transversally. The algebraic degeneracy of entire holomorphic curves in the complement

$$\mathbb{P}^n(\mathbb{C}) \setminus D$$

was established by Noguchi et al. [28]. For moving target versions, see [19, Theorem 1.6] and [20, Theorem 1.2]. Quantitatively, we establish the following defect relation.

Theorem 1.4 (The main theorem). *Let $\{D_i\}_{i=1}^{n+1}$ be $n+1$ hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ with total degrees $\sum_{i=1}^{n+1}\deg D_i \geqslant n+2$, in general position and intersecting transversally. Then for every algebraically nondegenerate entire holomorphic curve $f\colon \mathbb{C}\to\mathbb{P}^n(\mathbb{C})$, the following defect inequality holds*:

$$\sum_{i=1}^{n+1}\delta_f(D_i) < n+1. \tag{1.2}$$

Clearly, $\delta_f(D_i)=1$ if and only if

$$N_f(r,D_i)=o(T_f(r)), \quad r\to\infty, \tag{1.3}$$

which means that the curve f does not intersect D_i frequently. Therefore, (1.2) serves as a weak Second Main Theorem.

As a matter of fact, our initial motivation stems from the study of the case of 3 conics in $\mathbb{P}^2(\mathbb{C})$ [13,16]. Even in this seemingly simple case, the aforementioned methods appear to be ineffective.

Returning to our main theorem, we adopt an alternative geometric approach in which the number

$$n+1=\dim_{\mathbb{C}}\mathbb{P}^n(\mathbb{C})+1$$

of components of D plays a critical role. Let us now outline the proof. For simplicity, we assume that each hypersurface $D_i\subset\mathbb{P}^n(\mathbb{C})$ is defined by a homogeneous polynomial $Q_i\in\mathbb{C}[z_0,\dots,z_n]$ of equal degree d. Suppose, for contradiction, that (1.2) fails, i.e., all defect values attain their maximum:

$$\delta_f(D_i)=1, \quad i=1,\dots,n+1. \tag{1.4}$$

For the parabolic Riemann surface $\mathbb{C}\backslash f^{-1}(D)$, we employ an exhaustion function σ such that the weighted Euler characteristic $\mathfrak{X}_\sigma(r)$ is negligible:

$$\limsup_{r\to\infty}\frac{\mathfrak{X}_\sigma(r)}{T_{f,\sigma}(r)}=0 \tag{1.5}$$

compared with the parabolic order function $T_{f,\sigma}(r)$ (see Section 2).

The key idea is to introduce the auxiliary hypersurface $\mathcal{V}\subset\mathbb{P}^n(\mathbb{C})$, defined by the Jacobian

$$\det\frac{\partial(Q_1,\dots,Q_{n+1})}{\partial(z_0,\dots,z_n)},$$

which has degree $\sum_{i=0}^n d_i-(n+1)$. Such a hypersurface $\mathcal{V}$ was utilized in [2, p. 261] and [8] for $n=2$, and in [19, 20] for general n. Geometrically, $\mathcal{V}$ consists of the critical points of the endomorphism

$$F(z)=[Q_1(z):Q_2(z):\cdots:Q_{n+1}(z)]\colon \mathbb{P}^n(\mathbb{C})\to\mathbb{P}^n(\mathbb{C}).$$

Consequently, if the entire curve f intersects $\mathcal{V}$ at a point $P\in\mathbb{C}$, the composition $g:=F\circ f$ must be tangent to $\mathcal{W}:=F(\mathcal{V})$, meaning that the intersection multiplicity at P is at least 2.

For $\{D_i\}_{i=1}^{n+1}$ in general position, the hypersurface $\mathcal{V}$ is *in general position with* $\{D_i\}_{i=1}^{n+1}$, i.e., $\mathcal{V}$ and any n hypersurfaces among $\{D_i\}_{i=1}^{n+1}$ have empty intersection, *if and only if* $\{D_i\}_{i=1}^{n+1}$ intersect transversally. The *if* part is provided in [20, Section 5]. We prove the *only if* part in Lemma 4.1. Thus, under the presumed condition (1.4), we can apply a Second Main Theorem of Ru [33] to show that the intersection frequency of the holomorphic curve $\widetilde{f}:=f|_{\mathbb{C}\backslash f^{-1}(D)}$ with $\mathcal{V}$ must be high. This will contradict another fact, to be established in Section 2 following a strategy of Noguchi et al. [29], that the parabolic holomorphic curve $\widetilde{g}:=g|_{\mathbb{C}\backslash f^{-1}(D)}$ into the semi-abelian variety $(\mathbb{C}^*)^n\subset\mathbb{P}^n(\mathbb{C})$ cannot be tangent to the effective divisor $\mathcal{W}$ too frequently. For detailed proofs, see Sections 3 and 4.

2 Parabolic Nevanlinna theory in semi-Abelian varieties and projective spaces

A non-compact Riemann surface $\mathcal{Y}$ is called *parabolic* if it admits a smooth exhaustion function

$$\sigma\colon \mathcal{Y} \to [1,\infty)$$

such that $\tau = \log\sigma$ is harmonic outside a compact subset of $\mathcal{Y}$. Such σ is called a *parabolic exhaustion* function of $\mathcal{Y}$ (see [37, p. 3]).

In this article, we utilize an exhaustion function σ on $\mathcal{Y}$ that is harmonic outside a non-compact subset of $\mathcal{Y}$ while satisfying specific boundedness properties, as investigated in [30, pp. 32 and 33]. To streamline our discussion, we introduce the notion of *exhaustion functions with bounded anharmonicity.*

Definition 2.1. Let $\mathcal{Y}$ be a parabolic surface. A smooth exhaustion function $\sigma : \mathcal{Y} \to [1,\infty)$ is called *with bounded anharmonicity* if the smooth $(1,1)$-form $dd^c\tau = dd^c \log\sigma$ is of finite total mass on $\mathcal{Y}$.

Clearly, the property of σ having bounded anharmonicity is weaker than the property of $\tau = \log\sigma$ being harmonic outside a compact set.

From now on, we focus on a specific class of parabolic surfaces equipped with carefully chosen exhaustion functions. Let $\mathcal{E} = \{a_j\}_{j=1}^{\infty}$ be a discrete, countable set of points in $\mathbb{C}$, and let $\mathcal{Y} := \mathbb{C}\setminus\mathcal{E}$. Up to translation and dilation on $\mathbb{C}$, we may assume that $|a_j| > 2$ for all $j \geqslant 1$. We choose radii $r_j \in (0,1)$ shrinking to zero sufficiently fast, as specified later (see (2.4)). These radii must satisfy the following conditions:

- the discs $\mathbb{D}(a_j, 2r_j)$ for all j and the disc $\mathbb{D}_2$ are mutually disjoint;
- the sum $\sum_{j\geqslant 1} r_j < +\infty$.

Here, the condition (2.4) and the two conditions above are independent and must all be satisfied.

Let σ (see Section 3 or Appendix A for details) be a smoothing of the exhaustion function $\hat\sigma$ defined by

$$\log\hat\sigma := \log^+|z| + \sum_{j\geqslant 1} r_j\, \log^+\frac{r_j}{|z-a_j|}, \tag{2.1}$$

with $\sigma \geqslant \hat\sigma$ on $\mathcal{Y}$ and $\sigma = \hat\sigma$ outside the annuli

$$A\biggl(0,\frac{1}{2},\frac{3}{2}\biggr) \cup \bigcup_{j=1}^{\infty} A\biggl(a_j,\frac{1}{2}r_j,\frac{3}{2}r_j\biggr).$$

The smooth $(1,1)$-form $dd^c\tau = dd^c\log\sigma$ is of finite total mass on $\mathcal{Y}$. We remark that

(1) the function σ is not a parabolic exhaustion function since $dd^c\tau$ is not compactly supported;

(2) the surface $\mathcal{Y}$ is parabolic, due to [37, Lemma 10.1(g), p. 75] by taking $\phi\colon \mathcal{Y} \to \mathbb{C}^*$ as the restriction of a transcendental function sending $\mathcal{E}$ to 0.

The classical results in Nevanlinna theory, after some modifications, have analogues in the case of a parabolic surface $\mathcal{Y}$ with an exhaustion function σ of bounded anharmonicity (see [30]).

For every $r > 1$, we denote by

$$B_r^\sigma := \{z \in \mathcal{Y} : \sigma(z) < r\} \quad \text{and} \quad S_r^\sigma := \{z \in \mathcal{Y} : \sigma(z) = r\}$$

the open *parabolic ball* and the *parabolic sphere* of radius r, respectively (see Figure 1). By Sard's theorem, for almost every value $r \in \mathbb{R}_{>1}$, the sphere S_r^σ is smooth. We denote the Euler characteristic of B_r^σ by $\chi_\sigma(r)$, and we consider the induced length measure

$$d\mu_r := d^c\log\sigma|_{S_r^\sigma},$$

where $d^c := \frac{\sqrt{-1}}{4\pi}(\bar\partial - \partial)$. The *weighted Euler characteristic* $\mathfrak{X}_\sigma(r)$ is then defined by the logarithmic average

$$\mathfrak{X}_\sigma(r) := \int_1^r \chi_\sigma(t)\frac{dt}{t}, \quad r > 1.$$

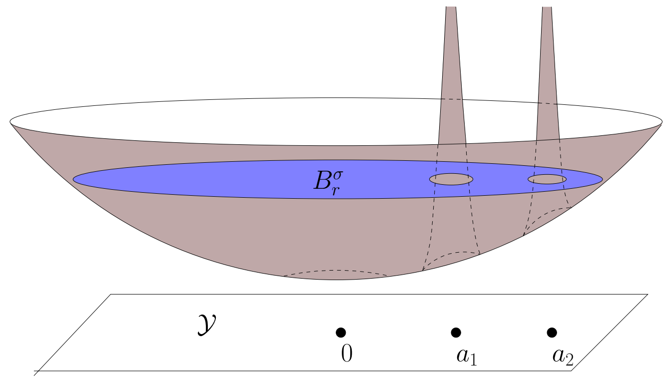

Figure 1 (Color online) The graph of σ and the σ-ball B_r^σ

Replacing the exhaustion $\mathbb{C} = \bigcup_{r>1} \mathbb{D}_r$ by $\mathcal{Y} = \bigcup_{r>1} B_r^\sigma$, one can develop Nevanlinna theory for parabolic Riemann surfaces (see [30, 37]). Let X be a compact complex manifold, and let L be a holomorphic line bundle on X equipped with a Hermitian metric $\|\cdot\|$ with the Chern $(1,1)$-form ω_L. Let E be an effective divisor defined by a global nonzero section s of L, normalized such that $\sup_{x\in X} \|s(x)\| = 1$.

In this article, we focus on the holomorphic map $\tilde{f} : \mathcal{Y} \to X$, which is the restriction of an entire curve $f : \mathbb{C} \to X$ on $\mathcal{Y}$. In the parabolic context, the standard notions in Nevanlinna theory are defined as follows:

(1) The order function is defined as

$$T_{\tilde{f},\sigma}(r, L) := \int_{t=1}^{r} \frac{dt}{t} \int_{B_t^\sigma} \tilde{f}^*\omega_L, \quad r > 1.$$

(2) The k-truncated counting function is defined as

$$N^{[k]}_{\tilde{f},\sigma}(r, E) := \int_{t=1}^{r} n^{[k]}_{\tilde{f},\sigma}(t, E)\frac{dt}{t}, \quad k \in \mathbb{N} \cup \{\infty\}, \quad r > 1,$$

where

$$n^{[k]}_{\tilde{f},\sigma}(t, E) := \sum_{z\in B_t^\sigma} \min\{k, \mathrm{ord}_z \tilde{f}^*E\}, \quad k \in \mathbb{N} \cup \{\infty\}, \quad t > 1.$$

(3) The proximity function is defined as

$$m_{\tilde{f},\sigma}(r, E) := \int_{S_r^\sigma} \log \frac{1}{\|s \circ \tilde{f}\|} d\mu_r, \quad r > 1.$$

In the important case of a nonconstant meromorphic function $\tilde{f} := [f_0 : f_1]\colon \mathcal{Y} \to \mathbb{P}^1(\mathbb{C})$, by taking $L := \mathscr{O}(1)$, $\omega_L := \omega_{\mathsf{FS}}$ and s a section vanishing at $\infty = [0 : 1]$, we see that the proximity function becomes

$$m_{\tilde{f},\sigma,\infty}(r) := \int_{S_r^\sigma} \log^+ |f_1/f_0| d\mu_r, \quad r > 1.$$

Define $N_{\tilde{f},\sigma}(r, E) := N^{[\infty]}_{\tilde{f},\sigma}(r, E)$ and $n_{\tilde{f},\sigma}(t, E) := n^{[\infty]}_{\tilde{f},\sigma}(t, E)$.

Since $\log \sigma \geqslant \log \hat{\sigma} \geqslant \log^+ |z|$, it follows that $B_t^\sigma \subset \mathbb{D}_t$ for any $t \geqslant 1$. Hence,

$$n^{[k]}_{\tilde{f},\sigma}(t, E) \leqslant n_f(t, E),$$

which implies

$$N_{\tilde{f},\sigma}(r, E) \leqslant N_f(r, E), \quad r > 1. \tag{2.2}$$

The classical First Main Theorem in Nevanlinna theory states that for an entire curve $f : \mathbb{C} \to X$ such that $f(\mathbb{C}) \not\subset \mathrm{Supp}(E)$, it holds that

$$N_f(r, E) \leqslant T_f(r, L) + O(1), \quad r > 1. \tag{2.3}$$

In (4.4), we show that when $r_j > 0$ shrink to zero sufficiently fast, e.g., when

$$\int_{\mathbb{D}(a_j, \frac{3}{2}r_j)} f^*\omega_L < 2^{-j}, \quad j \geqslant 1, \tag{2.4}$$

there exists a smoothing σ of $\hat{\sigma}$ with bounded anharmonicity such that $\sigma \geqslant \hat{\sigma}$ and

$$0 \leqslant T_f(r, L) - T_{\tilde{f},\sigma}(r, L) \leqslant \log r + O(1), \quad r > 1. \tag{2.5}$$

Combining (2.2), (2.3) and (2.5), we obtain the following useful inequality.

Theorem 2.2 (Parabolic First Main Theorem (inequality version)). *Let $\mathcal{Y} = \mathbb{C} \setminus \mathcal{E}$ be a parabolic surface. Let $\tilde{f}\colon \mathcal{Y} \to X$ be the restriction of an entire curve $f\colon \mathbb{C} \to X$. Suppose that $f(\mathcal{Y}) \not\subset \mathrm{Supp}(E)$. Let σ be a smooth exhaustion function on $\mathcal{Y}$ satisfying* (2.5). *Then*

$$N_{\tilde{f},\sigma}(r, E) \leqslant T_{\tilde{f},\sigma}(r, L) + \log r + O(1), \quad r > 1. \tag{2.6}$$

The following inequality is useful in proving the parabolic logarithmic derivative lemma. In the case where $X = \mathbb{P}^n(\mathbb{C})$ and $\omega_L = \omega_{\mathsf{FS}}$, define $T_{\tilde{f},\sigma}(r) := T_{\tilde{f},\sigma}(r, \omega_{\mathsf{FS}})$. For any $w \in \mathbb{P}^1(\mathbb{C})$ and $r \geqslant 1$, define

$$N_f(r, w) := \int_{t=1}^{r} n_f(t, w)\, \frac{dt}{t}, \quad N_{\tilde{f},\sigma}(r, w) := \int_{t=1}^{r} n_{\tilde{f},\sigma}(t, w) \frac{dt}{t},$$

where $n_f(t, w)$ (resp. $n_{\tilde{f},\sigma}(t, w)$) counts the solutions of $f(z) = w$ for $z \in \mathbb{D}_t$ (resp. $z \in B_t^{\sigma}$) with multiplicity.

Theorem 2.3 (Parabolic Nevanlinna's inequality). *Let $\mathcal{Y} = \mathbb{C} \setminus \mathcal{E}$ be a parabolic surface. Let $\tilde{f}\colon \mathcal{Y} \to \mathbb{P}^1(\mathbb{C})$ be the restriction of a nonconstant meromorphic function $f\colon \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$. Let σ be a smooth exhaustion function on $\mathcal{Y}$ with $\log\sigma \geqslant \log^+|z|$ and*

$$0 \leqslant T_f(r) - T_{\tilde{f},\sigma}(r) \leqslant \log r + O(1), \quad r > 1. \tag{2.7}$$

Then there exists a constant $C > 0$, independent of $w \in \mathbb{P}^1(\mathbb{C})$, such that

$$N_{\tilde{f},\sigma}(r, w) \leqslant T_{\tilde{f},\sigma}(r) + \log r + C, \quad r > 1. \tag{2.8}$$

Proof. The inequality follows from a combination of (2.2), (2.3), (2.7) and the classical Nevanlinna's inequality (see [27, Theorem 1.1.18]) that

$$N_f(r, w) \leqslant T_f(r) + C. \tag{2.9}$$

This completes the proof. □

To state a parabolic Second Main Theorem, the weighted Euler characteristic function naturally plays a crucial role. It is defined as (see [30, Definition 3.4])

$$\mathfrak{X}_{\sigma}^{+}(r) := \int_{S_r^{\sigma}} \log^+ \left| d\sigma\left(\frac{\partial}{\partial z}\right)\right|^2 d\mu_r.$$

Using the properties that the disks $\mathbb{D}(a_j, 2r_j)$ are disjoint and $\sum_{j\geqslant 1} r_j < +\infty$, we see from [30, Example (2), pp. 32 and 33] that

$$\mathfrak{X}_{\sigma}^{+}(r) = \mathfrak{X}_{\sigma}(r) + O(\log r) = N_{\mathcal{E}}(r) + O(\log r), \quad r > 1, \tag{2.10}$$

where $N_{\mathcal{E}}(r) := \int_{t=1}^{r} n_{\mathcal{E}}(t)\frac{dt}{t}$ for $r \geqslant 1$, and $n_{\mathcal{E}}(t)$ counts the number of points in $\mathcal{E} \cap \mathbb{D}_t$ for $t \geqslant 1$.

Using the parabolic Nevanlinna's inequality, one can prove the following important lemma.

Lemma 2.4 (Parabolic logarithmic derivative lemma [30, Remark 3.9, p. 28]). *Let $\mathcal{Y} = \mathbb{C} \setminus \mathcal{E}$ be a parabolic surface. Let $\tilde{f}\colon \mathcal{Y} \to \mathbb{P}^1(\mathbb{C})$ be the restriction of a nonconstant meromorphic function $f\colon \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$. Let σ be a smooth exhaustion function on $\mathcal{Y}$ of bounded anharmonicity that satisfies $\log\sigma \geqslant \log^+|z|$ and* (2.7)*. Then there exists a positive constant $C > 0$ such that*

$$\begin{aligned} m_{\frac{\tilde{f}'}{\tilde{f}},\sigma,\infty}(r) &\leqslant C(\log T_{\tilde{f},\sigma}(r) + \log r) + \mathfrak{X}_\sigma^+(r) \quad \| \\ &\leqslant C(\log T_{\tilde{f},\sigma}(r) + \log r) + N_{\mathcal{E}}(r) \quad \| \,. \end{aligned}$$

Proof. The proof requires modifying the approach in [30, Theorem 3.8, pp. 26–28] as outlined in [30, Remark 3.9, p. 28]. A key observation is that the inequality [30, (17), p. 27] no longer holds in our setting. Instead, the correct version, derived as a consequence of the parabolic Nevanlinna's inequality (2.8), is

$$\int_{t=1}^{r} \frac{dt}{t} \int_{B_t^\sigma} \tilde{f}^*\Omega = \int_{w\in\mathbb{C}} N_{\tilde{f},\sigma}(r,w)\Omega \leqslant \int_{\mathbb{C}} T_f(r)\Omega + \int_{\mathbb{C}} C\Omega$$

for some constant $C > 0$ independent of $r > 1$. Here, the auxiliary form

$$\Omega := \frac{1}{|w|^2(1+\log^2|w|)} \frac{\sqrt{-1}}{2\pi} dw \wedge d\overline{w},$$

introduced in [30, (15), p. 27], has finite volume on $\mathbb{C}$.

Notably, the right-hand side involves the classical $T_f(r)$ instead of the parabolic $T_{\tilde{f},\sigma}(r)$, due to (2.2) and (2.9). Consequently, we obtain

$$\int_{t=1}^{r} \frac{dt}{t} \int_{B_t^\sigma} \tilde{f}^*\Omega \leqslant C_0\, T_f(r)$$

for some constant $C_0 > 0$, as discussed in [30, (17) and (18), p. 27].

The last inequality in the proof of [30, Theorem 3.8, p. 28] becomes

$$\log^+ \frac{1}{r}\frac{d}{dr} \int_{B_r^\sigma} \tilde{f}^*\Omega \leqslant \delta\, \log r + (1+\delta)^2\, \log^+(C_0 T_f(r)) \quad \|,$$

and we deduce (see [30, Theorem 3.8, pp. 27 and 28])

$$m_{\frac{\tilde{f}'}{\tilde{f}},\sigma,\infty}(r) \leqslant (1+\delta)^2\,(\log^+ T_f(r)) + (1+\delta)\,\log r + \mathfrak{X}_\sigma^+(r) \quad \| \,.$$

We conclude by combining this with (2.10) and the fact that $\log^+ T_f(r) = O(\log^+ T_{\tilde{f},\sigma}(r))$, which follows from (2.7). □

Consequently, several results in the value distribution theory of entire holomorphic curves can be extended to the parabolic setting.

In [29], Noguchi et al. established a Second Main Theorem type estimate for k-jet liftings of algebraically nondegenerate entire holomorphic curves f into semi-abelian varieties. Their result features the optimal truncation level-one counting function and includes an error term of the form $\epsilon T_f(r)$, or equivalently, $o(T_f(r))$ (see [40, Lemma 1.5]). This result has several applications in studying the degeneracy of holomorphic curves [28, 29].

This remarkable result can be adapted to the parabolic setting. However, in this context, we must account for the weighted Euler characteristic $\mathfrak{X}_\sigma(r)$, which arises each time the logarithmic derivative lemma is applied. Therefore, from now on, we assume that

$$\limsup_{r\to\infty} \frac{\mathfrak{X}_\sigma(r)}{T_{\tilde{f},\sigma}(r)} = 0, \tag{2.11}$$

which is equivalent to

$$\limsup_{r\to\infty} \frac{\mathfrak{X}_\sigma^+(r)}{T_{\tilde{f},\sigma}(r)} = 0$$

and

$$\limsup_{r\to\infty}\frac{N_{\mathcal{E}}(r)}{T_{\tilde f,\sigma}(r)}=0.$$

For our main theorem, we only need to deal with parabolic holomorphic curves in $(\mathbb{C}^*)^n$. Nevertheless, we must use higher order jets and establish a Second Main Theorem type estimate, not only for divisors, but also for subvarieties of codimension greater than or equal to 2 (see [27, Subsection 2.4.1]). For the notions and the properties of logarithmic k-jet bundles, we refer the readers to [12, 26].

Under the assumption (2.11), we can translate the result of [27, Theorem 6.5.1] for the special case $A := (\mathbb{C}^*)^n$ in the parabolic context as follows.

Theorem 2.5. *Let $\mathcal{Y}=\mathbb{C}\setminus\mathcal{E}$ be a parabolic surface. Let $f:\mathbb{C}\to\mathbb{P}^n(\mathbb{C})$ be an algebraically nondegenerate entire curve such that $f(\mathcal{Y})\subset A:=(\mathbb{C}^*)^n\subset\mathbb{P}^n(\mathbb{C})$. Denote by $\tilde f$ the restriction of f on $\mathcal{Y}$. Let σ be a smooth exhaustion function on $\mathcal{Y}$ of bounded anharmonicity which satisfies $\log\sigma\geqslant\log^+|z|$ and (2.7). Assume (2.11). For an integer $k\geqslant 0$, denote by $J_k\tilde f$ the k-jet lifting of $\tilde f$ and by $X_k(\tilde f)$ the Zariski closure of $J_k\tilde f$ in the k-jet space $J_k(A)$. Let Z be an algebraic reduced subvariety of $X_k(\tilde f)$.*

(1) *There exists a compactification $\bar X_k(\tilde f)$ of $X_k(\tilde f)$ such that*

$$T_{J_k\tilde f,\sigma}(r,\omega_{\bar Z})\leqslant N^{[1]}_{J_k\tilde f,\sigma}(r,Z)+o(T_{\tilde f,\sigma}(r))\quad\|,$$

where $\bar Z$ denotes the closure of Z in $\bar X_k(\tilde f)$.

(2) *Assume furthermore that $\mathrm{codim}_{X_k(\tilde f)}Z\geqslant 2$, and then*

$$T_{J_k\tilde f,\sigma}(r,\omega_{\bar Z})=o(T_{\tilde f,\sigma}(r))\quad\|.$$

(3) *In the case where $k=0$ and Z is an effective divisor D on A, there exists a smooth compactification of A independent of $\tilde f$ such that*

$$T_{\tilde f,\sigma}(r,L(\overline{D}))\leqslant N^{[1]}_{\tilde f,\sigma}(r,D)+o(T_{\tilde f,\sigma}(r,L(\overline{D})))\quad\|.$$

The reader is referred to [27, Subsection 2.4] for standard notations in value distribution theory for coherent ideal sheaves. Combining this with the parabolic First Main Theorem (inequality version) (2.6), we obtain the following corollary.

Corollary 2.6. *Let $\mathcal{Y},f,\tilde f$ and σ be as above. Let D be an effective divisor on $A:=(\mathbb{C}^*)^n$. Then there exists a smooth compactification of A independent of $\tilde f$ such that*

$$N_{\tilde f,\sigma}(r,D)-N^{[1]}_{\tilde f,\sigma}(r,D)=o(T_{\tilde f,\sigma}(r,L(\overline{D})))\quad\|.$$

The proof of Theorem 2.5 will be developed later in this section by adapting the strategy of [29] with necessary modifications. As a first step, we directly extend [27, Lemma 4.7.1] to the parabolic context.

Lemma 2.7. *Let M be a complex projective manifold and let D be a reduced divisor on M. Let $\mathcal{Y}=\mathbb{C}\setminus\mathcal{E}$ be a parabolic surface. Let $f:\mathbb{C}\to M$ be an algebraically nondegenerate entire curve such that $f(\mathcal{Y})\not\subset D$. Denote by $\tilde f$ the restriction of f on $\mathcal{Y}$. Let σ be a smooth exhaustion function on $\mathcal{Y}$ of bounded anharmonicity which satisfies $\log\sigma\geqslant\log^+|z|$ and (2.7). Let ω be a logarithmic (along D) k-jet differential on M. Put $\xi:=\omega(J_k\tilde f)$. Then*

$$m_{\xi,\sigma}(r)\leqslant\mathfrak{S}_{\tilde f,\sigma}(r)+C\,\mathfrak{X}_\sigma(r)=o(T_{\tilde f,\sigma}(r))\quad\|,$$

where C is some positive constant and $\mathfrak{S}_{\tilde f,\sigma}(r)$ is a small term such that for any $\delta>0$,

$$\mathfrak{S}_{\tilde f,\sigma}(r)=O(\log^+T_{\tilde f,\sigma}(r))+\delta\log r\quad\|.$$

For an integer $k\geqslant 0$, let $J_k(A)$ denote the k-jet space of $A=(\mathbb{C}^*)^n$, which reads as

$$J_k(A)=A\times J_{k,A}=A\times\mathbb{C}^{nk}.$$

There is a natural A-action on $J_k(A)$ given by $a\colon (x,v)\to(x+a,v)$ for all $x\in A, v\in\mathbb{C}^{nk}$, where "+" is understood as multiplication. Denote by $J_k\tilde{f}$ the k-jet lifting of $\tilde{f}$ and by $X_k(\tilde{f})$ the Zariski closure of $J_k\tilde{f}$ in the k-jet space $J_k(A)$. Let $B:=\mathrm{St}_A(X_k(\tilde{f}))$ be the stabilizer group with respect to the natural A-action and let $q\colon A\to A/B$ be the quotient map. Then the jet projection method [27, Theorem 6.2.6] together with Lemma 2.7 yield $T_{q\circ\tilde{f},\sigma}(r)=o(T_{\tilde{f},\sigma}(r))$. Moreover, we can assume $\dim B>0$; otherwise we would get $T_{\tilde{f},\sigma}(r)=o(T_{\tilde{f},\sigma}(r))$, which is impossible.

We first establish a Second Main Theorem for jet liftings. Let Z be an algebraic reduced subvariety of $X_k(\tilde{f})$. Let $B^0=\mathrm{St}_A^0(X_k(\tilde{f}))$ denote the identity component of B. Then

$$\dim B^0>0 \quad\text{and}\quad T_{q_k^{B^0}\circ J_k\tilde{f},\sigma}(r)=o(T_{\tilde{f},\sigma}(r)) \quad \|, \tag{2.12}$$

where $q_k^{B^0}\colon J_k(A)\to J_k(A)/B^0\cong(A/B^0)\times J_{k,A}$ is the quotient map. This corresponds to [27, Equation (6.5.9)], and hence, we can translate [27, Theorem 6.5.6] to the parabolic setting as follows.

Lemma 2.8. *There exist a compactification $\bar{X}_k(\tilde{f})$ of $X_k(\tilde{f})$ and a positive integer ℓ_0 such that*

$$\begin{aligned}&m_{J_k\tilde{f},\sigma}(r,\bar{Z})=o(T_{\tilde{f},\sigma}(r)) \quad \|,\\ &T_{J_k\tilde{f},\sigma}(r,\omega_{\bar{Z}})\leqslant N^{[\ell_0]}_{J_k\tilde{f},\sigma}(r,Z)+o(T_{\tilde{f},\sigma}(r)) \quad \|,\end{aligned}$$

where $\bar{Z}$ denotes the closure of Z in $\bar{X}_k(\tilde{f})$.

Our next goal is to show that the "impact" of $J_k\tilde{f}$ on a subvariety of $X_k(\tilde{f})$ with codimension greater than or equal to 2 is relatively small.

Lemma 2.9. *Let $Z\subset X_k(\tilde{f})$ be a subvariety with* $\mathrm{codim}_{X_k(\tilde{f})}Z\geqslant 2$. *Then*

$$T_{J_k\tilde{f},\sigma}(r,\omega_{\bar{Z}})=o(T_{\tilde{f},\sigma}(r)) \quad \| . \tag{2.13}$$

In particular, one has

$$N_{J_k\tilde{f},\sigma}(r,Z)=o(T_{\tilde{f},\sigma}(r)) \quad \| . \tag{2.14}$$

Proof. This result is an analog of [27, Theorem 6.5.17]. Our proof follows the same lines, except for a necessary modification in the first reduction. We reduce to the case where A admits a splitting $A=B\times C$ for B and C being semi-abelian varieties of positive dimensions with

$$B\subset\mathrm{St}_A^0(X_k(\tilde{f})),\quad k\geqslant 0,\quad T_{q^B\circ\tilde{f},\sigma}(r)=o(T_{\tilde{f},\sigma}(r)) \quad \|,$$

where $q^B\colon A\to A/B=C$ denotes the projection to the second factor. To do this, we consider the set of all semi-abelian subvarieties $B\subset A$ such that

$$T_{q^B\circ\tilde{f},\sigma}(r)=o(T_{\tilde{f},\sigma}(r)) \quad \| .$$

We then use (2.12) and repeat the argument in the proof of [27, Theorem 6.5.17]. Note that since we only work with $A=(\mathbb{C}^*)^n$ instead of universal coverings of semi-abelian varieties, the result in [27, Lemma 6.5.25] automatically holds. By Lemma 2.8, it suffices to show that

$$N^{[1]}_{J_k\tilde{f},\sigma}(r,Z)=o(T_{\tilde{f},\sigma}(r)) \quad \| .$$

By induction on the dimension of Z, it suffices to check the above estimate for the nonsingular part Z^{ns} of Z. Following the same lines as in [27, 6.5.3], we can find a sequence $n(\ell)$ such that

$$\lim_{\ell\to\infty}\frac{n(\ell)}{\ell}=0$$

and

$$(\ell+1)\,N^{[1]}_{J_k\tilde{f},\sigma}(r,Z^{ns})\leqslant n(\ell)\,O(T_{\tilde{f},\sigma}(r))+o(T_{\tilde{f},\sigma}(r)) \quad \|,$$

which yields the required estimate. This finishes the proof of Lemma 2.9. □

Proof of Theorem 2.5. We follow the argument in [27, Subsection 6.5.4]. It suffices to consider the case where Z is a reduced Weil divisor on $X_k(\tilde{f})$ with the irreducible decomposition $Z = \sum_i Z_i$. Using Lemma 2.8, we have

$$\begin{aligned} T_{J_k\tilde{f},\sigma}(r,\omega_{\bar{Z}}) &\leqslant N^{[\ell_0]}_{J_k\tilde{f},\sigma}(r,Z) + o(T_{\tilde{f},\sigma}(r)) \quad \| \\ &\leqslant N^{[1]}_{J_k\tilde{f},\sigma}(r,Z) + \ell_0 \sum_{i<j} N^{[1]}_{J_k\tilde{f},\sigma}(r,Z_i\cap Z_j) \\ &\quad + \ell_0 \sum_i N^{[1]}_{J_{k+1}\tilde{f},\sigma}(r,J_1(Z_i)) + o(T_{\tilde{f},\sigma}(r)) \quad \| . \end{aligned} \tag{2.15}$$

Since $\operatorname{codim}_{X_k(\tilde{f})}(Z_i\cap Z_j) \geqslant 2$ for $i \neq j$, the second term on the right-hand side of (2.15) can be estimated by Lemma 2.9 as

$$\ell_0 \sum_{i<j} N^{[1]}_{J_k\tilde{f},\sigma}(r,Z_i\cap Z_j) = o(T_{\tilde{f},\sigma}(r)).$$

We now treat the third term of (2.15). We consider two cases depending on the position of $B^0_{k+1} := \mathrm{St}^0_A(X_{k+1}(\tilde{f}))$ with respect to $\mathrm{St}^0_A(Z_i)$.

Case (1) $B^0_{k+1} \not\subset \mathrm{St}^0_A(Z_i)$. We have (see [27, Lemma 6.6.50])

$$\operatorname{codim}_{X_{k+1}(\tilde{f})}(X_{k+1}(\tilde{f})\cap J_1(Z_i)) \geqslant 2,$$

where we can apply Lemma 2.9 to obtain

$$N^{[1]}_{J_{k+1}\tilde{f},\sigma}(r,J_1(Z_i)) = o(T_{\tilde{f},\sigma}(r)).$$

Case (2) $B^0_{k+1} \subset \mathrm{St}^0_A(Z_i)$. We consider the quotient map $q_k^{B^0_{k+1}} \colon X_k(\tilde{f}) \to X_k(\tilde{f})/B^0_{k+1}$. The image of Z_i under this map is contained in a divisor on $X_k(\tilde{f})/B^0_{k+1}$, and hence, we can argue as in [27, Theorem 6.5.6, Case (a)] to get

$$N^{[1]}_{J_{k+1}\tilde{f},\sigma}(r,J_1(Z_i)) \leqslant N_{J_{k+1}\tilde{f},\sigma}(r,J_1(Z_i)) = o(T_{\tilde{f},\sigma}(r)).$$

This finishes the proof of Theorem 2.5. □

A family $\{D_i\}_{i=1}^q$ of $q \geqslant n+2$ hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ is said to be *in general position* if the intersection of any $n+1$ hypersurfaces from this family is empty, i.e.,

$$\bigcap_{i\in I} D_i = \emptyset, \quad I \subset \{1,2,\ldots,q\}, \quad |I| = n+1.$$

In [33], Ru resolved a conjecture of Shiffman by extending the classical Cartan's Second Main Theorem to the setting of nonlinear targets. In the context of a parabolic surface with a smooth exhaustion function σ of bounded anharmonicity, his result can be adapted as follows.

Theorem 2.10. *Let $\mathcal{Y} = \mathbb{C} \setminus \mathcal{E}$ be a parabolic surface. Let $\{D_i\}_{i=1}^q$ be a family of $q \geqslant n+2$ hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$. For any algebraically nondegenerate holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, denote by $\tilde{f}$ the restriction of f on $\mathcal{Y}$. Let σ be a smooth exhaustion function on $\mathcal{Y}$ of bounded anharmonicity, which satisfies* $\log\sigma \geqslant \log^+|z|$ *and* (2.7). *Then there exists a positive constant C such that*

$$(q-n-1)\,T_{\tilde{f},\sigma}(r) \leqslant \sum_{i=1}^q \frac{N_{\tilde{f},\sigma}(r,D_i)}{\deg(D_i)} + C\,\mathfrak{X}_\sigma(r) + o(T_{\tilde{f},\sigma}(r)) \quad \| .$$

Here is an outline of the proof. First, we reduce the problem to the linear case, where the divisors D_i are hyperplanes. This reduction follows the same approach as in [33], which employs the filtration method introduced by Corvaja and Zannier [7]. The linear case is then established using the parabolic logarithmic derivative lemma, as detailed in [32, 38].

3 A smooth exhaustion function on some parabolic Riemann surface

In this section, we construct a piecewise smooth exhaustion function $\hat{\sigma}$ on the parabolic Riemann surface $\mathbb{C}\backslash\mathcal{E}$, where $\mathcal{E}=\{a_j\}_{j=1}^{\infty}$ is a discrete countable set of points on $\mathbb{C}$. Then we describe a smooth exhaustion function σ close to $\hat{\sigma}$. Details are presented in Appendix A.

Up to translation and dilation on $\mathbb{C}$, we may arrange a_j such that

$$2<|a_1|\leqslant|a_2|\leqslant|a_3|\leqslant\cdots.$$

Take $r_j\in(0,1)$ sufficiently small such that

- the discs $\mathbb{D}(a_j,2r_j)$ for all j and the disc $\mathbb{D}_2$ are mutually disjoint;
- the sum $\sum_{j\geqslant1}r_j<+\infty$.

Let $\mathcal{Y}:=\mathbb{C}\backslash\{a_j\}_{j=1}^{\infty}$ and define a piecewise smooth exhaustion function $\hat{\sigma}:\mathcal{Y}\to[1,+\infty)$ by

$$\hat{\sigma}(z):=\exp\left(\log^+|z|+\sum_{j=1}^{\infty}r_j\,\log^+\frac{r_j}{|z-a_j|}\right). \tag{3.1}$$

In other words,

$$\hat{\tau}:=\log\hat{\sigma}=\log^+|z|+\sum_{j=1}^{\infty}r_j\,\log^+\frac{r_j}{|z-a_j|}.$$

Obviously, the function $\hat{\tau}$

- takes value in $[0,+\infty)$;
- is continuous on $\mathcal{Y}$;
- is smooth, indeed harmonic, outside the circle $S(0,1):=\{|z|=1\}$ and the disjoint circles

$$S(a_j,r_j):=\{|z-a_j|=r_j\}.$$

By the Poincaré-Lelong formula (see, e.g., [27, Theorem 2.2.16]), it is clear that

$$dd^c\hat{\tau}=\frac{1}{2}\,\nu(0,1)+\frac{r_j}{2}\sum_{j=1}^{\infty}(\nu(a_j,r_j)-\delta_{a_j}) \tag{3.2}$$

is a distribution of order 0 and locally of finite mass. Here, $\nu(a_j,r_j)$ is the Haar measure on the circle $S(a_j,r_j)$.

Use the notations

$$B_r^{\hat{\sigma}}:=\{z\in\mathcal{Y}:\hat{\sigma}(z)<r\},\quad S_r^{\hat{\sigma}}:=\{z\in\mathcal{Y}:\hat{\sigma}(z)=r\}$$

for the $\hat{\sigma}$-ball of radius r and its boundary. For $r>0$, the boundary $S_r^{\hat{\sigma}}$ is a piece-wise smooth curve. It has

$$\#\{j:|a_j|+r_j<r\}+1$$

many connected components. Non-smooth points come from the following two cases (see Figure 2):

(1) When $r=|a_j|+r_j$ for some j, there is one non-smooth point which is the tangent point of $S(0,r)$ to $S(a_j,r_j)$.

(2) When $r\in(|a_j|-r_j,|a_j|+r_j)$ for some j, there are two non-smooth points which are the intersection points of $S(0,r)$ and $S(a_j,r_j)$.

Now we describe an exhaustion function σ of $\mathcal{Y}$ (see Figure 3). An explicit construction and the proofs of Lemmas 3.1 and 3.2 will be given in Appendix A.

Lemma 3.1. *There is a smooth exhaustion function $\sigma\geqslant\hat{\sigma}$ such that the difference $\sigma-\hat{\sigma}$ is supported on*

$$\mathrm{Supp}(\sigma-\hat{\sigma})\subset U:=\left(A\left(0,\frac{1}{2},\frac{3}{2}\right)\backslash\mathcal{E}\right)\cup\bigcup_{j=1}^{\infty}A\left(a_j,\frac{1}{2}r_j,\frac{3}{2}r_j\right),$$

where $A(a_j,\frac{1}{2}r_j,\frac{3}{2}r_j):=\{z\in\mathcal{Y}:\frac{1}{2}r_j\leqslant|z-a_j|\leqslant\frac{3}{2}r_j\}$ are pairwise disjoint annuli.

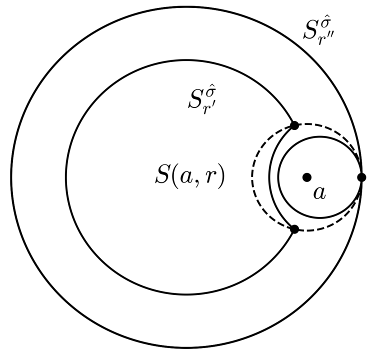

Figure 2 The non-smooth points on $S_{r'}^{\hat{\sigma}}$ of Case (1) and on $S_{r'}^{\hat{\sigma}}$ of Case (2)

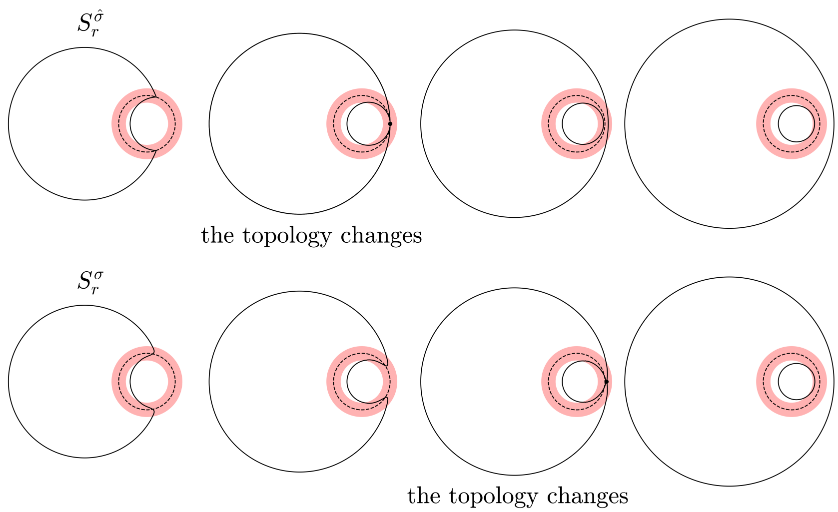

Figure 3 (Color online) The curves $S_r^{\hat{\sigma}}$ and S_r^{σ} as r increases

Let $B_r^\sigma := \{z \in \mathcal{Y} : \sigma(z) < r\}$ be the σ-ball of radius r. Then the lemma above implies $B_t^\sigma \subset \mathbb{D}_t$, $t \geqslant 1$ and

$$\mathbb{D}_t \backslash \left(\bigcup_{j=1}^{\infty} \mathbb{D}\left(a_j, \frac{3}{2} r_j\right)\right) \subset B_t^\sigma, \quad t \geqslant \frac{3}{2}.$$

The following lemma ensures that the parabolic logarithmic derivative lemma holds for σ (see [30, Remark 3.9]).

Lemma 3.2. *The smooth* $(1,1)$*-form* $dd^c \log \sigma$ *is of finite total mass on* $\mathcal{Y}$*.*

The proof of this lemma will be provided in Appendix A.

4 Proof of the main theorem

Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve, and let $D = \sum_{i=1}^{n+1} D_i$ be a simple normal crossing divisor on $\mathbb{P}^n(\mathbb{C})$. Let Q_i be the defining homogeneous polynomial of D_i with degree d_i. Let $F\colon \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C})$ be the endomorphism of degree $d = \operatorname{lcm}(d_1, \dots, d_{n+1})$ defined by

$$F(z) := [Q_1^{m_1}(z) : \cdots : Q_{n+1}^{m_{n+1}}(z)], \tag{4.1}$$

where $m_i = \frac{d}{d_i}$ for $1 \leqslant i \leqslant n+1$. By construction, F maps $\mathbb{P}^n(\mathbb{C}) \setminus D$ to $(\mathbb{C}^*)^n$. The critical points of F consist of the hypersurfaces D_i (if $m_i \geqslant 2$) and a hypersurface $\mathcal{V}$ of degree $\sum_{i=1}^{n+1} d_i - (n+1) > 0$ defined by

$$M(z) := \det \frac{\partial(Q_1, \ldots, Q_{n+1})}{\partial(z_0, \ldots, z_n)} = 0.$$

Lemma 4.1. *The hypersurface $\mathcal{V}$ is in general position with $\{D_i\}_{i=1}^{n+1}$ if and only if the hypersurfaces $\{D_i\}_{i=1}^{n+1}$ intersect transversally.*

Proof. The *if* part was proven in [20, Section 5]. For the *only if* part, without loss of generality, we may assume that there exists some $p \in D_1 \cap \cdots \cap D_n$ such that

- either p is a non-smooth point in some D_k, $1 \leqslant k \leqslant n$, i.e.,

$$\left(\frac{\partial Q_k}{\partial z_0}(p), \ldots, \frac{\partial Q_k}{\partial z_n}(p)\right) = (0, \ldots, 0);$$

- or the normal vectors $\{(\frac{\partial Q_k}{\partial z_0}(p), \ldots, \frac{\partial Q_k}{\partial z_n}(p))\}_{1 \leqslant k \leqslant n}$ of the tangent spaces are linearly dependent.

In both cases,

$$\operatorname{rank}\left(\frac{\partial Q_i}{\partial z_j}(p)\right)_{1 \leqslant i \leqslant n, 0 \leqslant j \leqslant n} < n,$$

i.e., each n-minor of the $n \times (n+1)$ matrix has determinant 0.

The point p has a homogeneous representation $[p_0 : \cdots : p_n]$. There is some $s \in \{0, \ldots, n\}$ such that $p_s \neq 0$. Following Guo-Sun-Wang's argument [20], by using the Euler formula $\sum_{j=0}^{n} \frac{\partial Q_i}{\partial z_j} z_j = d_i \cdot Q_i$, we have

$$p_s\, M(p) = \det \begin{pmatrix} \frac{\partial Q_1}{\partial z_0}(p) & \ldots & d_1 \cdot Q_1(p) & \ldots & \frac{\partial Q_1}{\partial z_n}(p) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial Q_{n+1}}{\partial z_0}(p) & \ldots & d_{n+1} \cdot Q_{n+1}(p) & \ldots & \frac{\partial Q_{n+1}}{\partial z_n}(p) \end{pmatrix}.$$

Noting that $Q_1(p) = \cdots = Q_n(p) = 0$ since $p \in D_1 \cap \cdots \cap D_n$, we obtain

$$p_s\, M(p) = (-1)^{n+s}\, d_{n+1} \cdot Q_{n+1}(p) \det\left(\frac{\partial Q_i}{\partial z_j}(p)\right)_{1 \leqslant i \leqslant n, 0 \leqslant j \leqslant n, j \neq s} = 0.$$

Thus, $M(p) = 0$, i.e., $\mathcal{V}$ and $\{D_i\}_{i=1}^{n}$ intersect at p. We conclude that they are not in general position (see Figure 4) . □

Set $g := F \circ f$. The image $F(\mathcal{V})$ is an algebraic variety, denoted by $\mathcal{W}$. It is clear that $T_g(r) = O(T_f(r))$.

As an illustrative example, consider the case of the projective plane, where D is the union of three curves D_1, D_2 and D_3 in $\mathbb{P}^2(\mathbb{C})$ in general position with total degree greater than or equal to 4. Then for every $z \in f^{-1}(\mathcal{V})$ except a finite set, one has $\operatorname{ord}_z g^*\mathcal{W} \geqslant \operatorname{ord}_z f^*\mathcal{V} + 1$ (see Figure 5).

Indeed, since $g = F \circ f$, one always has $\operatorname{ord}_z g^*\mathcal{W} \geqslant \operatorname{ord}_z f^*\mathcal{V}$. Thus, one only needs to exclude the possibility that $\operatorname{ord}_z g^*\mathcal{W} = \operatorname{ord}_z f^*\mathcal{V}$. In the simple case where $\operatorname{ord}_z g^*\mathcal{W} = \operatorname{ord}_z f^*\mathcal{V} = 1$, this means exactly that F has maximal rank at the point $p = f(z)$, which contradicts the definition of $\mathcal{V}$ as the critical set of F.

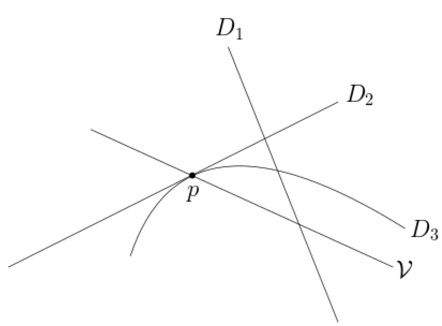

Figure 4 When D_2 and D_3 intersect non-transversally, $\mathcal{V}$, D_2 and D_3 are not in general position

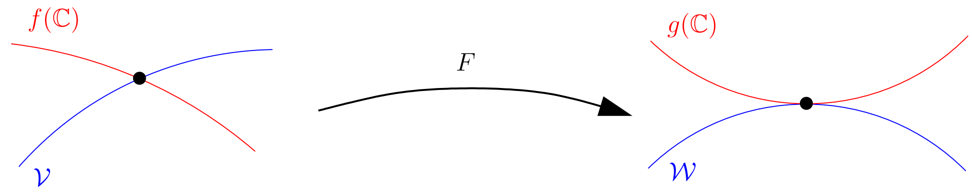

Figure 5 (Color online) If $f(\mathbb{C})$ intersects $\mathcal{V}$ at $f(z)$, then $g(\mathbb{C})$ must be tangent to $\mathcal{W}$ at $g(z)$

Proposition 4.2. *There exists a proper subvariety $\mathcal{Z}$ of $\mathcal{V}$ such that for every $z \in f^{-1}(\mathcal{V} \setminus \mathcal{Z})$, one has*

$$\mathrm{ord}_z f^*\mathcal{V} \leqslant \mathrm{ord}_z g^*\mathcal{W} - 1.$$

Proof. By our construction, the hypersurface $\mathcal{V}$ is contained in the support of the ramification divisor of the endomorphism $F : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C})$. Putting

$$\mathcal{Z}_1 = \mathcal{V} \cap \mathrm{Supp}\, D,$$
$$\mathcal{Z}_2 = \mathrm{Sing}(\mathcal{V}) \cup F^{-1}(\mathrm{Sing}(\mathcal{W})).$$

Let $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2$. Then for any point $p \in \mathcal{V} \setminus \mathcal{Z}$, there exist local coordinate systems $(x_1, \ldots, x_n)$ about p and $(y_1, \ldots, y_n)$ about $q = F(p)$ such that locally one has

$$\mathcal{V} = \{x_1 = 0\}, \quad \mathcal{W} = \{y_1 = 0\},$$
$$F(x_1, \ldots, x_n) =: (y_1, \ldots, y_n) = (x_1^m, x_2, \ldots, x_n).$$

Here, by our construction of $\mathcal{V}$, at the point p, we have the associated $m \geqslant 2$. Thus locally we have $F^*\mathcal{W} = m\,\mathcal{V}$. Consequently,

$$\mathrm{ord}_z g^*\mathcal{W} = \mathrm{ord}_z f^*(F^*\mathcal{W}) = m\,\mathrm{ord}_z f^*\mathcal{V} = \mathrm{ord}_z f^*\mathcal{V} + (m-1)\,\mathrm{ord}_z f^*\mathcal{V} \geqslant \mathrm{ord}_z f^*\mathcal{V} + 1$$

for every $z \in f^{-1}(\mathcal{V} \setminus \mathcal{Z})$. $\square$

Proof of the main theorem. Now put $\mathcal{E} := f^{-1}(D)$, which is a discrete countable set of points in $\mathbb{C}$. After a dilation and translation on $\mathbb{C}$, we can arrange $\mathcal{E} = \{a_j\}_{j=1}^{\infty}$ so that $2 < |a_1| \leqslant |a_2| \leqslant \cdots$. We remark that composing f with dilations or translations on $\mathbb{C}$ will not change the defects. Note that $\#\{\mathbb{D}_t \cap \mathcal{E}\}$ is exactly $n_f^{[1]}(t, D)$, which is finite. Denote by $\widetilde{f}$ and $\widetilde{g}$ the restrictions of f and g on $\mathcal{Y} := \mathbb{C} \setminus \mathcal{E}$, respectively. Consider the exhaustion function $\hat{\sigma}$ defined in the previous section. Take a smoothing σ of $\hat{\sigma}$ as in Appendix A. Denote by B_r^{σ} and S_r^{σ} the σ-ball of radius r and its boundary. The construction in Appendix A ensures $\sigma(z) \geqslant \hat{\sigma}(z) \geqslant |z|$ on $\mathcal{Y}$, and hence, $B_r^{\sigma} \subset \mathbb{D}_r$ and

$$\mathfrak{X}_{\sigma}(r) \leqslant N_f^{[1]}(r, D) + O(\log r), \quad r > 1. \tag{4.2}$$

Suppose on the contrary that (1.2) does not hold, i.e., $\delta_f(D_j) = 1$ for $j = 1, 2, \ldots, n+1$. Then $N_f(r, D) = o(T_f(r))$, and the weighed Euler characteristic $\mathfrak{X}_{\sigma}(r)$ satisfies

$$\limsup_{r \to \infty} \frac{\mathfrak{X}_{\sigma}(r)}{T_f(r)} = 0. \tag{4.3}$$

Since

$$B_t^{\sigma} \subset \mathbb{D}_t, \quad t \geqslant 1$$

and

$$\mathbb{D}_t \setminus \left(\bigcup_{j=1}^{\infty} \mathbb{D}\left(a_j, \frac{3}{2} r_j\right) \right) \subset B_t^{\sigma}, \quad t \geqslant \frac{3}{2},$$

one has

$$
\begin{aligned}
0 \leqslant T_f(r) - T_{\widetilde{f},\sigma}(r) &\leqslant \int_{t=\frac{3}{2}}^{r} \frac{dt}{t} \int_{(\bigcup_{j=1}^{\infty} \mathbb{D}(a_j, \frac{3}{2} r_j)) \cap \mathbb{D}_t} f^*\omega + O(1) \\
&\leqslant \int_{t=1}^{r} \frac{dt}{t} \int_{\bigcup_{j=1}^{\infty} \mathbb{D}(a_j, \frac{3}{2} r_j)} f^*\omega + O(1) \\
&= \int_{t=1}^{r} \frac{dt}{t} \bigg(\sum_{j=1}^{\infty} \int_{\mathbb{D}(a_j, \frac{3}{2} r_j)} f^*\omega \bigg) + O(1), \quad r \geqslant 1.
\end{aligned}
$$

Recall (3.1) that the radius $r_j > 0$ can be chosen arbitrarily small. For our purpose, for each $j \geqslant 1$, we choose $r_j > 0$ sufficiently small so that

$$
\int_{\mathbb{D}(a_j, \frac{3}{2} r_j)} f^*\omega < 2^{-j}.
$$

Hence, the above estimate yields

$$
0 \leqslant T_f(r) - T_{\widetilde{f},\sigma}(r) \leqslant \log r + O(1), \quad r \geqslant 1. \tag{4.4}
$$

This, together with (4.3), implies

$$
\limsup_{r \to \infty} \frac{\mathfrak{X}_\sigma(r)}{T_{\widetilde{f},\sigma}(r)} = 0.
$$

Thus, the technical assumption (2.11) is satisfied, allowing us to utilize all the established results in the parabolic setting. First, by applying Theorem 2.10 to the $q = n+2$ hypersurfaces $D_1, \ldots, D_{n+1}, \mathcal{V}$, we obtain

$$
T_{\widetilde{f},\sigma}(r) \leqslant \frac{N_{\widetilde{f},\sigma}(r, \mathcal{V})}{\deg \mathcal{V}} + o(T_{\widetilde{f},\sigma}(r)) \quad \| . \tag{4.5}
$$

Next, applying Corollary 2.6 to $\widetilde{g}$, we get

$$
N_{\widetilde{g},\sigma}(r, \mathcal{W}) - N^{[1]}_{\widetilde{g},\sigma}(r, \mathcal{W}) = o(T_{\widetilde{g},\sigma}(r)) \quad \| . \tag{4.6}
$$

On the other hand, Theorem 2.5 and Proposition 4.2 imply

$$
N_{\widetilde{f},\sigma}(r, \mathcal{V}) \leqslant N_{\widetilde{g},\sigma}(r, \mathcal{W}) - N^{[1]}_{\widetilde{g},\sigma}(r, \mathcal{W}) + o(T_{\widetilde{f},\sigma}(r)) \quad \| . \tag{4.7}
$$

By combining (4.5)–(4.7), we derive

$$
\begin{aligned}
T_{\widetilde{f},\sigma}(r) &\leqslant \frac{N_{\widetilde{f},\sigma}(r, \mathcal{V})}{\deg \mathcal{V}} + o(T_{\widetilde{f},\sigma}(r)) \quad \| \\
&\leqslant \frac{N_{\widetilde{g},\sigma}(r, \mathcal{W}) - N^{[1]}_{\widetilde{g},\sigma}(r, \mathcal{W})}{\deg \mathcal{V}} + o(T_{\widetilde{f},\sigma}(r)) \quad \| \\
&= o(T_{\widetilde{g},\sigma}(r)) + o(T_{\widetilde{f},\sigma}(r)) \quad \|,
\end{aligned}
$$

which leads to a contradiction. This completes the proof of the main theorem. □

Remark 4.3. In the case where $f\colon \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ is an algebraically nondegenerate holomorphic curve and $\mathcal{C}$ is the collection of two lines and one conic in $\mathbb{P}^2(\mathbb{C})$, in a private note, Noguchi obtained a Second Main Theorem of the form

$$
T_f(r) \leqslant C\, N_f(r, \mathcal{C}) + [N^{[2]}_f(r, \mathcal{V}) - N^{[1]}_f(r, \mathcal{V})] + o(T_f(r)) \quad \|,
$$

where $\mathcal{V}$ is the critical curve of the endomorphism defined as above, and $C > 0$ is some constant. Although the right-hand side of the above inequality involves a quantity depending on $\mathcal{V}$ (which actually counts the number of tangent points of f and $\mathcal{V}$), this term is negligible when f omits $\mathcal{C}$.

Remark 4.4. Our result can be extended to the case of entire holomorphic curves into algebraic varieties of log-general type X with $\overline{q}(X) = \dim X$ by a similar argument.

Acknowledgements Song-Yan Xie was supported by National Key R&D Program of China (Grant Nos. 2023YFA1010500 and 2021YFA1003100) and National Natural Science Foundation of China (Grant Nos. 12288201 and 12471081). Dinh Tuan Huynh was supported by the Vietnam Ministry of Education and Training (Grant No. B2025-DHH-02). A part of this article was written while he was visiting the Vietnam Institute for Advanced Study in Mathematics. Ruiran Sun acknowledges the support of the Centre de Researches Mathématiques (CRM), Montreal, during his CRM-postdoctoral fellowship and thanks McGill University for its hospitality. Zhangchi Chen was supported by the Science and Technology Commission of Shanghai Municipality (Grant No. 22DZ2229014), the Shanghai Sailing Program (Grant No. 24YF2709900), the Shanghai Pujiang Program (Grant No. 24PJA023), the Labex Centre Européen pour les Mathématiques, la Physique et leurs Interactions (Grant No. ANR-11-LABX-0007-01), and the Project Quantization, Singularities, and Holomorphic Dynamics (Grant No. ANR-21-CE40-0016). We are deeply indebted to Julien Duval for introducing us to the problem of seeking a Second Main Theorem for three conics in the projective plane and for sharing fruitful ideas, including the construction of the hypersurface $\mathcal{V}$. We also thank Junjiro Noguchi for his inspiring ideas and private notes. Special thanks go to Min Ru for his thoughtful comments, helpful suggestions, and for bringing references [8,19,20] to our attention. We are grateful to Xiangyu Zhou for his perceptive remarks during a seminar talk and to Qiming Yan for inviting Zhangchi Chen to present a seminar talk at Tongji University, as well as pointing out that the main theorem holds with truncation level-one by applying a parabolic version of [41]. Finally, we thank the referees for their detailed comments and constructive suggestions, which have greatly improved the clarity and quality of our manuscript.

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Appendix A A detailed construction of one smooth exhaustion function

We provide an explicit construction of [30, Example (2), pp. 32 and 33], precisely, a smooth exhaustion function σ on the parabolic surface $\mathcal{Y} := \mathbb{C}\backslash\{a_j\}_{j=1}^{\infty}$ which satisfies Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. Define

$$h(r) := \begin{cases} 0, & r \leqslant \dfrac{3}{4}, \\ \dfrac{1}{4\pi}\dfrac{1}{1+\mathrm{e}^{\frac{r-1}{(r-1)^2-1/16}}}, & \dfrac{3}{4} < r < \dfrac{5}{4}, \\ \dfrac{1}{4\pi}, & r \geqslant \dfrac{5}{4}. \end{cases}$$

The function $h(r)$ (see Figure A.1) is bounded and agrees with $\frac{1}{4\pi}\mathbf{1}_{r>1}$ outside $[\frac{3}{4}, \frac{5}{4}]$.

By symmetry, for $r \geqslant \frac{5}{4}$, we have the integration

$$\int_0^r 4\pi h(s)ds = \int_0^r \mathbf{1}_{s>1}ds = r-1.$$

However, we get the integration

$$\int_0^r 4\pi h(s)\frac{ds}{s} < \int_0^r \mathbf{1}_{s>1}\frac{ds}{s} = \log r$$

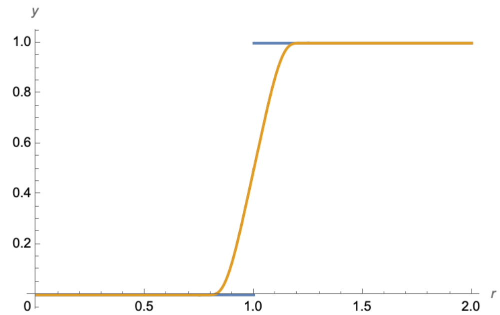

Figure A.1 (Color online) Graphs of $y = h(r)$ and $y - \frac{1}{4\pi}\mathbf{1}_{r>1}$

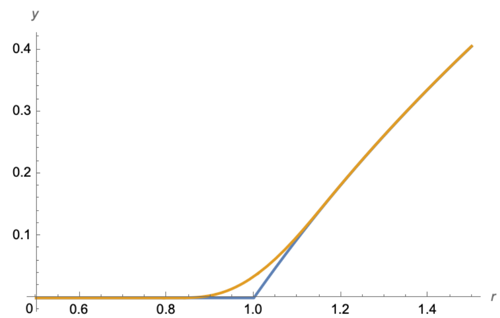

Figure A.2 (Color online) Graphs of $y = H(r)$ and $y = \log^+ r$

since $1/s$ is strictly decreasing on $(0, +\infty)$. We need a small translation $c \in (0, \frac{1}{2})$ to ensure that the primitive

$$H(r) := 4\pi \int_0^r (h(s-c))\frac{ds}{s}$$

agrees with $\log^+ r = \int_0^r \mathbf{1}_{r>1} \frac{ds}{s}$ outside $[\frac{1}{2}, \frac{3}{2}]$.

Thus we get a smoothing $H(r)$ (see Figure A.2) of $\log^+ r$ with

$$0 \leqslant H(r) - \log^+ r \leqslant \log^+ \frac{3}{2} < \frac{1}{2}.$$

Together with the monotonicity of H, one has $H(r) = \log^+ r$ when $H(r) \geqslant \log \frac{3}{2}$.

Define

$$\tau := H(|z|) + \sum_{j=1}^{\infty} r_j H\left(\left|\frac{r_j}{z - a_j}\right|\right), \quad \sigma := \exp(\tau).$$

Then $\sigma \geqslant \hat{\sigma}$ and the difference $\sigma - \hat{\sigma}$ is supported on

$$\mathrm{Supp}(\sigma - \hat{\sigma}) \subset U := \left(A\left(0, \frac{1}{2}, \frac{3}{2}\right) \backslash \mathcal{E}\right) \cup \bigcup_{j=1}^{\infty} A\left(a_j, \frac{1}{2}r_j, \frac{3}{2}r_j\right),$$

where $A(a_j, \frac{1}{2}r_j, \frac{3}{2}r_j)$ are pairwise disjoint. □

Proof of Lemma 3.2. In polar coordinates $z = a + re^{\mathrm{i}\theta}$ for some $a \in \mathbb{C}$, for a smooth function ϕ, one has (see [27, p. 2])

$$d^c\phi = \frac{1}{4\pi}\left(r\frac{\partial\phi}{\partial r}d\theta - \frac{1}{r}\frac{\partial\phi}{\partial\theta}dr\right).$$

The smooth $(1,1)$-form $dd^c \log\sigma$ is supported on U since $\log\sigma$ is harmonic elsewhere. On the annulus $A(0, \frac{1}{2}, \frac{3}{2})$, in polar coordinates $z = re^{\mathrm{i}\theta}$, one has

$$dd^c H(|z|) = d\left(\frac{r}{4\pi}\frac{\partial H(r)}{\partial r}d\theta\right) = d(h(r-c)d\theta) = O(1)dr \wedge \theta = O(1)rdr \wedge \theta$$

since $r \in [\frac{1}{2}, \frac{3}{2}]$ is bounded. Thus $dd^c H(|z|)$ is of finite mass on $A(0, \frac{1}{2}, \frac{3}{2}) \backslash \mathcal{E}$.

On the annulus $A(a_j, \frac{1}{2}r_j, \frac{3}{2}r_j)$, in polar coordinates $z = a_j + re^{\mathrm{i}\theta}$, one has

$$\begin{aligned} dd^c \sum_{j=1}^{\infty} r_j H\left(\left|\frac{r_j}{z-a_j}\right|\right) &= d\left(\frac{r}{4\pi}\frac{\partial r_j H(\frac{r_j}{r})}{\partial r} d\theta\right) \\ &= d\left(-r_j h\left(\frac{r_j}{r} - c\right) d\theta\right) = O(r_j) dr \wedge \theta = O(1) r dr \wedge \theta. \end{aligned}$$

Since

$$\sum_{j=1}^{\infty} r_j^2 \leqslant \sum_{j=1}^{\infty} r_j < +\infty,$$

the support U is of finite Lebesgue measure. We conclude that $dd^c \log \sigma$ is of finite mass. $\square$