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Universal Entire Curves in Projective Spaces with Slow Growth

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Abstract
We construct explicit universal entire curves $h : \mathbb{C} \to \mathbb{CP}^n$ whose Nevanlinna characteristic functions grow slower than any preassigned transcendental growth rate. Moreover, we can make h to be hypercyclic for translation operations along any given countable directions.

Keywords Universality · Entire curves · Hypercyclicity · Nevanlinna characteristic functions · Nevanlinna theory · Rational approximations

Mathematics Subject Classification 32A22 · 30D35 · 47A16 · 41A20 · 32H30

1 Introduction

About one century ago, in the space of entire functions, Birkhoff [2] constructed some *universal* ones $h : \mathbb{C} \to \mathbb{C}$, which can approximate *any* entire function g by

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translations of h. Precisely, given any compact set K in $\mathbb{C}$, there is a sequence $(a_i)_{i\geqslant 1}$ of complex numbers such that $h(\bullet + a_i)$ converges to $g(\bullet)$ uniformly on K.

Such universality also appears in the space $\mathscr{M}_n$ of entire curves $f : \mathbb{C} \to \mathbb{CP}^n$ in projective n-spaces, endowed with the topology of uniform convergence on compact sets. Let $\mathscr{U}_n$ denote the subset of $\mathscr{M}_n$ consisting of *universal entire curves* defined likewise.

In this paper, we answer a question of Dinh–Sibony [3, Problem 9.1] in an extended version.

Question 1.1 *Find minimal growth rate of the Nevanlinna characteristic function $T_h(r)$ for $h \in \mathscr{U}_n$, where*

$$T_h(r) := \int_1^r \frac{dt}{t} \int_{|z|<t} h^* \omega_{\mathsf{FS}}. \tag{1}$$

In the formula, ω_{FS} is the Fubini–Study form on $\mathbb{CP}^n$. The Nevanlinna characteristic function $T_f(r)$ measures the complexity of f. For instance, f is a rational function if and only if $T_f(r) \approx O(1) \cdot \log\ r$ as $r \to +\infty$ (cf. e.g. [8, Theorem 2.5.28]).

One partial result in this direction was obtained in [7], in which some meromorphic function f was constructed for approximating any entire functions by its translations, having Nevanlinna characteristic function $T_f(r) \approx q(r) \cdot (\log\ r)^2$ for some preassigned positive continuous function $q(r)$ tending to infinity as $r \to +\infty$.

Any universal entire curve f must be transcendental, which guarantees that [8, 9]

$$\lim_{r\to+\infty} T_f(r)/\log r = +\infty.$$

In this paper we provide the following optimal answer to Dinh-Sibony's Question 1.1.

Theorem 1.2 *For any $n \geqslant 1$, given any positive continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ tending to infinity $\lim_{r\to+\infty} \phi(r) = +\infty$, there exists some universal entire curve h with slow growth*

$$T_h(r) \leqslant \phi(r) \cdot \log\ r, \quad \forall\, r \geqslant 1. \tag{2}$$

In history, Birkhoff's discovery of universal entire functions foreshadowed a new branch of functional analysis called hypercyclic operators theory [1, 6]. Now, we briefly introduce the key notions.

A continuous linear operator $\mathsf{T} : X \to X$ acting on a topological vector space X (over $\mathbb{C}$ or $\mathbb{R}$) is called *hypercyclic* if there is some element $x \in X$ whose orbit $\{\mathsf{T}^n(x)\}_{n\geqslant 1}$ is dense in X. Such a vector x is called *hypercyclic* for T.

From today's hindsight, Birkhoff's result can be rephrased as follows: for any nonzero complex number a, the translation operator $\mathsf{T}_a(f)(z) := f(z+a)$ is hypercyclic on the space $\mathscr{H}(\mathbb{C})$ of entire functions, endowed with the topology of uniform convergence on compact sets.

In this paper, we show the existence of universal entire curves f with slow growth, which are moreover hypercyclic simultaneously for all nontrivial translations along given countable directions. One key ingredient is a reminiscent of [4, Theorem 8].

Theorem 1.3 *For any $n \geqslant 1$, given any positive continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ tending to infinity, and given any countable set $E \subset [0, 2\pi)$, there exists some universal entire curve h satisfying*

- *small growth rate $T_h(r) \leqslant \phi(r) \cdot \log\, r$, for all $r \geqslant 1$;*
- *h is hypercyclic for T_a for any nonzero complex number a with argument in E.*

Our proof is based on insight of Oka manifolds theory (cf. [5]) and Nevanlinna theory. Precisely, in our setting $\mathbb{CP}^n$ is a special Oka manifold with a "large" open subset $\mathbb{C}^n$ having complex vector space structure. The idea is, first, to construct some countable meromorphic discs $\{g^{[k]} : \mathbb{D}_{R_k} \to \mathbb{C}^n\}_{k \geqslant 1}$, each being a rational map having designed approximation property and decaying to zero near infinity $\lim_{z\to\infty} g^{[k]}(z) = \vec{0} \in \mathbb{C}^n$. This step can be accomplished by using Runge's approximation theorem, or by Lemma 2.1. Secondly, we "patch" the discs $\{g^{[k]}\}_{k\geqslant 1}$ together by sophisticated translations, repetitions, and infinite summation.
Convention We denote by $\mathbb{D}_r$ the disc centered at the origin with radius $r > 0$, and by $\mathbb{D}(a, r)$ the disc centered at $a \in \mathbb{C}$ with radius r.

2 Preparations

2.1 Nevanlinna Theory

Let $[Z_0 : \cdots : Z_n]$ be homogeneous coordinates of $\mathbb{CP}^n$. Denote by $H_0 := \{Z_0 = 0\}$ the first coordinate hyperplane. Let $h : \mathbb{C} \to \mathbb{CP}^n$ be an entire curve not contained in H_0, with reduced representation $h = [h_0 : h_1 : \cdots : h_n]$. Denote by $n_h(r, H_0)$ the number of zeros of h_0 in $\{|z| \leqslant r\}$, counting multiplicities. The *counting function* $N_h(r, H_0)$ of h with respect to H_0 is

$$N_h(r, H_0) := \int_{t=1}^{r} \frac{n_h(t, H_0)}{t}\, \mathrm{d}t,$$

and the *proximity function* $m_h(r, H_0)$ is

$$m_h(r, H_0) := \int_{\theta=0}^{2\pi} \log \sqrt{1 + |h_1/h_0|^2(re^{i\theta}) + \cdots + |h_n/h_0|^2(re^{i\theta})}\, \frac{\mathrm{d}\theta}{2\pi}.$$

By Nevanlinna theory (cf. [9]), Shimizu–Ahlfors' version (1) of Nevanlinna's characteristic function can be rewritten as [8, Theorem 2.3.31]

$$T_h(r) = m_h(r, H_0) + N_h(r, H_0) - m_h(1, H_0). \tag{3}$$

2.2 Runge-Type Approximations

Runge's approximation theorem (1885) states that any holomorphic function defined on a neighborhood of a compact set $K \subset \mathbb{C}$ can be uniformly approximated by rational functions with poles outside K. For our purpose, we need the following analogue.

Lemma 2.1 *For any $f \in \mathscr{M}_n$, for any error bound $\epsilon > 0$, for any $N > 0$, there exists some rational map*

$$\gamma = [p_0 : p_1 : \cdots : p_n] \in \mathscr{M}_n,$$

where all $p_i \in \mathbb{C}[z]$ for $0 \leqslant i \leqslant n$, such that

(i) *the Fubini-Study distance $d_{\mathsf{FS}}\big(\gamma(z), f(z)\big) < \epsilon$ on $\overline{\mathbb{D}_N}$;*
(ii) *for each $j \neq 0$, $\deg(p_0) > \deg(p_j)$, whence*

$$\left|p_j(z)/p_0(z)\right| = O\left(1/|z|\right), \quad z \to \infty.$$

Proof Take a reduced representation $f = [f_0 : \cdots : f_n]$, where each f_j is holomorphic on $\mathbb{C}$. By rescaling, we can assume that $|f_0|^2 + \cdots + |f_n|^2 \geqslant 1$ on $\overline{\mathbb{D}_N}$.

Geometrically, the Fubini-Study distance between two points $[a_0 : \cdots : a_n]$ and $[b_0 : \cdots : b_n]$ is, up to multiplying by some positive constant, the angle between two complex vectors $\vec{a} = (a_0, \ldots, a_n)$ and $\vec{b} = (b_0, \ldots, b_n) \in \mathbb{C}^{n+1}$. For $\vec{a}$ away from the origin, say $||\vec{a}|| \geqslant 1$, and for $\vec{b}$ sufficiently near to $\vec{a}$, say $||\vec{b} - \vec{a}|| < \mu \ll 1$, we have

$$d_{\mathsf{FS}}([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) < \epsilon.$$

Therefore, we approximate each f_j by some polynomial p_j such that $|p_j(z) - f_j(z)| < \frac{\mu}{2\sqrt{n+1}}$ uniformly on $\overline{\mathbb{D}_N}$. Whence the requirement (i) is satisfied.

For the requirement (ii), we can start with a nonzero polynomial $p_0(z)$, and then replace it by $p_0(z)\frac{(z+M)^Q}{M^Q}$ for some large $M, Q \gg 1$, so that $\left|p_0(z)\frac{(z+M)^Q}{M^Q} - p_0(z)\right| < \frac{\mu}{2\sqrt{n+1}}$ on $\overline{\mathbb{D}_N}$. □

3 Construction

We now prove Theorem 1.3 by an explicit construction.

Let $E = \{\theta_v\}_{v \geqslant 1} \subset [0, 2\pi)$ be a countable set of angles. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive continuous function tending to infinity $\lim\limits_{r \to +\infty} \phi(r) = +\infty$. After replacing $\phi(r)$ by $\hat{\phi}(r) := \min_{t \geqslant r} \phi(t)$, we can assume that ϕ is nondecreasing.

3.1 Preparing Model Curves Over Discs

We select a countable and dense subfield of $\mathbb{C}$, say $\mathbb{Q}_c := \{z = x + \sqrt{-1}y \, : \, x, y \in \mathbb{Q}\}$. We will make use of the following countable subset of rational curves

$$\begin{aligned}\mathscr{R} := \{[p_0(z) : p_1(z) : \cdots : p_n(z)] \, : \, p_0, p_1, \ldots, p_n \in \mathbb{Q}_c[z];\\ \deg(p_0) > \deg(p_j), \forall\, j \neq 0\},\end{aligned} \tag{4}$$

which can be enumerated as

$$\left(\gamma^{[k]} = [p_0^{[k]}(z) : p_1^{[k]}(z) : \cdots : p_n^{[k]}(z)]\right)_{k\geqslant 1}.$$

By Lemma 2.1, noting that $\mathbb{Q}_c \subset \mathbb{C}$ is dense, we can find a sequence of elements in $\mathscr{R}$ to approximate any given entire curve uniformly on compact sets $\overline{\mathbb{D}_N}$.

Since $p_0^{[k]} \not\equiv 0$, we can rewrite

$$\gamma^{[k]}(z) = [1 : \gamma_1^{[k]}(z) : \cdots : \gamma_n^{[k]}(z)], \quad \gamma_j^{[k]}(z) := p_j^{[k]}(z)/p_0^{[k]}(z).$$

Note that $\left|\gamma_j^{[k]}(z)\right| = O(1/|z|)$ as $z \to \infty$, since $\deg(p_0^{[k]}) > \deg(p_j^{[k]})$. Choose a large radius $\eta_k > 0$ such that $\left|\gamma_j^{[k]}(z)\right| \leqslant 2^{-k}$ for all $|z| > \eta_k$ and for all $j = 1, \ldots, n$.

A natural idea is constructing $h = [1 : h_1 : \cdots : h_n]$ where $h_j = \sum_{k\geqslant 1} \gamma_j^{[k]}(z-a_k)$ for some sequence $(a_k)_{k\geqslant 1}$ of complex numbers growing sufficiently fast

$$1 \ll |a_1| \ll |a_2| \ll |a_3| \ll \cdots,$$

so that

- on each disc $\overline{\mathbb{D}(a_k, \eta_k)}$, the error term $\sum_{\ell\neq k} \gamma_j^{[\ell]}(z - a_\ell)$ is negligible

$$\Big|\sum_{\ell\neq k} \gamma_j^{[\ell]}(z - a_\ell)\Big| \leqslant 2^{-k+1}, \quad \forall\, j = 1, \ldots, n, \quad \forall\, k \geqslant 1;$$

- outside all these discs, the sum h_j has norm bounded by 1.

However, this h is not obviously universal due to the presence of error terms. To prove universality, we make the following manipulations.

The Cartesian product

$$\mathscr{A} := \mathscr{R} \times \mathbb{Z}_+ \times E = \Big\{\big((\gamma^{[k]}, u, \theta_v\big);\ k, u, v \in \mathbb{Z}_+\Big\}$$

is countable. We enumerate $\mathscr{A}$, for instance by lexicographic order of the index $(k, u, v) \in \mathbb{Z}_+^3$, as

$$\mathscr{A} = (A_k)_{k\geqslant 1}, \quad A_k := (g^{[k]}, w_k, \rho_k).$$

For any $\gamma \in \mathscr{R}$, for any $\theta \in E$, there is an infinite subsequence $A_{k_m} = (g^{[k_m]}, w_{k_m}, \rho_{k_m})$ for $m \geqslant 1$ such that $g^{[k_m]} = \gamma$, $\rho_{k_m} = \theta$, and w_{k_m} tends to infinity as $m \to +\infty$.

Let

$$h(z) := [1 : h_1(z) : \cdots : h_n(z)], \quad h_j(z) := \sum_{k=1}^{+\infty} g_j^{[k]}(z - a_k).$$

for some $a_k \in \mathbb{C}$ to be chosen subsequently by a sophisticated algorithm (13). The following lemmas show that if the sequence $(|a_k|)_{k \geqslant 1}$ grows fast enough, we have the following properties:

(1) (convergence) h_j is well-defined meromorphic function for each $j = 1, \ldots, n$;
(2) (universality) h is a universal;
(3) (growth rate) $T_h(r) \leqslant \phi(r) \cdot \log\ r$, for all $r \geqslant 1$.

3.2 Proof of Convergence and Universality

Since $\left|g_j^{[k]}(z)\right| = O(1/|z|)$ for all $1 \leqslant j \leqslant n$, there are some $\delta_k > 0$ and $C_k > 0$ such that $\left|g_j^{[k]}(z)\right| < \frac{C_k}{|z|}$ for any $|z| > \delta_k$. Set $R_k := \delta_k + 2^k\, C_k$. Hence

$$\left|g_j^{[k]}(z)\right| < \frac{C_k}{|z|} < 2^{-k} \cdot \frac{R_k}{|z|} < 2^{-k}, \quad \forall\, |z| > R_k, \quad \forall\, 1 \leqslant j \leqslant n. \tag{5}$$

We demand that all the discs $\mathbb{D}(a_k, R_k)$ for $k \geqslant 1$ are sufficiently far way from each other, in order to have the following good estimates.

Lemma 3.1 *If*

$$|a_k| - |a_{k-1}| - R_k - R_{k-1} > (R_1 + \cdots + R_{k-1})(k-1)2^k, \quad \forall\, k \geqslant 2, \tag{6}$$

then for any $j \in \{1, \ldots, n\}$*:*

1. *outside* $\bigcup_{\ell} \mathbb{D}(a_\ell, R_\ell)$*, the function* $h_j(z) = \sum_{k=1}^{+\infty} g_j^{[k]}(z - a_k)$ *has bound* $|h_j(z)| < 1$*;*
2. *in each disc* $\overline{\mathbb{D}(a_k, R_k)}$*, the error term* $\epsilon_j^{[k]}(z) := \sum_{\ell \neq k} g_j^{[\ell]}(z - a_l)$ *has bound* $\left|\epsilon_j^{[k]}\right| < 2^{-k+1}$.

Proof The first estimate follows by summing up (5) for $k \geqslant 1$. Now, we prove the second estimate.

For each $\ell < k$, the distance between $\mathbb{D}(a_\ell, R_\ell)$ and $\overline{\mathbb{D}(a_k, R_k)}$ is at least $R_\ell\,(k-1)\,2^k$ by (6). Hence by (5) we have

$$\left|g_j^{[\ell]}(z - a_\ell)\right| < 2^{-\ell} \frac{R_\ell}{R_\ell\,(k-1)\,2^k} < \frac{2^{-k}}{k-1}, \quad \forall\, z \in \mathbb{D}(a_k, R_k), \quad \forall\, \ell < k.$$

Lastly, using the above estimate for $\ell < k$ and using (5) for $\ell > k$, we receive

$$\left|\epsilon_j^{[k]}(z)\right| \leqslant \sum_{\ell<k} \left|g_j^{[\ell]}(z - a_\ell)\right| + \sum_{\ell>k} \left|g_j^{[\ell]}(z - a_\ell)\right| < \sum_{\ell<k} \frac{2^{-k}}{k-1} + \sum_{\ell>k} 2^{-\ell} = 2^{-k} + 2^{-k}.$$

□

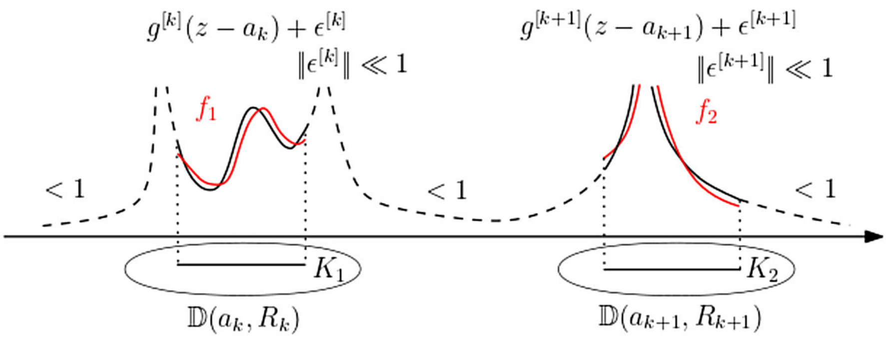

Fig. 1 The dotted universal curve h, locally expressed by $g+\epsilon$, approximates a curve f_1 on K_1 and a curve f_2 (intersecting H_0) on K_2

Consequently, for each $j \in \{1, \ldots, n\}$, the infinite sum $h_j(z)$ is a well-defined meromorphic function, and all its poles lie in $\bigcup_k \mathbb{D}(a_k, R_k)$.

Lemma 3.2 *If the condition* (6) *holds, then h is universal.*

Proof It suffices to prove that for any entire curve $f \in \mathscr{M}_n$, for any error bound $\epsilon > 0$, and for any compact set K in $\mathbb{C}$, there is some translation T_c given by large $c \in \mathbb{C}$, such that

$$d_{\mathsf{FS}}\big(f(z), \mathsf{T}_c(h)(z)\big) = d_{\mathsf{FS}}\big(f(z), h(z+c)\big) < \epsilon, \quad \forall\, z \in K.$$

Take a large disc $\overline{\mathbb{D}_N}$ containing K. By Lemma 2.1, recalling (4), we can find some rational curve $\gamma = [p_0 : p_1 : \cdots : p_n] \in \mathscr{R}$ such that

$$d_{\mathsf{FS}}\,(f(z), \gamma(z)) < \frac{\epsilon}{2}, \quad \forall\, z \in \overline{\mathbb{D}_N}.$$

By rescaling, we can moreover assume that $\sum_{j=0}^{n} |p_j(z)|^2 \geqslant 1$ on $\overline{\mathbb{D}_N}$.

By our construction, γ repeats infinitely many times in $\big(g^{[k]}\big)_{k\geqslant 1}$. Take $k \gg 1$ such that

- the model curve $g^{[k]} = \gamma = [1 : p_1/p_0 : \cdots : p_n/p_0] =: [1 : g_1^{[k]} : \cdots : g_n^{[k]}]$;
- the error term $\Big|\epsilon_j^{[k]}(z)\Big| \leqslant \frac{\delta}{M}$ on $\overline{\mathbb{D}(a_k, R_k)}$ for all $j \in \{1, \ldots, n\}$, where $M := \max_{z\in\overline{\mathbb{D}_N}} \{|p_0(z)|\}$, and where $\delta > 0$ is a sufficiently mall number such that

$$d_{\mathsf{FS}}([\vec{a}], [\vec{b}]) < \epsilon/2, \quad \forall\, \vec{a}, \vec{b} \in \mathbb{C}^{n+1}, \|\vec{a}\| \geqslant 1, \|\vec{b}-\vec{a}\| < \delta;$$

- the radius $R_k > N$.

Hence $\left|\epsilon_j^{[k]}(z+a_k)\right| \leqslant \frac{\delta}{M}$ on $\overline{\mathbb{D}_{R_k}}$. Rewrite

$$\begin{aligned} h(z+a_k) &= [1 : h_1(z+a_k) : \cdots : h_n(z+a_k)] \\ &= [1 : g_1^{[k]}(z) + \epsilon_1^{[k]}(z+a_k) : \cdots : g_n^{[k]}(z) + \epsilon_n^{[k]}(z+a_k)] \\ &= [p_0(z) : p_1(z) + \epsilon_1^{[k]}(z+a_k)\, p_0(z) : \cdots : p_n(z) + \epsilon_n^{[k]}(z+a_k)\, p_0(z)], \end{aligned}$$

where $\left|\epsilon_j^{[k]}(z+a_k) \cdot p_0(z)\right| \leqslant \frac{\delta}{M} \cdot M = \delta$ on $\overline{\mathbb{D}_N}$. Since $\sum_{j=0}^{n} |p_j(z)|^2 \geqslant 1$, using the argument in the proof of Lemma 2.1, this sufficiently small δ ensures that

$$d_{\mathsf{FS}}\,(h(z+a_k), \gamma(z)) < \frac{\epsilon}{2}, \quad \forall\, z \in \overline{\mathbb{D}_N}.$$

Hence $d_{\mathsf{FS}}\,(h(z+a_k), f(z)) < \epsilon$ on $\overline{\mathbb{D}_N}$. □

3.3 Estimate of the Growth Rate

Now, we prove that if the sequence $(|a_k|)_{k \geqslant 1}$ grows fast enough, (2) holds.

Denote by $n_{k,j}$ the number of poles of the rational function $g_j^{[k]}$, counting with multiplicities. Let $n_k := \sum_{j=1}^{n} n_{k,j}$.

We demand that $|a_k| > R_k + 1$ for all $k \geqslant 1$.

Lemma 3.3 *If*

$$\begin{cases} |a_k| - |a_{k-1}| - R_k - R_{k-1} > (R_1 + \cdots + R_{k-1})(k-1)2^k, & \forall\, k \geqslant 2, \\ \phi(|a_k| - R_k) > (n_1 + \cdots + n_k + \sqrt{n+1})\, \frac{\log(|a_k|+R_k)}{\log(|a_k|-R_k)}, & \forall\, k \geqslant 1, \end{cases} \tag{7}$$

then $T_h(r) \leqslant \phi(r) \cdot \log\, r$, *for all* $r \geqslant \max\{|a_1| - R_1, e\}$.

Proof By (3), for $r \geqslant 1$, we have

$$T_h(r) = N_h(r, H_0) + m_h(r, H_0) - m_h(1, H_0) \leqslant N_h(r, H_0) + m_h(r, H_0). \tag{8}$$

Case 1: There is some $k \in \mathbb{Z}_+$ such that $|a_k| + R_k < r < |a_{k+1}| - R_{k+1}$. In this case the circle $\{|z| = r\}$ does not intersect with any disc $\overline{\mathbb{D}(a_\ell, R_\ell)}$, $\ell \geqslant 1$.

By Lemma 3.1, on this circle $|h_j(z)| \leqslant 1$ for all $j \in \{1, \ldots, n\}$. Hence the proximity function

$$m_h(r, H_0) \leqslant \sqrt{n+1}. \tag{9}$$

Although the expression $h(z) = [1 : h_1(z) : \cdots : h_n(z)]$ is not a reduced representation, nevertheless, we have the clear estimate $n_h(r, H_0) \leqslant n_1 + \cdots + n_k$. Therefore

$$N_h(r, H_0) \leqslant (n_1 + \cdots + n_k) \cdot \log\, r. \tag{10}$$

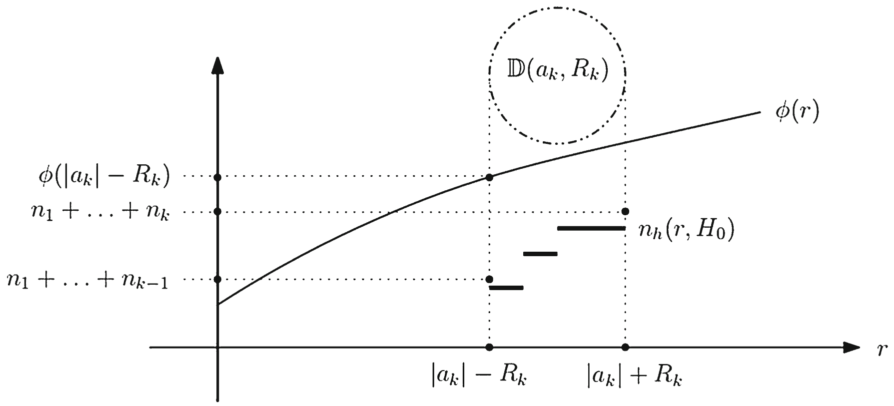

Fig. 2 When $(|a_k|)_{k\geqslant 1}$ grows sufficiently fast, $n_h(r, H_0)$ is smaller than $\phi(r)$

Summing up (9) and (10), and using (8), we get

$$\begin{aligned} T_h(r) &\leqslant (n_1 + \cdots + n_k) \cdot \log\, r + \sqrt{n+1} \\ \text{[use (7) and } r \geqslant e] \quad &< \phi(r) \cdot \log\, r. \end{aligned} \tag{11}$$

Case 2: There is some $k \in \mathbb{Z}_+$ such that $\big|r - |a_k|\big| \leqslant R_k$. In this case, the circle $\{|z| = r\}$ intersects only one disc $\overline{\mathbb{D}(a_k, R_k)}$.

By definition, $T_h(r)$ is nondecreasing. So

$$\begin{aligned} T_h(r) &\leqslant T_h(|a_k| + R_k) \\ \text{[use (11)]} \quad &\leqslant (n_1 + \cdots + n_k + \sqrt{n+1}) \cdot \log(|a_k| + R_k) \\ \text{[use (7)]} \quad &< \phi(|a_k| - R_k) \cdot \log(|a_k| - R_k) \\ \text{[use } |a_k| - R_k \leqslant r] \quad &\leqslant \phi(r) \cdot \log\, r. \end{aligned}$$

□

Now, we estimate the growth rate for $r \in [1, |a_1| - R_1]$. The following Lemma tells that the Nevanlinna characteristic function $T_h(r)$ of an "almost constant" curve h has very slow growth for small r.

Lemma 3.4 *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive continuous nondecreasing function. For any $r_0 > 1$, there is $\epsilon_0 > 0$ such that if $h = [1 : h_1(z) : \cdots : h_n(z)]$ for some meromorphic functions h_j with $|h_j(z)| \leqslant \epsilon_0$ on $\overline{\mathbb{D}_{r_0+1}}$, then $T_h(r) \leqslant \phi(r) \cdot \log\, r$, for all $r \in [1, r_0]$.*

Proof By Cauchy's integral formula

$$h_j'(z) = \frac{1}{2\pi\, i} \int_{|\zeta|=r_0+1} \frac{h_j(\zeta)}{(\zeta - z)^2}\, \mathrm{d}\zeta, \quad \forall\, z \in \overline{\mathbb{D}_{r_0}},$$

we receive the derivative estimate $|h'_j(z)| \leqslant (r_0+1)\,\epsilon_0$ on $\overline{\mathbb{D}_{r_0}}$. The Fubini-Study from on the affine chart $\mathbb{C}^n \subset \mathbb{CP}^n$ with coordinates $(z_1, \ldots, z_n) \mapsto [1 : z_1 : \cdots : z_n]$, is

$$\begin{aligned}\omega_{\mathsf{FS}} &= \frac{i}{2\pi}\partial\bar{\partial}\log\Big(1+\sum\nolimits_{j=1}^{n}|z_j|^2\Big)\\ &= \frac{i}{2\pi}\left(\frac{\sum_{j=1}^{n}\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j}{1+\sum_{j=1}^{n}|z_j|^2}-\frac{\sum_{j=1}^{n}\bar{z}_j\,\mathrm{d}z_j}{1+\sum_{j=1}^{n}|z_j|^2}\wedge\frac{\sum_{j=1}^{n}z_j\,\mathrm{d}\bar{z}_j}{1+\sum_{j=1}^{n}|z_j|^2}\right).\end{aligned}$$

The pull-back of the Fubini-Study form is

$$h^*\omega_{\mathsf{FS}} = \left(\frac{\sum_{j=1}^{n}|h'_j|^2}{1+\sum_{j=1}^{n}|h_j|^2}-\frac{\sum_{j=1}^{n}\overline{h_j}h'_j}{1+\sum_{j=1}^{n}|h_j|^2}\,\frac{\sum_{j=1}^{n}h_j\overline{h_j}'}{1+\sum_{j=1}^{n}|h_j|^2}\right)\frac{i}{2\pi}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}.$$

Here, the coefficient

$$\begin{aligned}&\left(\frac{\sum_{j=1}^{n}|h'_j|^2}{1+\sum_{j=1}^{n}|h_j|^2}-\frac{\sum_{j=1}^{n}\overline{h_j}h'_j}{1+\sum_{j=1}^{n}|h_j|^2}\,\frac{\sum_{j=1}^{n}h_j\overline{h_j}'}{1+\sum_{j=1}^{n}|h_j|^2}\right)\\ &\quad\leqslant \frac{\sum_{j=1}^{n}|h'_j|^2}{1+\sum_{j=1}^{n}|h_j|^2}\leqslant n(r_0+1)^2\,\epsilon_0^2\end{aligned}$$

on $\overline{\mathbb{D}_{r_0}}$. Hence for any $r \in [1, r_0]$, we have

$$\begin{aligned}T_h(r) &\leqslant \int_{t=1}^{r}\frac{\mathrm{d}t}{t}\int_{|z|\leqslant r_0}h^*\omega_{\mathsf{FS}}\\ &\leqslant \int_{t=1}^{r}\frac{\mathrm{d}t}{t}\int_{|z|\leqslant r_0}n\,(r_0+1)^2\,\epsilon_0^2\,\frac{i}{2\pi}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} = n\,r_0^2(r_0+1)^2\epsilon_0^2\,\log\,r.\end{aligned}$$

We complete the proof by taking $\epsilon_0 > 0$ sufficiently small such that $n\,r_0^2(r_0+1)^2\epsilon_0^2 \leqslant \phi(1)$. □

Lemma 3.5 *Take large $r_0 > e$ such that $\phi(r)\cdot\log\,r > \sqrt{n+1}$, for all $r > r_0$. Set*

$$\epsilon_0 = \min\left\{1, \sqrt{\tfrac{\phi(1)}{n\,r_0^2\,(r_0+1)^2}}\right\}.$$

Under the condition (7) and, if moreover

$$|a_k| > R_k/\epsilon_0 + r_0 + 1, \quad \forall\, k \geqslant 1, \tag{12}$$

then $T_h(r) \leqslant \phi(r)\cdot\log\,r$, for all $r \in [1, R_1 - |a_1|)$.

Proof. Recalling (5)

$$\left|g_j^{[k]}(z)\right| < 2^{-k}\cdot\frac{R_k}{|z|}, \quad \forall\,|z| > R_k,$$

noting that

$$|z - a_k| \geqslant R_k/\epsilon_0 \geqslant R_k, \quad \forall\, |z| \leqslant r_0 + 1,$$

we receive

$$\left| g_j^{[k]}(z - a_k) \right| \leqslant 2^{-k} \, \frac{R_k}{|z - a_k|} \leqslant 2^{-k} \, \frac{R_k}{R_k/\epsilon_0} = 2^{-k} \, \epsilon_0, \quad \forall\, |z| \leqslant r_0 + 1.$$

Summing up the above inequality for $k \geqslant 1$, we get $|h_j(z)| \leqslant \epsilon_0$ on $\overline{\mathbb{D}_{r_0+1}}$. By Lemma 3.4, $T_h(r) \leqslant \phi(r) \cdot \log\, r$, for $r \in [1, r_0]$.

For the remaining $r \in (r_0, \max\{|a_1| - R_1, e\})$, the proximity function $m_h(r, H_0) \leqslant \sqrt{n+1}$ since $|h_j(z)| \leqslant 1$ by Lemma 3.1, and the counting function $N_h(r, H_0) = 0$ by our construction. Thus we conclude the proof by (3)

$$T_h(r) \leqslant N_h(r, H_0) + m_h(r, H_0) \leqslant \sqrt{n+1} < \phi(r) \cdot \log\, r. \qquad \square$$

3.4 End of the Proof

Now we summarize the first restrictions on the sequence $(a_k)_{k \geqslant 1}$:

$$\begin{cases} |a_k| > R_k/\epsilon_0 + r_0 + 1, & \forall\, k \geqslant 1, \\ |a_k| - |a_{k-1}| - R_k - R_{k-1} > (R_1 + \cdots + R_{k-1})(k-1)2^k, & \forall\, k \geqslant 2, \\ \phi(|a_k| - R_k) > (n_1 + \cdots + n_k + \sqrt{n+1}) \, \frac{\log(|a_k| + R_k)}{\log(|a_k| - R_k)}, & \forall\, k \geqslant 1. \end{cases} \tag{13}$$

In the first inequality, ϵ_0 and r_0 are defined in Lemma 3.5. Moreover, we demand that each complex number

$$a_k \in \mathbb{Z}_+ \cdot e^{\sqrt{-1}\rho_k}, \quad \forall\, k \geqslant 1,$$

has the argument ρ_k.

Clearly, such a_k can be chosen subsequently for $k = 1, 2, 3, \ldots$, since

$$\lim_{|a_k| \to +\infty} (n_1 + \cdots + n_k + \sqrt{n+1}) \, \frac{\log(|a_k| + R_k)}{\log(|a_k| - R_k)} < +\infty = \lim_{r \to +\infty} \phi(r).$$

Thus, the entire curve h is universal with slow growth (2) in Theorem 1.2 by the previous arguments.

Lastly, h is hypercyclic for any translation operator $\mathsf{T}_{e^{\sqrt{-1}\theta}}$ with $\theta \in E$, since θ repeats infinitely many time in $\{\rho_k\}_{k \geqslant 1}$. Using the argument of a result [4, Theorem 8], we conclude that h is simultaneously hypercyclic for any T_a where $a \in \mathbb{R}_+ \cdot e^{\sqrt{-1}\theta}$ has argument θ.

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