A SYMMETRY METHOD FOR KEY VANISHING LEMMAS AND THE 2-JET RIEMANN–ROCH THRESHOLDS

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Abstract. This manuscript reports on an ongoing project. The finite full-column-rank computations underlying the Key Vanishing Lemmas isolated below are currently undergoing exact computer-assisted verification. Conditional on successful completion of that verification, the arguments presented here would establish the two positivity endpoints of Demailly's invariant two-jet Riemann–Roch calculation: every very general surface in $\mathbb{P}^3$ of degree 15 would be Kobayashi hyperbolic, and the complement of every general smooth plane curve of degree 11 would be Kobayashi hyperbolic. At the logarithmic endpoint the same conditional argument would also give a Second Main Theorem with coefficient 1422; the preceding work treated all degrees $d \geqslant 12$.

The argument separates an infinite geometric range from a finite algebraic obstruction. An $\mathcal{O}(3)$ slanted-vector-field package combines a quadratic acceleration with a classical matrix lift. Their incompatible actions on the invariant fibre ring, together with the common-twist, prime-ideal, and relative-family interfaces, exclude the dependence alternative for every weight $m \geqslant 3$. This improves the differentiation range from $t \geqslant 8$ to $t \geqslant 4$ and leaves, beyond the twist-one ranges, only five compact and thirteen logarithmic systems.

The residual systems are reduced by symmetry. A two-term Fermat perturbation carries an involution, so a hypothetical negatively twisted two-jet differential has an invariant or anti-invariant component. Imposing this eigencondition before elimination adds exact linear constraints without adding unknowns. The required full-column-rank statements are isolated in the Key Vanishing Lemmas as finite computational targets. This paper records their finite reduction and proof-to-certificate interface. Their exact computer-assisted verification is in progress; the corresponding modular certificates and reproduction files remain to be generated, released, and independently checked.

1. Introduction

1.1. **Motivation and background.** Kobayashi's conjecture predicts that a general hypersurface $H \subset \mathbb{P}^n$ of degree at least $2n-1$ is hyperbolic. In dimension two the conjectural compact threshold is therefore 5. The jet-differential approach begins with Bloch and Green–Griffiths and takes its invariant form in Demailly's work [1, 9, 4]. A negatively twisted invariant jet differential is an algebraic differential equation satisfied by every entire curve. The difficulty is to produce enough independent equations, not merely one.

For surfaces, symmetric differentials do not provide the required starting point [22]; the present strategy therefore passes to invariant two-jets. For a smooth surface $X_d \subset \mathbb{P}^3$, Demailly's Riemann–Roch calculation has leading term

$$\frac{d(4d^2-68d+154)}{648}\, m^4;$$

for a smooth plane curve of degree d, the logarithmic leading term is

$$\frac{4d^2-51d+90}{648}\, m^4.$$

The first positive integer degrees in the respective general-type ranges are 15 and 11 [4, Section 13, formula (13.8), Corollary 13.9, and formula (13.16)]. These are the classical invariant two-jet Riemann–Roch positivity thresholds addressed here.

The route to these endpoints has two parallel histories. For compact surfaces, McQuillan obtained hyperbolicity in degree at least 36; Demailly–El Goul lowered the bound to 21 using Miyaoka's

Date: August 25, 2026.

2020 *Mathematics Subject Classification.* 32Q45, 32H25, 14J70, 14F10, 14Q20.

Key words and phrases. Kobayashi hyperbolicity, entire curves, jet differentials, symmetry method, Key Vanishing Lemma.

two-jet zero-locus method (see also Lu–Yau), and Păun lowered it to 18 using slanted vector fields [17, 7, 6, 5, 20]. For complements of plane curves, Siu–Yeung proved the high-degree theorem, El Goul reached degree 15, and Rousseau reached degree 14 [24, 8, 21]. In both settings the remaining obstruction is a finite strip in which the zero-locus inequality is too weak and the available differentiation no longer preserves negative twist.

The recent Hou–Huynh–Merker–Xie series reorganised this finite strip around Key Vanishing Lemmas. The first paper found effective exponent bounds and a finite algebraic encoding in the three-component model [12]. This was the decisive finiteness step: once the bounds are explicit, the remaining vanishing question belongs to exact polynomial algebra. The second paper developed the one-component theory, including the direct-image, irreducible-factor, and chart-divisibility interfaces. Its public version gives the compact and logarithmic bounds 17 and 12 [13]. The first public version of the present symmetry paper proves the compact bound $d \geqslant 16$ [3]; that degree-16 proof is not reproduced or used here. The present revision adds two ingredients at the remaining endpoints: symmetry is imposed before sparse elimination, and pole-three differentiation removes the entire high-twist range before a matrix is built. Together, they would reach degrees 15 and 11 once the residual full-rank assertions have been verified. The pole-three argument is not obtained from the pole-three matrix-lift directions extracted from Păun's classical construction alone. Those directions supply the semisimple operator, but by themselves they do not eliminate the Wronskian dependence of a preserved irreducible factor. The new quadratic-acceleration fields provide the missing locally nilpotent translation in the Wronskian variable; only the combination of the two types of fields yields the all-weight contradiction used here.

The new Key Vanishing reduction uses a two-term Fermat perturbation invariant under an involution exchanging two coordinates. Any hypothetical section has an eigencomponent for the induced $\mathbb{Z}/2\mathbb{Z}$ action. The eigencondition adds exact linear equations without enlarging the coefficient space, turning the residual systems into highly overdetermined sparse matrices. The authors' `C++` implementation is currently being used to perform exact modular elimination block by block. A full-column-rank certificate over one good finite field, once generated and independently checked, would prove full rank over $\mathbb{Q}$. Once archived and reproduced, this exact computation would become part of the proof, not a floating-point experiment.

The accompanying pole-three construction has a complementary role. Beyond the twist-one ranges, it reduces the original high-weight lists to five compact and thirteen logarithmic cases; it is not the source of the Key Vanishing Lemmas themselves. This division of labour—geometry for the high-twist range and exact algebra for the residual strip—would close both endpoint arguments once the residual full-rank assertions have been verified.

Scope and priority. The priority asserted here is at the level of global section spaces. Merker gave an algorithm generating invariant jet algebras in arbitrary dimension and arbitrary jet order [19]; we make no claim to originate that broader invariant-theoretic problem. The distinct finite reduction developed in the present series uses effective exponent and denominator bounds to realise the relevant negatively twisted compact and logarithmic global H^0 spaces as kernels of explicit finite matrices [12, 13]. The symmetry reduction was publicly announced in the first version of the present work, arXiv:2606.19155v1, submitted on 17 June 2026 [3]. This revision develops that reduction, adds the global pole-three differentiation theorem, and reduces the conditional degree-15 and degree-11 endpoints to the finite full-rank assertions underlying Lemma 1.3, whose exact computer-assisted verification is currently in progress. Both the symmetry method and the pole-three strategy are ideas of the authors. More precisely, the original contributions are the new quadratic-acceleration fields and their global two-operator prime-ideal argument, together with the symmetry reduction imposed before exact elimination. The classical matrix lifts are an indispensable input, but they are not the new pole-three mechanism proved in this paper.

1.2. Main results.

The conditional endpoint argument has three inputs: the finite Key Vanishing statements isolated below, whose exact computer-assisted verification is in progress, the all-weight pole-three differentiation theorem of Section 2, and the established zero-locus and foliation arguments recalled in Section 5. We first state the geometric conclusions. Here *general* means outside a proper Zariski-closed subset, whereas *very general* means outside a countable union of proper Zariski-closed subsets.

The following theorem, and the degree-11 Second Main Theorem stated afterward, are conditional on successful completion of the ongoing exact computer-assisted verification of the full-rank assertions underlying Lemma 1.3; the precise status is recorded immediately after that lemma.

Theorem 1.1 (Two-jet endpoint theorem)**.** *Assume that the ongoing exact computer-assisted full-rank verification successfully establishes both clauses of Lemma 1.3. Then:*

(1) *Every very general surface $X \subset \mathbb{P}^3$ of degree 15 is Kobayashi hyperbolic.*
(2) *For every general smooth plane curve $C \subset \mathbb{P}^2$ of degree 11, the complement $\mathbb{P}^2 \setminus C$ is Kobayashi hyperbolic.*

The first assertion is the new conditional compact endpoint that would follow from this revision once the pending verification succeeds. The public version of [13] treats every degree $d \geqslant 17$, while the public first version of the present symmetry paper treats every degree $d \geqslant 16$ [3]. Together with the conditional degree-15 endpoint derived below, these results would give the compact range $d \geqslant 15$. No degree-16 proof is reproduced or used in the degree-15 argument.

Here $T_f(r)$ denotes the Nevanlinna characteristic, $N_f^{[1]}(r, C)$ the level-one truncated counting function, and the symbol $\|$ means that the estimate holds outside a set of finite Lebesgue measure.

At the logarithmic endpoint the argument would, subject to the same pending verification, give more than hyperbolicity: it would also give the following quantitative statement.

Theorem 1.2 (Second Main Theorem at degree 11)**.** *Assume that the ongoing exact computer-assisted full-rank verification successfully establishes the logarithmic clause of Lemma 1.3. Then, for a general smooth plane curve $C \subset \mathbb{P}^2$ of degree 11 and every algebraically nondegenerate entire curve $f : \mathbb{C} \to \mathbb{P}^2$ whose image is not contained in C, one has*

$$T_f(r) \leqslant 1422\, N_f^{[1]}(r, C) + o\big(T_f(r)\big) \ \|.$$

For general smooth plane curves of every degree $d \geqslant 12$, a Second Main Theorem with an explicit degree-dependent constant C_d was already proved in the preceding paper [13]. Theorem 1.2 would, upon successful completion of the pending verification, supply precisely the previously missing endpoint $d = 11$.

The sole outstanding finite computational task behind both conditional endpoint theorems is the exact computer-assisted verification of the next statement. Twist monotonicity would then supply vanishing at every more negative twist with the same weight. In the current version, the following lemma records the finite computational targets and is not claimed to have been verified.

Lemma 1.3 (Key Vanishing Lemmas)**.** (1) *If $X \subset \mathbb{P}^3$ is a general surface of degree 15, then*

$$H^0(X, E_{2,m} T_X^* \otimes \mathcal{O}_X(-t)) = 0$$

for

$$(m, t) = (3, 1), (4, 1), (5, 1), (6, 1), (7, 2), (8, 3), (9, 3), (10, 3), (11, 3).$$

(2) *If $C \subset \mathbb{P}^2$ is a general smooth curve of degree 11, then*

$$H^0\big(\mathbb{P}^2, E_{2,m} T_{\mathbb{P}^2}^*(\log C) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)\big) = 0$$

for $3 \leqslant m \leqslant 9$ and $t = 1$, and for

$$(m, t) = (10, 2), (11, 2), (12, 2), (13, 2), (14, 2),$$
$$(m, t) = (15, 3), (16, 3), (17, 3), (18, 3), (19, 3), (20, 3), (21, 3), (22, 3).$$

We use Lemma 1.3 only as a conditional computational target. Sections 3 and 4 record the mathematical reduction and the proof-to-certificate interface, while the exact full-rank computations are currently in progress. The present version does not claim that the listed Key Vanishing assertions have been verified. The modular certificates have yet to be generated, released, and independently checked. After completion of the computation, the corresponding source, certificates, and reproduction instructions will be deposited in a computational archive on Song-Yan Xie's homepage.[1]

Pending completion of the exact computer-assisted verification, generation and release of the certificates, and independent reproduction, all results using Lemma 1.3 remain conditional on

[1] https://xiesongyan.github.io/

successful verification of the underlying full-rank assertions. No later argument introduces another computational vanishing assumption: Section 2 treats the infinite high-twist range geometrically, Sections 3 and 4 reduce exactly the displayed finite cases, and Section 5 combines these two inputs with the classical geometric theorems.

Remark 1.4 (The multi-foliation route from very general to general)**.** The compact theorem above is stated for very general surfaces because its last step uses the classical Clemens incidence argument [2]. Without a uniform bound for the degrees of rational and elliptic curves, that argument must range over countably many Hilbert polynomials and therefore removes a countable union of closed subsets.

The sharp direct-image theorem developed in the first papers of the series does more than produce a single jet equation: it spreads the relevant irreducible factors in algebraic families over the hypersurface parameter space; compare [12, Proposition 6.1]. In forthcoming work, Song-Yan Xie and Shengyuan Zhao use these families to construct a finite-type multi-foliation structure. A rational or elliptic curve on a surface in the family is forced to be invariant under that structure. Invariance supplies a uniform degree bound for such curves. Once the degrees are bounded, only finitely many incidence families remain, and Clemens's argument yields one proper algebraic exceptional set. This upgrades "very general" to "general" (equivalently, to a nonempty Zariski-open parameter set).

This forthcoming theorem is not used anywhere in the present proof. Its role here is conceptual: an analytic multi-foliation extracted from the direct-image package supplies precisely the algebraic boundedness input missing from the classical argument. The compact conclusion of this paper would therefore admit the announced upgrade once that theorem is public and its hypotheses have been checked in the present family, without changing the ongoing Key Vanishing verification.

1.3. **Finite reduction at the two-jet thresholds.** At degree 15 the compact zero-locus threshold is 17/66; at degree 11 the logarithmic threshold is 13/96. For each weight m, let t_m be the least integral twist on the large-slope side of the corresponding threshold. A pole-order-seven field preserves negative twist only when $t_m \geqslant 8$. Hence it leaves the finite strips

$$t_m^{\mathrm{cpt}} = \left\lceil \frac{17m}{66} \right\rceil, \quad 3 \leqslant m \leqslant 27, \qquad t_m^{\log} = \left\lceil \frac{13m}{96} \right\rceil, \quad 3 \leqslant m \leqslant 51.$$

to a separate Key Vanishing verification task: the upper bounds 27 and 51 are exactly the last weights for which $t_m \leqslant 7$. At those endpoints, the raw ansatz has 78 250 775 compact unknowns for $(m,t) = (27,7)$ and 33 956 835 logarithmic unknowns for $(m,t) = (51,7)$, before the larger equation sets and elimination fill-in are formed. These systems are prohibitive for the present exact sparse-elimination implementation. The $\mathcal{O}(3)$ package resolves this scale obstruction geometrically: it removes every target with $t_m \geqslant 4$ before any matrix is constructed.

The pole-three argument changes the problem before elimination begins. The reduction has three steps. First, Theorem 2.27, through Corollaries 2.30 and 2.29, disposes of every target with $t_m \geqslant 4$. This removes the compact weights $12 \leqslant m \leqslant 27$ and the logarithmic weights $23 \leqslant m \leqslant 51$ without constructing any matrix.

Second, use monotonicity in the twist. If $H^0(E \otimes \mathcal{O}(-s)) = 0$, then $H^0(E \otimes \mathcal{O}(-t)) = 0$ for every $t \geqslant s$: otherwise multiplication by a nonzero form of degree $t - s$ would produce a section at twist $-s$. Once verified, the twist-one targets would therefore cover the compact weights $3 \leqslant m \leqslant 6$ and the logarithmic weights $3 \leqslant m \leqslant 9$; in particular, they would also cover the logarithmic twist-two targets at $m = 8, 9$.

Third, what remains to be verified is exactly the finite rank calculation formulated in this paper. In the compact clause, it consists of the twist-two case at $m = 7$ and the twist-three range $8 \leqslant m \leqslant 11$. In the logarithmic clause, it consists of the twist-two range $10 \leqslant m \leqslant 14$ and the twist-three range $15 \leqslant m \leqslant 22$. Thus pole order three converts the inaccessible high-weight problem into five compact and thirteen logarithmic systems. Lemma 1.3 is the unique location of the full case lists; the computational sections reduce its two clauses separately to the pending full-rank assertions now undergoing exact computer-assisted verification.

Remark 1.5 (Size of the new finite systems)**.** The pole-three package replaces every high-twist calculation by geometry. After the twist-one ranges, it leaves only the five compact and thirteen logarithmic pairs displayed above. The number of raw coefficient unknowns can be recomputed directly from the finite ansatz. In the compact case put

$$D_k = (d-2)m - 2(d-1)k - t.$$

Then

$$N_d^{\mathrm{cpt}}(m,t) = \sum_{k=0}^{\lfloor m/3 \rfloor} (m-3k+1)\left\{\binom{D_k+3}{3} - \binom{D_k-d+3}{3}\right\}, \tag{1.1}$$

where a binomial coefficient is zero when its upper entry is smaller than the lower one. The subtraction imposes the normal form $\deg_x K_{j,k} < d$. For the logarithmic ansatz put

$$D_{j,k} = (d-2)m - (2d-3)k + j - t;$$

then

$$N_d^{\log}(m,t) = \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{j=0}^{m-3k} \binom{D_{j,k}+2}{2}. \tag{1.2}$$

Consequently the largest surviving degree-15 compact system is $(m,t) = (11,3)$ with 2 732 550 raw unknowns, while the largest surviving degree-11 logarithmic system is $(22,3)$ with 1 383 586 raw unknowns. These are raw ansatz counts—after the compact quotient normal form, but before the symmetry and holomorphicity equations are imposed. They count columns of the eventual exact matrix, not its rows or the elimination fill-in. Symmetry adds rows, not unknowns. Thus even after pole order seven has been replaced by pole order three, the surviving Key Vanishing systems press against the present limit of exact computation. The $\mathcal{O}(3)$ argument is precisely what moves the problem from an inaccessible regime to a difficult but tractable one.

1.4. **The symmetry method: overcoming computational barriers.** The two clauses of Lemma 1.3 use the same specialisation principle as [13]. It is enough to prove vanishing on one smooth special fibre; upper semicontinuity then gives vanishing on a nonempty Zariski-open parameter set. The outstanding issue is therefore not specialisation, but the exact computer-assisted verification now in progress on the chosen fibre.

The local descriptions in [13, Propositions 3.1 and 4.1] write a hypothetical negatively twisted invariant two-jet differential in a finite polynomial normal form; see also [12]. Its coefficient polynomials $K_{j,k}$ are the unknowns, and regularity on the remaining charts gives a homogeneous linear system over $\mathbb{Q}$. The present method keeps this finite encoding but inserts the symmetry equations before the matrix is assembled.

Choose the special fibre to be invariant under an involution σ that exchanges two coordinates. If its section space were nonzero, the induced finite-dimensional $\mathbb{Z}/2\mathbb{Z}$ representation would contain an eigensection with eigenvalue $\varepsilon = \pm 1$. Multiplication by a coordinate hyperplane embeds the negatively twisted section into the untwisted bundle. Because that hyperplane is not required to be σ-invariant, the eigenidentity becomes the covariance identity derived in Sections 3 and 4. Cancelling its two known hyperplane factors leaves linear equations in the same coefficients $K_{j,k}$: symmetry adds constraints, not variables.

The resulting matrix has rational, and after clearing denominators integral, entries. A full-column-rank certificate modulo a prime p, once generated and checked, would prove full column rank over $\mathbb{Q}$, because a nonzero maximal minor modulo p is already a nonzero integer minor. The nullity check will record whether a kernel remains, and the prime check will record the exact field in which the rank was computed. The pending full-rank assertions underlying Lemma 1.3 are precisely these computational targets; the later certificate sections specify the data still to be generated and archived for reproduction.

The pole-three argument and the symmetry argument play complementary roles. The former removes every target with $t \geqslant 4$ before computation; the latter reduces the remaining low-twist cases to the pending rank assertions currently undergoing exact computer-assisted verification. The exact count in Remark 1.5 shows that the largest new systems have 2 732 550 compact and 1 383 586 logarithmic raw unknowns. Thus the gain is not a heuristic speedup of the former high-weight matrices: those matrices no longer occur.

This separation is also conceptual: differentiation controls an infinite range, whereas symmetry addresses a finite exact kernel problem. Within the present Riemann–Roch mechanism, passing below the classical two-jet thresholds would require a new source of jet differentials—for example, a workable three-jet theory—together with new Key Vanishing statements. Richer finite automorphism groups may then replace the involution used here.

1.5. **Structure of the paper.** Section 2 constructs the two pole-three slanted-field packages and proves the all-weight differentiation theorem, including the compact and logarithmic relative-family extensions, stationary saturation, common-twist and prime-ideal interfaces. Section 3 gives the logarithmic symmetry reduction and its finite certificate interface, while Section 4 gives the compact counterpart. Section 5 records the resulting conditional compact and logarithmic endpoint deductions and, subject to the pending Key Vanishing verification, derives the coefficient 1422. Finally, Section 6 discusses further perspectives.

2. Low-pole Slanted Vector Fields

This section supplies the low-pole differentiation input. Generating the full tangent bundle of the vertical two-jet space leads to pole order 7; the conditional endpoint arguments require less. We instead pair a pole-three quadratic acceleration, which translates the Wronskian, with a semisimple direction from the classical compact and logarithmic matrix lifts of Păun [20, Lemma 1.1] and Rousseau [21, Lemma 5]. The correcting length-three finite-difference fields are the corresponding specialization of Merker's construction [18, Section 3]. These ingredients yield the required theorem only after the new acceleration, common-twist cancellation, and two-operator independence argument are combined.

This distinction is essential. The pole-three matrix-lift directions extracted from the classical package are indispensable for the semisimple step, but they are not sufficient by themselves for the present dependence problem: they do not force a preserved prime factor to lose its Wronskian variable. The acceleration fields constructed below are new fields of pole order three. Their induced action $S_3(u,v)\partial_W$ supplies exactly that missing implication, after which the classical matrix direction completes the contradiction.

The section is organised around one contradiction. In the invariant fibre ring $K[u,v,W]$, stability under the acceleration eliminates the Wronskian variable W; stability under the semisimple operator then forces an irreducible binary form to have degree at most 2. This is impossible for the jet factors of weight at least 3.

The proof realises this algebraic picture in its dependency order. We first prove the contradiction in the whole unlocalised invariant ring. We then establish the common line-bundle twist and show that a relative irreducible divisor, after removal of its stationary part, defines a height-one prime ideal. The compact and logarithmic constructions come next, followed by the single incidence open on which both operators are available. The final subsection assembles these inputs into the differentiation theorem. Every divisibility statement is therefore an identity of polynomial derivations, not an inference from one pointwise tangent-vector value.

2.1. **The invariant fibre ring and the abstract two-operator contradiction.** Let K be a field of characteristic zero. On a two-dimensional fibre write the first- and second-jet variables as (x,y) and (X,Y) and put

$$W = xY - yX.$$

The reparametrization weights are

$$\mathrm{wt}(x) = \mathrm{wt}(y) = 1, \qquad \mathrm{wt}(W) = 3.$$

Lemma 2.1 (Unlocalised invariant ring)**.** *One has*

$$\ker\bigl(x\partial_X + y\partial_Y\bigr) = K[x,y,W] \subset K[x,y,X,Y].$$

Proof. After localising at x, use x, y, X, W as coordinates. The derivation is $x\partial_X$, hence its kernel is $K[x,y,W]_x$. If a polynomial in the original four variables were $x^{-N}b(x,y,W)$ with $N>0$ minimal, reduction modulo x would say that $b(0,y,-yX)$ vanishes. The map $K[y,W] \to K[y,X,Y]$, $W \mapsto -yX$, is injective, so b would be divisible by x, a contradiction. □

The following two elementary lemmas are the algebraic core of the construction. It is essential that they are applied in the unlocalised graded ring, not in a rational coordinate ring of a Semple chart.

Lemma 2.2 (Locally nilpotent divisibility)**.** *Let $0 \neq S_3 \in K[x,y]$ be homogeneous of degree 3 and set $\delta = S_3\partial_W$. If a nonzero weighted-homogeneous $P \in K[x,y,W]$ satisfies $\delta P \in (P)$, then*

$$\partial_W P = 0.$$

Proof. Put $R = K[x,y,W]$. The assumption means that

$$\delta P = QP \qquad \text{for some } Q \in R.$$

The derivation $\delta = S_3\partial_W$ has weight zero. If $\delta P \neq 0$, then both P and δP are weighted homogeneous of the same weight. Writing $Q = \sum_{j\geqslant 0} Q_j$ as a sum of weighted-homogeneous components, the equality

$$\delta P = \sum_{j\geqslant 0} Q_j P$$

and the fact that R is a graded domain show that $Q_j = 0$ for every $j \neq 0$. Hence $Q \in R_0$. Since

$$\mathrm{wt}(x) = \mathrm{wt}(y) = 1, \qquad \mathrm{wt}(W) = 3$$

is a positive grading, $R_0 = K$. Thus $Q = \lambda \in K$; the same conclusion is immediate when $\delta P = 0$, with $\lambda = 0$.

If $r = \deg_W P$, then $\delta^{r+1}P = 0$, because S_3 is independent of W. On the other hand, $\delta(\lambda) = 0$, so induction in the equation $\delta P = \lambda P$ gives

$$\delta^{r+1}P = \lambda^{r+1}P.$$

Consequently $\lambda = 0$. Finally $S_3\partial_W P = 0$, and the facts that $S_3 \neq 0$ and R is a domain imply $\partial_W P = 0$. □

Lemma 2.3 (Semisimple binary rigidity)**.** *Let $T \in \mathfrak{sl}_2(K)$ satisfy $\det T \neq 0$. If a homogeneous irreducible $P \in K[x,y]$ satisfies $\mathcal{D}_T(P) \in (P)$ for the induced linear derivation, then $\deg P \leqslant 2$.*

Proof. Let $m = \deg P$. Degree gives $\mathcal{D}_T(P) = \lambda P$ with $\lambda \in K$. Since $T \in \mathfrak{sl}_2(K)$ and $\det T \neq 0$, its characteristic polynomial has two distinct roots $\alpha, -\alpha$ over a splitting field L/K of degree at most two. Choose eigenlinear forms u, v over L. Then

$$\mathcal{D}_T = \alpha(u\partial_u - v\partial_v), \qquad \alpha \neq 0.$$

The monomials $u^r v^{m-r}$, $0 \leqslant r \leqslant m$, have the pairwise distinct eigenvalues $\alpha(2r-m)$. Therefore the λ-eigenspace in $\mathrm{Sym}^m(L^2)$ is one-dimensional, and for some $c \in L^\times$ one has

$$P = c\,u^r v^{m-r}.$$

If both eigenlines are defined over K, this factorisation is defined over K up to nonzero scalars. The irreducibility of P then forces $m = 1$. Suppose instead that L/K is quadratic and let σ be its nontrivial Galois involution. It interchanges the two eigenlines, so the Galois orbit of either eigenline has size two. Since P is defined over K, the multiset of its linear factors is σ-invariant. Hence the multiplicities of u and v agree: $r = m - r$. It follows that

$$P = c(uv)^{m/2}.$$

The product uv descends, up to a scalar, to the irreducible quadratic whose zero set is the two-element Galois orbit of eigenlines. Irreducibility of P therefore forces $m/2 = 1$. Thus $m \leqslant 2$ in all cases. □

Proposition 2.4 (Abstract two-operator contradiction)**.** *Let*

$$R = K[u,v,W], \qquad \mathrm{wt}(u) = \mathrm{wt}(v) = 1, \quad \mathrm{wt}(W) = 3,$$

where K has characteristic zero. Let $P \in R$ be irreducible and weighted homogeneous of weight $m \geqslant 3$. Suppose that the prime ideal (P) is stable under two weight-zero derivations

$$\delta_{\mathrm{acc}} = S_3(u,v)\partial_W, \qquad \delta_{\mathrm{ss}} = \mathcal{D}_T + R_3(u,v)\partial_W,$$

where $0 \neq S_3 \in K[u,v]_3$, $R_3 \in K[u,v]_3$, and $T \in \mathfrak{sl}_2(K)$ has nonzero determinant. Then these assumptions are contradictory.

Proof. Stability under δ_{acc} and Lemma 2.2 give $P \in K[u,v]$. Hence the $R_3\partial_W$ term annihilates P, and stability under δ_{ss} becomes

$$\mathcal{D}_T(P) \in (P).$$

The weighted degree of P is now its ordinary binary degree. Lemma 2.3 gives $m \leqslant 2$, contrary to $m \geqslant 3$. □

2.2. **The common-twist and prime-ideal interfaces.** We next record explicitly the two interfaces that are often hidden in the phrase "evaluate the slanted field at a point." Let V_d be the vector space of degree-d equations, let $V_d \setminus \{0\}$ be its coefficient cone, and write $[a] \in \mathbb{P}(V_d)$ for the corresponding projective parameter. A cone field

$$\sum_{\alpha} q_\alpha(a,z)\partial_{a_\alpha}$$

whose coefficient vector is homogeneous of degree e in a scales by λ^{e-1} under $a \mapsto \lambda a$. Modulo the coefficient Euler field, it therefore descends with parameter twist $\mathcal{O}_{\mathbb{P}(V_d)}(e-1)$. In particular, a field whose parameter coefficients are linear in a is parameter-untwisted, whereas a constant translation has parameter twist $\mathcal{O}_{\mathbb{P}(V_d)}(-1)$.

Fix a coefficient linear form $\ell \in H^0(\mathbb{P}(V_d), \mathcal{O}_{\mathbb{P}(V_d)}(1))$ and work on $\{\ell \neq 0\}$. For multi-indices $\alpha, \gamma \in \mathbb{N}^3$ satisfying $\gamma \leqslant \alpha$ componentwise, $|\gamma| = 3$, and $|\alpha| \leqslant d$, define the raw third-difference field

$$D^{[3]}_{\alpha,\gamma} = \sum_{\eta \leqslant \gamma} (-1)^{|\eta|} \binom{\gamma}{\eta} z^\eta \frac{\partial}{\partial a_{\alpha-\eta}}. \tag{2.1}$$

It has no base or jet component. Applied to the universal polynomial in a dummy variable Y, it gives the exact identity

$$D^{[3]}_{\alpha,\gamma}\big(F(Y,a)\big) = Y^{\alpha-\gamma}(Y-z)^\gamma. \tag{2.2}$$

Thus its value, gradient, and Hessian in Y vanish at $Y = z$, which is precisely tangency to the three vertical two-jet equations. Its coefficients have degree at most three in z and cone degree zero in the parameters, so

$$D^{[3]}_{\alpha,\gamma} \in T_{J_2^v} \otimes \mathcal{O}_{\mathrm{base}}(3) \boxtimes \mathcal{O}_{\mathbb{P}(V_d)}(-1).$$

These are the order-three fields in the finite-difference package of [18, Section 3, "Higher lengths"]. If $\Phi_{\alpha,\gamma}(Y,z) := Y^{\alpha-\gamma}(Y-z)^\gamma$, then

$$\mathrm{span}_K\{\Phi_{\alpha,\gamma}\} = \mathfrak{m}_z^3 \cap K[Y]_{\leqslant d}. \tag{2.3}$$

Indeed, the Taylor monomials $(Y-z)^\nu$, with $3 \leqslant |\nu| \leqslant d$, form a basis of the right-hand side. Choose $\gamma \leqslant \nu$ with $|\gamma| = 3$ and expand $(Y-z)^{\nu-\gamma}$ in powers of Y; every resulting term is one of the displayed $\Phi_{\alpha,\gamma}$.

Lemma 2.5 (Common-twist cancellation with coefficient homogeneity)**.** *Let V be a parameter-untwisted field of base twist $\mathcal{O}(3)$ whose cone parameter component is represented by a polynomial $Q_V(Y;z,a)$ of degree at most d in Y and degree one in a. Suppose that, over an incidence function field K,*

$$j_z^2 Q_V = 0.$$

On the chart $\ell \neq 0$, put $\overline{Q}_V = Q_V/\ell$. Choose $c_i \in K$ such that

$$\overline{Q}_V = \sum_i c_i \Phi_i \quad \textit{in } \mathfrak{m}_z^3 \cap K[Y]_{\leqslant d},$$

where the Φ_i are third-difference polynomials, and set

$$\mathcal{V} = \ell V - \sum_i c_i \ell^2 D_i.$$

Then $\mathcal{V}$ is an actual local section of

$$T_{J_2^v} \otimes \mathcal{O}_{\mathrm{base}}(3) \boxtimes \mathcal{O}_{\mathbb{P}(V_d)}(1),$$

not merely a subtraction of tangent-vector values. If the prime ideal (P) is stable under V and every D_i, it is stable under $\mathcal{V}$; after cancellation, evaluation leaves the residual derivation on the whole unlocalised fibre ring $K[x, y, W]$.

Proof. The parameter component of V has cone degree one, whereas that of each D_i has cone degree zero. The identity

$$Q_V = \ell \sum_i c_i \Phi_i$$

therefore has degree one on both sides. Multiplication by ℓ puts V in parameter twist $\mathcal{O}(1)$, while multiplication by ℓ^2 changes the twist of D_i from $\mathcal{O}(-1)$ to $\mathcal{O}(1)$. Every summand has the same base twist $\mathcal{O}(3)$ as well. The functions c_i have coefficient degree zero: they are rational functions on the projective incidence space, not additional line-bundle sections. After shrinking the incidence open around its generic point, all finitely many c_i are regular. Thus the displayed formula is a genuine local section of the asserted bundle.

The cone parameter component of ℓV is $\ell Q_V = \ell^2 \sum_i c_i \Phi_i$, exactly the parameter component of the sum being subtracted. The cancellation is consequently an identity over the chosen incidence open. Ideal stability is closed under multiplication by regular functions and finite sums, so $\mathcal{V}(P) \in (P)$. Finally, the D_i have no jet component. After fixing the base and parameter variables, the residual action of $\mathcal{V}$ is therefore a derivation of the whole polynomial ring $K[x, y, W]$; the variables x, y, W have not been specialised or inverted. □

Convention 2.6 (Projective normalisation)**.** On the chart $\ell \neq 0$, every occurrence of the universal equation in the moving Hessian and third-order tensors below means the normalised equation F/ℓ (and, in the logarithmic setting, f/ℓ). Consequently all moving coefficients are rational functions of projective degree zero on the incidence space. The explicit powers of ℓ in Lemma 2.5 account for the entire parameter twist; no additional coefficient twist is hidden in the tensor notation.

Definition 2.7 (Nonstationary irreducible factor)**.** Let ω be an invariant two-jet differential, viewed as a section on the second Demailly–Semple stage, and write

$$\operatorname{div}(\omega) = \sum_j n_j Z_j + b\Gamma_2$$

with the Z_j prime divisors different from the stationary divisor Γ_2. By a nonstationary irreducible factor we mean the invariant factor associated by the standard Demailly–El Goul reduction to one reduced prime Cartier component Z_j. Thus the factorisation is made in the local regular rings of the Semple tower. After trivialising its line bundle and choosing a tangent frame on the regular jet locus, the same factor is represented by an irreducible weighted-homogeneous polynomial in $K[x, y, W]$. No localisation in a first-jet variable or in W is part of this definition.

2.2.1. *Relative compact jet-divisor families.* We now supply the relative-family input needed to apply the preceding function-field argument to a jet differential initially chosen on one surface. The construction follows Siu [23, pp. 558–559], the compact extension argument of [13, Proposition 5.4], and the relative-divisor construction in [13, §6.2]. The following use of the direct-image theorem—to pass from fibrewise jet-direction loci and irreducible factors to algebraic data over the parameter space—was suggested by Professor Yum-Tong Siu during the Tianyuan Conference on Several Complex Variables in the summer of 2024; see the Acknowledgement below.

Theorem 2.8 (Generic extension of fibrewise sections)**.** *Let $\tau\colon \mathcal{Z} \to S$ be a projective flat morphism of finite presentation, with S integral and noetherian, and let $\mathcal{L}$ be a locally free sheaf on $\mathcal{Z}$. There is a dense Zariski open $S^\circ \subset S$ on which $\tau_*\mathcal{L}$ is locally free and the base-change map*

$$(\tau_*\mathcal{L}) \otimes \kappa(s) \longrightarrow H^0(\mathcal{Z}_s, \mathcal{L}_s)$$

is an isomorphism for every $s \in S^\circ$. Consequently, after replacing S° by a Zariski open neighbourhood of s, every section of $\mathcal{L}_s$ extends to a regular relative section.

Proof. After generic freeness and shrinking the base, cohomology and base change applies to $\mathcal{L}$ and makes the displayed map an isomorphism; see [11, Chapter III, §12, especially p. 288]. A vector in the fibre of the resulting locally free sheaf $\tau_*\mathcal{L}$ extends over an affine trivializing neighbourhood of s. Its corresponding relative section restricts to the prescribed fibrewise section. □

Lemma 2.9 (Relative residual Cartier divisors)**.** *Let $\tau\colon \mathcal{Y} \to S$ be projective, flat and of finite presentation, where S and $\mathcal{Y}$ are integral noetherian schemes and the fibres of τ are geometrically integral. Let $\mathcal{L}$ be an invertible sheaf and let $s \in H^0(\mathcal{Y}, \mathcal{L})$ restrict to a nonzero section on every fibre. Then its zero scheme $D = Z(s)$ is an effective Cartier divisor, and*

$$D \longrightarrow S$$

is projective, flat and of finite presentation. If one fibre D_{s_0} is geometrically integral, then, after shrinking S around s_0, every fibre of $D \to S$ is geometrically integral.

More generally, let $\Gamma \subset \mathcal{Y}$ be a relative prime effective Cartier divisor whose fibres Γ_σ are integral, with canonical section s_Γ. Fix $b \geqslant 0$ and suppose that

$$\widetilde{s} \in H^0(\mathcal{Y}, \mathcal{L}(-b\Gamma)) , \qquad s = s_\Gamma^b \widetilde{s},$$

and that, for every $\sigma \in S$,

$$\widetilde{s}_\sigma \neq 0, \qquad \widetilde{s}_\sigma|_{\Gamma_\sigma} \neq 0.$$

Then the same conclusions apply to the residual divisor $Z(\widetilde{s})$, and b is the full order of s_σ along Γ_σ on every fibre.

Proof. After a local trivialisation of $\mathcal{L}$, the section s is represented by a regular function h. Since $\mathcal{Y}$ is integral and h is nonzero on the generic fibre, h is a nonzerodivisor; hence D is an effective Cartier divisor. The two terms on the left in

$$0 \longrightarrow \mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_{\mathcal{Y}} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

are flat over S. On a fibre $\mathcal{Y}_\sigma$, which is integral, $s_\sigma \neq 0$, so multiplication by s_σ remains injective. The fibrewise local criterion for flatness therefore makes $\mathcal{O}_D$ flat over S. Finite presentation follows from the fact that D is Cartier, and projectivity follows because it is closed in $\mathcal{Y}$. These are the standard relative-effective-Cartier hypotheses; see [25, Tags 062Y and 056P].

For a projective flat morphism of finite presentation, geometric reducedness and geometric irreducibility, hence geometric integrality, are open after shrinking around a geometrically integral fibre; see [25, Tags 0574 and 0559]. This proves the first assertion. For the last assertion, the equality $s = s_\Gamma^b \widetilde{s}$ multiplies a section of $\mathcal{L}(-b\Gamma)$ by the canonical section of $\mathcal{O}(b\Gamma)$ and returns a section of $\mathcal{L}$. Applying the first part to $\widetilde{s}$ gives the residual statement. Fibrewise, let g define Γ_σ and let h represent $\widetilde{s}_\sigma$ near its generic point. The restriction condition says $h \notin (g)$, so $s_\sigma = g^b h$ has order exactly b and $Z(h)$ has no stationary component. $\square$

Remark 2.10 (Fibrewise stationary order can jump)**.** The restriction condition in Lemma 2.9 is essential. In the local model

$$s = x(t + xy), \qquad \Gamma = \{x = 0\},$$

with parameter t, the order along Γ is one on the generic fibre, whereas $s|_{t=0} = x^2 y$ has order two. Dividing only by the generic factor x leaves $t + xy$, whose special fibre xy still contains Γ_0. Accordingly, the relative constructions below first remove the full stationary order on the chosen fibre and then extend that residual section.

Lemma 2.11 (No negatively twisted factors below weight three)**.** *Let $t \geqslant 1$.*

(1) *If $X \subset \mathbb{P}^3$ is a smooth surface, then*

$$H^0(X, E_{2,m}T_X^* \otimes \mathcal{O}_X(-t)) = 0, \qquad m = 1, 2.$$

(2) *If $C \subset \mathbb{P}^2$ is a smooth irreducible curve, then*

$$H^0\big(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log C) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)\big) = 0, \qquad m = 1, 2.$$

Consequently every positive-slope irreducible factor selected below has weight $m \geqslant 3$.

Proof. Below weight three the invariant fibre ring has no Wronskian generator, so $E_{2,m}T_X^* = \operatorname{Sym}^m \Omega_X^1$ for $m = 1, 2$, and likewise in the logarithmic case. Sakai's vanishing theorem gives $H^0(X, \operatorname{Sym}^m \Omega_X^1) = 0$ for a smooth surface in $\mathbb{P}^3$ [22]. Multiplication by a nonzero section of $\mathcal{O}_X(t)$ injects the negatively twisted space into this zero space.

For the logarithmic assertion put $A = \Omega_{\mathbb{P}^2}^1$, $E = \Omega_{\mathbb{P}^2}^1(\log C)$, and let $i : C \hookrightarrow \mathbb{P}^2$. The residue sequence

$$0 \longrightarrow A \longrightarrow E \xrightarrow{\text{res}} i_*\mathcal{O}_C \longrightarrow 0$$

gives $H^0(E(-t)) = 0$: the Euler sequence gives $H^0(A(-t)) = 0$, while $H^0(C, \mathcal{O}_C(-t)) = 0$.

For $\operatorname{Sym}^2 E$, choose local coordinates with $C = \{z = 0\}$ and write $e = dz/z$, $f = dw$. The intrinsic second residue

$$\mathrm{res}_2 : \operatorname{Sym}^2 E \longrightarrow i_*\mathcal{O}_C, \qquad A_0e^2 + B_0ef + C_0f^2 \longmapsto A_0|_C$$

is surjective. If K_2 is its kernel, the local generators ze^2, ef, f^2 give exact sequences

$$0 \longrightarrow K_{\mathbb{P}^2} \longrightarrow A \otimes E \longrightarrow K_2 \longrightarrow 0,$$
$$0 \longrightarrow K_2 \longrightarrow \operatorname{Sym}^2 E \xrightarrow{\mathrm{res}_2} i_*\mathcal{O}_C \longrightarrow 0.$$

After twisting by $\mathcal{O}(-t)$, the residue sequence tensored with A and the Euler sequence give $H^0(A \otimes E(-t)) = 0$. Moreover $H^1(K_{\mathbb{P}^2}(-t)) = H^1(\mathcal{O}_{\mathbb{P}^2}(-3-t)) = 0$. The two displayed sequences therefore give $H^0(K_2(-t)) = 0$ and then $H^0(\operatorname{Sym}^2 E(-t)) = 0$.

Finally, the maximal-slope factor used below has integral twist $t > 0$. Weights one and two have just been excluded, hence $m \geqslant 3$. □

Let $B_d^{\mathrm{sm}} \subset \mathbb{P}(V_d)$ be the open parametrizing smooth degree-d surfaces, let

$$\rho\colon \mathcal{X} \longrightarrow B_d^{\mathrm{sm}}$$

be the universal surface, and write

$$\rho_2\colon \mathcal{X}_2 \longrightarrow B_d^{\mathrm{sm}}, \qquad \pi_{2,0}\colon \mathcal{X}_2 \longrightarrow \mathcal{X}$$

for its relative second Demailly–Semple stage and its projection. Its relative stationary divisor is a smooth projective relative prime effective Cartier divisor with integral fibres; it is denoted by $\Gamma_2 \subset \mathcal{X}_2$. If $h = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathcal{X}}$, put

$$\mathscr{L}_{m,t} = \mathcal{O}_{\mathcal{X}_2}(m) \otimes \pi_{2,0}^* h^{-t}.$$

For an open $U \subset B_d^{\mathrm{sm}}$ we use $\mathcal{X}_U = \rho^{-1}(U)$ and $\mathcal{X}_{2,U} = \rho_2^{-1}(U)$.

Proposition 2.12 (Compact relative irreducible jet-divisor family)**.** *Let $d \geqslant 15$, let $a_0 \in B_d^{\mathrm{sm}}$ be very general, and suppose that*

$$0 \neq \omega_0 \in H^0\Big(X_{a_0}, E_{2,M}T^*_{X_{a_0}} \otimes \mathcal{O}_{X_{a_0}}(-T)\Big), \qquad M, T \geqslant 1.$$

After selecting a nonstationary irreducible factor ω_{1,a_0} of ω_0, write s_Γ for the canonical section of $\mathcal{O}(\Gamma_2)$. There are integers $m \geqslant 3$, $t \geqslant 1$ and $b_0 \geqslant 0$, with $t/m \geqslant T/M$, a dense Zariski open $a_0 \in U \subset B_d^{\mathrm{sm}}$, and a regular residual section

$$\bar\omega \in H^0\big(\mathcal{X}_{2,U}, \mathscr{L}_{m,t}(-b_0\Gamma_2)|_{\mathcal{X}_{2,U}}\big)$$

whose central restriction is

$$\bar\omega_{a_0} = s_{\Gamma,a_0}^{-b_0}\omega_{1,a_0},$$

and such that, on every fibre,

$$\bar\omega_a \neq 0, \qquad \bar\omega_a|_{\Gamma_{a,2}} \neq 0.$$

Set

$$\omega_1 = s_\Gamma^{b_0}\bar\omega \in H^0\big(\mathcal{X}_{2,U}, \mathscr{L}_{m,t}|_{\mathcal{X}_{2,U}}\big), \qquad \mathscr{Z} = Z(\bar\omega).$$

Then ω_{1,a_0} is the selected factor, $\mathscr{Z}$ is an effective Cartier divisor, and

$$\operatorname{div}(\omega_1) = \mathscr{Z} + b_0\Gamma_2, \tag{2.4}$$

and the morphism

$$\mathscr{Z} \longrightarrow U$$

is projective and flat with geometrically integral fibres. In particular, $\mathscr{Z}_a = Z(\bar\omega_a)$ is precisely the irreducible and reduced stationary-saturated zero divisor of $\omega_{1,a}$.

Proof. Decompose the Cartier divisor of ω_0 on $X_{a_0,2}$ as in Definition 2.7. The standard Demailly–El Goul reduction selects a prime component different from $\Gamma_{a_0,2}$ and its reduced invariant factor ω_{1,a_0}, of weight m and twist $-t$. Additivity of weight and base twist under multiplication permits the usual maximal-slope choice, for which

$$\frac{t}{m} \geqslant \frac{T}{M}.$$

The resulting residual zero divisor is irreducible and reduced. This is precisely the compact factor construction in [13, Proposition 5.4]; in particular, it is a factor on the Semple tower rather than in a localised affine jet chart.

Let s_{Γ,a_0} be the canonical section of the stationary divisor on the chosen fibre, and define its full fibrewise stationary order and residual section by

$$b_0 = \mathrm{ord}_{\Gamma_{a_0,2}}(\omega_{1,a_0}), \qquad \bar{\omega}_{a_0} = s_{\Gamma,a_0}^{-b_0}\omega_{1,a_0}.$$

Thus $\bar{\omega}_{a_0}$ is a regular section of $\mathscr{L}_{m,t}(-b_0\Gamma_2)|_{X_{a_0,2}}$, its restriction to $\Gamma_{a_0,2}$ is nonzero, and its zero scheme is exactly the selected reduced prime divisor.

The universal smooth surface is projective and smooth over B_d^{sm}, and its relative Semple tower is obtained by two projective bundle constructions. Consequently $\mathcal{X}_2 \to B_d^{\mathrm{sm}}$ is projective, smooth and of finite presentation, with geometrically integral fibres. In particular, all ambient hypotheses of Lemma 2.9 hold after every base restriction below.

Apply Theorem 2.8 to this relative Semple tower and the residual line bundle $\mathscr{L}_{m,t}(-b_0\Gamma_2)$. Since a_0 is very general, it lies in the generic base-change locus for the resulting triple (m,t,b_0): there are only countably many such triples, so all their generic loci may be included in the very-general choice. After shrinking to a Zariski open neighbourhood U of a_0, $\bar{\omega}_{a_0}$ therefore extends to a regular relative section $\bar{\omega}$.

The stationary divisor is a smooth projective family with integral fibres. The zero scheme of $\bar{\omega}|_{\Gamma_2}$ has fibre dimension strictly smaller than $\dim \Gamma_{a_0,2}$ at a_0. Upper semicontinuity of fibre dimension permits a further shrinking such that

$$\bar{\omega}_a|_{\Gamma_{a,2}} \neq 0 \quad \text{for every } a \in U.$$

In particular $\bar{\omega}_a \neq 0$ on every fibre. Define $\omega_1 = s_\Gamma^{b_0}\bar{\omega}$ and $\mathscr{Z} = Z(\bar{\omega})$. The first assertion of Lemma 2.9, applied to $\bar{\omega}$, shows that $\mathscr{Z} \to U$ is projective and flat. Its final assertion, applied to $\omega_1 = s_\Gamma^{b_0}\bar{\omega}$, shows that no $\Gamma_{a,2}$ is a component of $\mathscr{Z}_a$ and that (2.4) has full fibrewise stationary order b_0 on every fibre.

By construction, the scheme-theoretic fibre $\mathscr{Z}_{a_0} = Z(\bar{\omega}_{a_0})$ is the reduced prime divisor selected above; over $\mathbb{C}$ this is geometric integrality. The geometric-integrality clause of Lemma 2.9 allows one last shrinking around a_0 so that every $\mathscr{Z}_a$ is geometrically integral. This is the relative divisor $Z + b_0\Gamma$ of [13, §6.2], now with the factorisation locus and every flatness hypothesis explicit. ☐

On the regular vertical two-jet space, the section ω_1 is equivalently a regular, reparametrization-equivariant weighted-homogeneous polynomial map of weight m,

$$P\colon J_2^v(\mathcal{X}_U) \longrightarrow p^*\,\mathrm{pr}_1^*\,\mathcal{O}_{\mathbb{P}^3}(-t),$$

where $p\colon J_2^v(\mathcal{X}_U) \to \mathcal{X}_U$ is the natural projection. The natural lift from the regular vertical two-jet space lands in $\mathcal{X}_{2,U}\backslash\Gamma_2$. Hence the pullback of s_Γ is nowhere vanishing and, after a local trivialisation of $\mathcal{O}(\Gamma_2)$, is represented by a unit. Consequently P generates the same ideal as the restriction of $\bar{\omega}$ and hence defines the stationary-saturated residual divisor. If $\bar{P}$ represents $\bar{\omega}$ locally, then $P = g\bar{P}$, where g represents $s_\Gamma^{b_0}$ and is a unit, and $V(P) = gV(\bar{P}) + V(g)\bar{P}$. Thus reconstruction does not change divisibility modulo the residual ideal, and P_a is a genuine holomorphic family of the type required in Lemma 2.16 and Theorem 2.27; its weight is recorded by the reparametrization character, or equivalently by the factor $\mathcal{O}_{\mathcal{X}_{2,U}}(m)$ above.

Corollary 2.13 (Compact generic-fibre prime alternative)**.** *In the situation of Proposition 2.12, after possibly shrinking U, one of the following holds.*

(1) *The image $\pi_{2,0}(\mathscr{Z})$ is a proper algebraic subset of $\mathcal{X}_U$; then an entire curve whose nonstationary second lift lies in $\mathscr{Z}_a$ is already algebraically degenerate in X_a.*

(2) *The divisor $\mathscr{Z}$ dominates $\mathcal{X}_U$. If K is the function field of the universal compact incidence and a rational tangent frame is chosen at its generic point, then the restriction of P gives a weighted-homogeneous polynomial*

$$0 \neq P_K \in K[x, y, W]$$

for which (P_K) is a prime ideal in the whole unlocalised invariant ring.

The second conclusion remains valid after replacing the incidence space by any smaller dominant incidence open, in particular the simultaneous open of Lemma 2.26.

Proof. If the projection is not dominant, the image of any curve constrained by $\mathscr{Z}_a$ lies in its projection to X_a. Properness of $\pi_{2,0}$ and upper semicontinuity of fibre dimension allow us to shrink U so that this projection is proper for every remaining a; this is the first alternative.

Assume instead that $\mathscr{Z}$ dominates $\mathcal{X}_U$. Since U is integral and $\mathscr{Z} \to U$ is flat with geometrically integral fibres, $\mathscr{Z}$ is integral. Localising its coordinate ring at the nonzero functions of the integral base $\mathcal{X}_U$ shows that the generic fibre of

$$\mathscr{Z} \longrightarrow \mathcal{X}_U$$

is integral as well.

Let K be the function field of the universal compact incidence. A rational tangent frame and trivialisations of the tautological and base line bundles turn the restriction of ω_1 into a weighted-homogeneous element

$$P_K \in R := K[x, y, W].$$

Via the nowhere-vanishing pullback of $s_\Gamma^{b_0}$ on the regular jet locus, P_K is also a local generator of the ideal defined by the stationary-saturated section $\bar{\omega}$. Lemma 2.1 identifies R before any localisation in x, y or W. Scheme-theoretically, the generic residual divisor on the regular jet locus is $V(P_K)$ and is integral and reduced.

For completeness, suppose $P_K = AB$ with A, B of positive weighted degree. The factors can be chosen weighted homogeneous. Each defines a codimension-one subset of $\operatorname{Spec} R$. The omitted locus $x = y = 0$ has codimension two, so neither factor can disappear when one passes to the regular Semple locus. Clearing only coefficient-field denominators would therefore split the integral generic residual divisor, a contradiction. Thus P_K is irreducible. Reducedness of that divisor says equivalently that P_K is square-free. Finally, R is a unique factorisation domain, so the irreducible element P_K is prime. Hence (P_K) is a height-one prime ideal in the whole unlocalised ring. Passing to a smaller dominant incidence open changes neither K nor this conclusion. □

2.2.2. *Relative logarithmic jet-divisor families.* We now construct the logarithmic analogue of Proposition 2.12. Besides extending the factor itself, we remove its full stationary multiplicity before passing to the incidence function field. This is the step which turns the phrase "irreducible and reduced modulo $\Gamma_{a,2}$" into a prime ideal in the whole, unlocalised invariant fibre ring. The relative divisor construction is the logarithmic counterpart of the $Z + b\Gamma$ construction in [13, §6.2].

Put

$$V_d^{(2)} = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)), \qquad B_d^{\log,\mathrm{sm}} \subset \mathbb{P}(V_d^{(2)})$$

for the open parametrizing smooth plane curves, and let

$$\mathscr{Q} = \mathbb{P}^2 \times B_d^{\log,\mathrm{sm}}, \qquad \mathscr{C} = \{(z, a) : f_a(z) = 0\} \subset \mathscr{Q}.$$

The relative logarithmic tangent bundle

$$\mathscr{V}_0 = T_{\mathscr{Q}/B_d^{\log,\mathrm{sm}}}(-\log \mathscr{C})$$

is locally free. Its two-stage relative logarithmic Demailly–Semple tower will be denoted by

$$\rho_2^{\log} \colon \mathscr{P}_2^{\log} \longrightarrow B_d^{\log,\mathrm{sm}}, \qquad \pi_{2,0}^{\log} \colon \mathscr{P}_2^{\log} \longrightarrow \mathscr{Q}.$$

Write

$$\mathscr{D}_2^{\log} = (\pi_{2,0}^{\log})^{-1}(\mathscr{C})$$

for the pulled-back logarithmic boundary and $\Gamma_2^{\log} \subset \mathscr{P}_2^{\log}$ for the smooth projective relative prime effective Cartier stationary divisor, whose fibres are integral. With $h_{\log} = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^2}(1)$, set

$$\mathscr{L}_{m,t}^{\log} = \mathcal{O}_{\mathscr{P}_2^{\log}}(m) \otimes (\pi_{2,0}^{\log})^* h_{\log}^{-t}.$$

For $U \subset B_d^{\log,\mathrm{sm}}$, a subscript U denotes base change to U.

Proposition 2.14 (Logarithmic relative irreducible jet-divisor family)**.** *Let $d \geqslant 11$. Fix one pair (m,t) with $m \geqslant 3$, $t \geqslant 1$, and a nonstationary irreducible factor*

$$0 \neq \eta_{1,a_0} \in H^0\big(\mathbb{P}^2, E_{2,m} T_{\mathbb{P}^2}^*(\log C_{a_0}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)\big)$$

occurring in the construction of [13, Proposition 5.5]*, and write*

$$b_0 = \mathrm{ord}_{\Gamma_{a_0,2}^{\log}}(\eta_{1,a_0}).$$

Assume that a_0 lies in the dense cohomology-and-base-change locus for the residual line bundle $\mathscr{L}_{m,t}^{\log}(-b_0\Gamma_2^{\log})$. Write $s_\Gamma^{\log}$ for the canonical stationary section. There are a Zariski open neighbourhood $a_0 \in U \subset B_d^{\log,\mathrm{sm}}$ and a regular residual section

$$\bar{\eta} \in H^0\left(\mathscr{P}_{2,U}^{\log}, \mathscr{L}_{m,t}^{\log}(-b_0\Gamma_2^{\log})|_{\mathscr{P}_{2,U}^{\log}}\right)$$

whose central restriction is

$$\bar{\eta}_{a_0} = (s_{\Gamma,a_0}^{\log})^{-b_0} \eta_{1,a_0},$$

and such that

$$\bar{\eta}_a \neq 0, \qquad \bar{\eta}_a|_{\Gamma_{a,2}^{\log}} \neq 0 \quad \textit{for every } a \in U.$$

Set

$$\eta_1 = (s_\Gamma^{\log})^{b_0} \bar{\eta}, \qquad \mathscr{Z}^{\log} = Z(\bar{\eta}).$$

Then η_{1,a_0} is the prescribed factor and $\mathscr{Z}^{\log}$ is an effective Cartier divisor. Moreover,

$$\mathrm{div}(\eta_1) = \mathscr{Z}^{\log} + b_0\Gamma_2^{\log}, \tag{2.5}$$

and the morphism

$$\mathscr{Z}^{\log} \longrightarrow U$$

is projective and flat with geometrically integral fibres. Thus $\mathscr{Z}_a^{\log} = Z(\bar{\eta}_a)$ is exactly the irreducible and reduced stationary-saturated zero divisor of $\eta_{1,a}$.

If the factor is extracted from a section of weight M and twist $-T$, the usual maximal-slope choice may be made so that $t/m \geqslant T/M$. For any fixed finite collection of residual factor types (m,t,b_0), all conclusions hold on the intersection of finitely many nonempty Zariski opens.

Proof. Since $\mathscr{C} \to B_d^{\log,\mathrm{sm}}$ is a smooth relative divisor, $\mathscr{V}_0$ is a vector bundle and the relative logarithmic Semple tower is an iterated projective bundle. In particular, $\rho_2^{\log}$ is projective, smooth and of finite presentation, with geometrically integral fibres. On every fibre the logarithmic direct-image formula gives

$$(\pi_{2,0}^{\log})_* \mathcal{O}_{\mathscr{P}_{2,a}^{\log}}(m) = E_{2,m} T_{\mathbb{P}^2}^*(\log C_a).$$

Proposition 5.5 of [13] supplies, for a fixed pair (m,t), a nonempty parameter open on which the factor selected in the sense of Definition 2.7 has an irreducible and reduced zero divisor modulo $\Gamma_{a,2}^{\log}$. Thus the factor is first chosen as a prime Cartier component on the logarithmic Semple tower; its weighted-polynomial representative is taken only afterwards. Let $s_{\Gamma,a_0}^{\log}$ be the stationary section on the chosen fibre and remove its full fibrewise order before extending:

$$b_0 = \mathrm{ord}_{\Gamma_{a_0,2}^{\log}}(\eta_{1,a_0}), \qquad \bar{\eta}_{a_0} = (s_{\Gamma,a_0}^{\log})^{-b_0} \eta_{1,a_0}.$$

This is a regular section of $\mathscr{L}_{m,t}^{\log}(-b_0\Gamma_2^{\log})$ on the central fibre, its restriction to $\Gamma_{a_0,2}^{\log}$ is nonzero, and its zero scheme is the selected reduced prime divisor.

By the base-change assumption, Theorem 2.8 extends $\bar{\eta}_{a_0}$ to a regular relative section $\bar{\eta}$ after an algebraic shrinking. The relative stationary divisor is a smooth projective family with integral fibres.

At a_0, the fibre dimension of $Z(\bar\eta|_{\Gamma_2^{\log}})$ is strictly smaller than $\dim \Gamma^{\log}_{a_0,2}$. Upper semicontinuity of fibre dimension therefore permits a further shrinking such that

$$\bar\eta_a|_{\Gamma^{\log}_{a,2}} \neq 0 \quad \text{for every } a \in U.$$

Thus every $\bar\eta_a$ is nonzero. Define

$$\eta_1 = (s_\Gamma^{\log})^{b_0}\bar\eta, \qquad \mathscr{Z}^{\log} = Z(\bar\eta).$$

The first assertion of Lemma 2.9, applied to $\bar\eta$, shows that $\mathscr{Z}^{\log} \to U$ is projective and flat. Its final assertion, applied to $\eta_1 = (s_\Gamma^{\log})^{b_0}\bar\eta$, excludes a stationary component on every fibre and gives (2.5) with full fibrewise order b_0.

The scheme-theoretic fibre $\mathscr{Z}^{\log}_{a_0} = Z(\bar\eta_{a_0})$ is exactly the selected reduced prime divisor, hence geometrically integral over $\mathbb{C}$. The geometric-integrality clause of Lemma 2.9 permits one last shrinking so that every fibre is geometrically integral. The slope assertion is the same maximal-slope factor choice used in [13, Proposition 5.5]. Because only a fixed residual type, or later a finite list of types, is involved, no complement of a countable union is used here. □

Set

$$\mathscr{J}^{\log}_{2,U} := J_2^v(\mathscr{Q}_U, -\log \mathscr{C}_U).$$

The regular logarithmic jet lift avoids $\Gamma_2^{\log}$. Hence $(s_\Gamma^{\log})^{b_0}$ canonically trivialises the pullback of $\mathcal{O}(b_0\Gamma_2^{\log})$ there. Via this identification, let $P^{\log}$ denote the regular polynomial representative of the residual generator $\bar\eta$; equivalently, it represents the reconstructed section η_1 under the same identification. It is equivariant under reparametrization and defines a map

$$P^{\log}\colon \mathscr{J}^{\log}_{2,U} \longrightarrow p^*h_{\log}^{-t}$$

of weight m. Its principal ideal is the restriction of the saturated residual ideal. Thus the Semple-tower saturation will be performed before, and independently of, the unlocalised polynomial-ring argument below.

Lemma 2.15 (Logarithmic stationary saturation, boundary separation and primality)**.** *In the situation of Proposition 2.14, let $\mathscr{I}_{\eta_1}$ be the ideal sheaf of $\operatorname{div}(\eta_1)$ and let $\mathscr{I}_\Gamma$ be the ideal sheaf of $\Gamma_2^{\log}$. Then*

$$\mathscr{I}_{\eta_1} : \mathscr{I}_\Gamma^\infty = \mathscr{I}_{\mathscr{Z}^{\log}} = (\bar\eta). \tag{2.6}$$

Here $(\bar\eta)$ denotes the image ideal obtained from the residual section after a local trivialisation of its line bundle; this notation is independent of the chosen trivialisation. After possibly shrinking U, exactly one of the following alternatives holds.

1. *The image $\pi_{2,0}^{\log}(\mathscr{Z}^{\log})$ is a proper algebraic subset of $\mathscr{Q}_U$. Then an entire curve in $\mathbb{P}^2 \setminus C_a$ whose nonstationary logarithmic second lift is contained in $\mathscr{Z}_a^{\log}$ is algebraically degenerate.*
2. *The divisor $\mathscr{Z}^{\log}$ dominates $\mathscr{Q}_U$. Let $\Omega_U^{\log}$ be the restriction to U of the dominant simultaneous interior in Lemma 2.26, and let*

   $$L = \mathbb{C}(\Omega_U^{\log}) = \mathbb{C}(\mathscr{Q}_U).$$

   After choosing a rational logarithmic tangent frame at its generic point, the polynomial map $P^{\log}$, or equivalently the restriction of the saturated residual ideal, gives a weighted-homogeneous polynomial

   $$0 \neq P_L^{\log} \in L[u,v,W], \qquad \mathrm{wt}(u) = \mathrm{wt}(v) = 1, \quad \mathrm{wt}(W) = 3,$$

   and $(P_L^{\log})$ is a prime ideal in the whole polynomial ring $L[u,v,W]$.

In the second alternative, let V be a regular relative logarithmic common-twist field over a dense open of U. If $V(P^{\log})$ vanishes at the generic point of the relative residual divisor $\mathscr{Z}^{\log}$, then, after passage to L,

$$V(P_L^{\log}) \in (P_L^{\log}) \quad \textit{in } L[u,v,W]. \tag{2.7}$$

Only coefficient functions on the incidence base are inverted in this statement; no element of positive degree in $L[u,v,W]$, and in particular neither W nor a first-jet variable, is inverted. Moreover, *in the dominant alternative the integral residual divisor has no component supported in the pulled-back logarithmic boundary* $\mathscr{D}_2^{\log}$. *Restriction to* $f \neq 0$ *therefore loses no height-one component of its defining ideal.*

Proof. Equation (2.5) gives

$$\mathscr{I}_{\eta_1} = \mathscr{I}_{\mathscr{Z}^{\log}} \mathscr{I}_{\Gamma}^{b_0}.$$

We verify the saturation stalkwise. The logarithmic Semple tower is smooth, so its local rings are regular domains. At a stalk, trivialise all relevant line bundles, write $\mathscr{I}_{\Gamma} = (g)$ and $\mathscr{I}_{\mathscr{Z}^{\log}} = (h)$, and represent $\bar{\eta}$ by h and η_1 by $g^{b_0}h$. The residual divisor is integral and has no component equal to $\Gamma_2^{\log}$. Consequently (h) is prime at every stalk along the residual divisor and $g \notin (h)$; multiplication by every power of g is injective in the domain $\mathcal{O}/(h)$. At stalks away from the residual divisor the calculation is trivial. Thus

$$(g^{b_0}h) : (g^n) = \begin{cases} (g^{b_0-n}h), & 0 \leqslant n \leqslant b_0, \\ (h), & n \geqslant b_0. \end{cases}$$

Taking the increasing union over n removes exactly the full stationary multiplicity b_0 and nothing from the residual divisor. This proves (2.6), including its scheme-theoretic multiplicity claim.

The morphism $\pi_{2,0}^{\log}$ is proper. If the image of $\mathscr{Z}^{\log}$ is not dense, it is closed and proper; the base projection of any lifted curve contained in this divisor lies in that proper subset. Upper semicontinuity of fibre dimension allows us to shrink U so that the same conclusion holds fibrewise. This proves the first alternative.

Assume now that $\mathscr{Z}^{\log}$ dominates $\mathscr{Q}_U$. Since U is integral and $\mathscr{Z}^{\log} \to U$ is flat with geometrically integral fibres, $\mathscr{Z}^{\log}$ is integral. A dominant morphism of integral schemes has an integral generic fibre: on affine charts its coordinate ring is obtained from a domain by localising at the nonzero elements of the base. Consequently the generic fibre of

$$\mathscr{Z}^{\log} \longrightarrow \mathscr{Q}_U$$

is integral. It meets the regular logarithmic jet locus, since the removed stationary divisor cannot contain the dominant residual divisor.

The pulled-back boundary $\mathscr{D}_2^{\log}$ maps into $\mathscr{C}_U \subsetneq \mathscr{Q}_U$. Since $\mathscr{Z}^{\log}$ is integral and dominates $\mathscr{Q}_U$, it is not contained in that boundary and cannot have a separate boundary component. Thus passage to the interior $f \neq 0$ preserves its generic height-one divisor. The further equations $p = 0$ and $\det \mathcal{H} = 0$ also define proper subsets of the incidence base; their inverses are coefficient-field elements and not fibre-variable localisations.

On $\Omega_U^{\log}$ the base boundary equation f, the chart coefficient $p = f_1$, and $\det \mathcal{H}$ are nonzero. Their inverses are therefore elements of the coefficient field L. A rational logarithmic tangent frame and a trivialisation of the residual line bundle turn the generator $\bar{\eta}$, equivalently $P^{\log}$ on the regular jet locus, into $P_L^{\log}$. Equation (2.6) says that its zero divisor on the regular locus is the restriction of $\mathscr{Z}^{\log}$. By Lemma 2.1, before any localisation in fibre variables the invariant ring is exactly $L[u, v, W]$.

The locus $u = v = 0$, on which the passage from affine jets to the regular Semple chart is not defined, has codimension two in $\operatorname{Spec} L[u, v, W]$. Hence no divisor, and in particular no positive-degree polynomial factor, can be supported entirely in that omitted locus.

If $P_L^{\log} = AB$ with A and B of positive weighted degree, the factors may be chosen weighted homogeneous. Clearing only their base denominators would split the integral generic fibre into two effective divisors. No factor can disappear into $\Gamma_2^{\log}$, by the saturation identity, and no factor can disappear into $\mathscr{D}_2^{\log}$, because we are in the dominant interior alternative. This is impossible. Hence $P_L^{\log}$ is irreducible. Its divisor is reduced, so there is no repeated factor. Because $L[u, v, W]$ is a unique factorisation domain, $(P_L^{\log})$ is a height-one prime; equivalently, $P_L^{\log}$ is irreducible and square-free in the whole unlocalised ring.

Finally compare the residual generator with the reconstructed section. After choosing compatible local trivialisations on the regular jet locus, the pullback of $s_\Gamma^{\log}$ is represented by a unit and the product rule gives

$$V(\eta_1) = (s_\Gamma^{\log})^{b_0} V(\bar{\eta}) + b_0 (s_\Gamma^{\log})^{b_0-1} V(s_\Gamma^{\log}) \bar{\eta}.$$

Consequently

$$(s_{\Gamma}^{\log})^{-b_0} V(\eta_1) \equiv V(\bar{\eta}) \pmod{(\bar{\eta})},$$

so differentiating the reconstructed section gives exactly the same ideal-membership test as differentiating the residual generator. Replacing a local generator P by $P' = gP$, with g a unit from a different line-bundle trivialisation, gives $V(P') = gV(P) + V(g)P$. Hence the ideal-membership assertion is independent of every chosen trivialisation.

Under the relative-containment hypothesis in the statement, $V(P_L^{\log})$ vanishes at the generic point of the integral residual divisor. It therefore lies in the height-one prime generated by $P_L^{\log}$. Both polynomials belong to the unlocalised ring, so the membership holds in $L[u, v, W]$ itself and gives (2.7). □

2.2.3. *Prime-ideal preservation.*

Lemma 2.16 (Dependence implies prime-ideal preservation)**.** *Let P_a be a holomorphic family of weighted-homogeneous two-jet polynomials on a Zariski open parameter set. After removing the full stationary multiplicity, assume that at the generic incidence point the residual equation P generates a height-one prime ideal in the whole unlocalised invariant ring*

$$R = K[u, v, W].$$

Let V be a regular common-twist slanted field for which $V(P)$ belongs to R. If the residual divisor $Y = \{P = 0\}$ is a component of the zero divisor of the derivative, then

$$V(P) \in (P)$$

in R. Consequently, if every available derivative contains Y as a divisorial component, the prime ideal (P) is stable under every field in the package.

Proof. At the generic point of Y, vanishing of the derivative gives

$$V(P) \in (P)R_{(P)}.$$

Thus there is an element $s \in R \setminus (P)$ such that $sV(P) \in (P)$. Since (P) is prime and $s \notin (P)$, it follows directly that $V(P) \in (P)$ in R itself. This proves membership without inverting a first-jet variable or the Wronskian.

The assertion is also independent of a local line-bundle trivialisation. Indeed, another local generator has the form $P' = gP$ for a unit g, and

$$V(P') = gV(P) + V(g)P.$$

Hence $V(P') \in (P')$ if and only if $V(P) \in (P)$. Applying the argument to each field gives the final assertion. □

Remark 2.17 (What the acceleration operator actually proves)**.** The distinction between pointwise evaluation and ideal preservation is essential. If one merely evaluates a field at a fixed jet point, the resulting equality can at most say that the specialised coefficients of the Wronskian terms vanish there. Lemma 2.2 applies only after Lemmas 2.5 and 2.16 have upgraded the dependence assumption to the polynomial identity

$$S_3(x, y)\partial_W P \in (P)$$

over the incidence function field and on the whole unlocalised fibre ring. At that stage it proves $P \in K[x, y]$, so P is genuinely a first-jet differential at the generic incidence point, not merely Wronskian-free at one fixed point. It does not yet prove $P = 0$: the semisimple matrix operator of Lemma 2.3 is the second, indispensable step which rules out an irreducible binary form of degree $m \geqslant 3$.

When P_a has base twist $-t$, an $\mathcal{O}(3)$ field produces a section of base twist $-(t-3)$. Thus for $t \geqslant 4$ both P_a and $V(P_a)$ remain negatively twisted and are annihilated along every entire curve by the fundamental vanishing theorem. This is the precise numerical point at which the low-pole construction enters the hyperbolicity argument.

Lemma 2.18 (Induced action on the Wronskian)**.** *Let $x = (u, v)^\mathsf{T}$ be a first jet, let $a = (X, Y)^\mathsf{T}$ be a second jet, and put $W = \det(x, a)$. Suppose a reparametrization-equivariant derivation has*

$$\delta x = Tx, \qquad \delta a = Ta + B(x, x),$$

where T is a 2×2 matrix and $B : \mathrm{Sym}^2 K^2 \to K^2$ is symmetric bilinear. Then

$$\delta W = (\operatorname{tr} T)W + R_3(u, v), \qquad R_3(u, v) = \det\bigl(x, B(x, x)\bigr). \tag{2.8}$$

In particular, a traceless matrix direction may add a nonzero binary cubic; the abstract two-operator argument does not require this cubic to vanish.

Proof. Differentiate the determinant and use

$$\det(Tx, a) + \det(x, Ta) = (\operatorname{tr} T)\det(x, a).$$

The remaining term $\det(x, B(x, x))$ is homogeneous of degree three. □

Lemma 2.19 (Second Semple boundary chart)**.** *Let a reparametrization-equivariant derivation act on two-dimensional jet coordinates (x, y, X, Y), and put $W = xY - yX$. Suppose*

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix}, \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \delta W = (\operatorname{tr} T)W + S_3(x, y),$$

where S_3 is a binary cubic; the case $S_3 = 0$ is allowed. On the first Semple chart $x \neq 0$, set

$$p = \frac{y}{x}, \qquad q = \frac{W}{x^3}.$$

Then

$$\begin{aligned} \delta p &= c + (d - a)p - bp^2, \\ \delta q &= S_3(1, p) + (d - 2a - 3bp)q. \end{aligned}$$

On the chart at the second-stage infinity, $r = q^{-1}$, one has

$$\delta r = -(d - 2a - 3bp)r - S_3(1, p)r^2.$$

In particular the field is holomorphic along $r = 0$. The analogous formulas on $y \neq 0$, obtained from

$$\widetilde{p} = \frac{x}{y}, \qquad \widetilde{q} = -\frac{W}{y^3},$$

give the same conclusion on the other first-stage chart.

Proof. The first identity follows from

$$\delta p = \frac{x\delta y - y\delta x}{x^2}.$$

Using $\delta x/x = a + bp$ and the homogeneity of S_3, direct differentiation gives

$$\delta q = \frac{\delta W}{x^3} - 3\frac{W}{x^3}\frac{\delta x}{x} = S_3(1, p) + (a + d - 3a - 3bp)q,$$

which is the asserted formula. Since $\delta r = -q^{-2}\delta q$, the formula for r follows. The y-chart calculation is identical. Thus an affine term in q becomes a quadratic term in r, while a linear term in q remains linear in r; neither creates a pole at the second Semple boundary. □

2.3. **Compact case.** On an affine chart of the universal hypersurface write

$$F(z, a) = \sum_{|\alpha| \leqslant d} a_\alpha z^\alpha.$$

The vertical two-jet equations are

$$\begin{aligned} E_0 &= F, \\ E_1 &= dF(\xi^{(1)}), \\ E_2 &= dF(\xi^{(2)}) + d^2F(\xi^{(1)}, \xi^{(1)}). \end{aligned}$$

We recall the compact matrix-lift package of Păun [20, Lemma 1.1]. The uniform formula below is an explicit interpolation presentation of that classical lemma and will be used only to select its required low-coefficient direction.

Proposition 2.20 (Păun's compact matrix-lift interpolation)**.** *For every constant matrix $A \in M_3(\mathbb{C})$, put*

$$B_A(Y,z) = A(Y-z), \qquad G_A(Y;z,a) = -dF_Y\big(B_A(Y,z)\big).$$

Expand $G_A = \sum_\alpha g_\alpha(z,a)Y^\alpha$ and define

$$V_A = \sum_\alpha g_\alpha(z,a)\partial_{a_\alpha} + \sum_{r=1}^{2}(A\xi^{(r)})\cdot\partial_{\xi^{(r)}}.$$

Then V_A is tangent to $E_0 = E_1 = E_2 = 0$. Its parameter coefficients are linear in a and have z-degree at most 1, and its projective prolongation has pole order at most 3. In particular it defines a reparametrization-equivariant, parameter-untwisted slanted field with base twist $\mathcal{O}_{\mathbb{P}^3}(3)$ and extends holomorphically over the second Semple boundary.

Proof. The polynomial G_A has Y-degree at most d. At $Y = z$ its Taylor data through order two are

$$\begin{aligned} G_A(z) &= 0,\\ dG_A(z)(u) &= -dF_z(Au),\\ d^2G_A(z)(u,v) &= -d^2F_z(u,Av) - d^2F_z(v,Au). \end{aligned}$$

Consequently

$$V_A(E_0) = 0, \qquad V_A(E_1) = -dF(A\xi^{(1)}) + dF(A\xi^{(1)}) = 0,$$

and

$$\begin{aligned} V_A(E_2) = &-dF(A\xi^{(2)}) - 2d^2F(\xi^{(1)},A\xi^{(1)})\\ &+ dF(A\xi^{(2)}) + 2d^2F(\xi^{(1)},A\xi^{(1)}) = 0. \end{aligned}$$

This proves tangency without deleting any low multi-index block: missing shifted monomials simply occur with coefficient zero in the expansion of G_A. Applying A simultaneously to the first and second jets commutes with the triangular action of the two-jet reparametrization group, proving equivariance.

For completeness, let $\widehat{z} = \phi(z)$ be a projective chart change. The prolonged jet coordinates satisfy

$$\widehat{\xi}^{(1)} = D\phi\,\xi^{(1)}, \qquad \widehat{\xi}^{(2)} = D\phi\,\xi^{(2)} + D^2\phi(\xi^{(1)},\xi^{(1)}).$$

Since the base point is fixed by V_A, their variations are

$$\begin{aligned} \delta\widehat{\xi}^{(1)} &= D\phi\,A\xi^{(1)},\\ \delta\widehat{\xi}^{(2)} &= D\phi\,A\xi^{(2)} + 2D^2\phi(\xi^{(1)},A\xi^{(1)}). \end{aligned}$$

For a projective transition $\phi_i = z_i/z_\nu$, the entries of $D\phi$ and $D^2\phi$ have denominators of order at most 2 and 3, respectively. The parameter correction has z-degree at most 1. Thus no component has pole order exceeding 3.

It remains to check the second Semple boundary rather than only the affine jet charts. Choose a rational tangent frame and eliminate the normal jet. The prolonged action has

$$\delta(x,y) = T(x,y), \qquad \delta(X,Y) = T(X,Y) + B\big((x,y),(x,y)\big)$$

for a symmetric bilinear map B; its quadratic term comes from the Hessian of the frame change and from normal elimination. Lemma 2.18 therefore gives

$$\delta(x,y) = T(x,y), \qquad \delta W = (\operatorname{tr}T)W + S_3(x,y)$$

for a binary cubic S_3. Lemma 2.19 gives

$$\delta q = S_3(1,p) + c_1(p)q, \qquad \delta r = -c_1(p)r - S_3(1,p)r^2.$$

Both expressions are regular at $q < \infty$ and at $r = 0$. This completes the projective and Semple-stage extension check. □

Let $C : \mathrm{Sym}^2\mathbb{C}^3 \to \mathbb{C}^3$ be constant and prescribe

$$\delta\xi^{(1)} = 0, \qquad \delta\xi^{(2)} = C(\xi^{(1)}, \xi^{(1)}).$$

Its parameter correction starts from

$$Q_0(Y; z, a) = -\frac{1}{2}\sum_{r,j,k} C^r_{jk} F_r(Y, a)(Y_j - z_j)(Y_k - z_k).$$

At $Y = z$ it has value and gradient zero and Hessian

$$d^2 Q_0 = -dF \circ C.$$

Although Q_0 can have degree $d+1$ in Y, write its top part as $\sum_{|\alpha|=d+1} h_\alpha Y^\alpha$, choose $\gamma(\alpha) \leqslant \alpha$ with $|\gamma(\alpha)| = 3$, and subtract

$$\sum_{|\alpha|=d+1} h_\alpha (Y - z)^{\gamma(\alpha)} Y^{\alpha - \gamma(\alpha)}.$$

The subtracted polynomial belongs to $\mathfrak{m}_z^3$, has the same top homogeneous part, and does not change the Taylor two-jet. We obtain $Q_C = \sum q_\alpha(z, a) Y^\alpha$ of degree at most d, linear in a, with $\deg_z q_\alpha \leqslant 3$.

Proposition 2.21 (Compact quadratic acceleration). *The vector field*

$$V_C = \sum_\alpha q_\alpha(z, a)\partial_{a_\alpha} + C(\xi^{(1)}, \xi^{(1)}) \cdot \partial_{\xi^{(2)}}$$

is tangent to $E_0 = E_1 = E_2 = 0$, is reparametrization equivariant, and homogenises with base twist $\mathcal{O}_{\mathbb{P}^3}(3)$. It extends holomorphically over the second Semple stage.

Proof. The first two tangency identities are $Q_C(z) = dQ_C(z) = 0$. For the third,

$$V_C(E_2) = dQ_C(\xi^{(2)}) + d^2 Q_C(\xi^{(1)}, \xi^{(1)}) + dF\big(C(\xi^{(1)}, \xi^{(1)})\big) = 0.$$

Quadratic homogeneity proves equivariance. Under a projective chart change $\widehat{z} = \phi(z)$, the base point and first jet are fixed by V_C, so variation of

$$\widehat{\xi}^{(2)} = D\phi\, \xi^{(2)} + D^2\phi(\xi^{(1)}, \xi^{(1)})$$

is simply $D\phi\, C(\xi^{(1)}, \xi^{(1)})$. The entries of $D\phi$ have pole order at most 2, while the parameter coefficients have base degree at most 3; hence the stated $\mathcal{O}(3)$ homogenization clears every component.

For the Semple boundary calculation, write $C(\xi^{(1)}, \xi^{(1)}) = (A_1(x, y), A_2(x, y))$ in a tangent frame. Then

$$\delta x = \delta y = 0, \qquad \delta X = A_1(x, y), \qquad \delta Y = A_2(x, y),$$

and therefore

$$\delta W = xA_2(x, y) - yA_1(x, y) =: S_3(x, y).$$

The polynomial S_3 is a binary cubic. Lemma 2.19, with $T = 0$, gives

$$\delta q = S_3(1, p), \qquad \delta r = -S_3(1, p) r^2.$$

This is regular on both second-stage charts and proves extension across their common boundary. $\square$

At a smooth incidence point choose a rational tangent frame e_2, e_3 and take $C(e_2, e_2) = e_3$, all other components being zero. Since $\operatorname{im} C \subset \ker dF$, the Taylor two-jet of the parameter correction vanishes and can be removed by Lemma 2.5. For

$$x = \xi_2^{(1)}, \quad y = \xi_3^{(1)}, \quad X = \xi_2^{(2)}, \quad Y = \xi_3^{(2)}, \quad W = xY - yX,$$

the third-difference fields have no jet component, so the residual field satisfies

$$\delta x = \delta y = \delta X = 0, \qquad \delta Y = x^2.$$

Consequently

$$\delta W = (\delta x)Y + x\delta Y - (\delta y)X - y\delta X = x^3,$$

and its action on the whole unlocalised invariant ring is

$$\delta_{\mathrm{acc}} = x^3 \partial_W. \tag{2.9}$$

It remains to select the semisimple direction from Păun's matrix-lift package on the same incidence open. Let H be the restriction of d^2F to the tangent plane and assume $\det H \neq 0$. Put

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \operatorname{adj}(H)J.$$

Then

$$T^{\mathsf{T}}H + HT = 0, \qquad \operatorname{tr} T = 0, \qquad \det T = \det H \neq 0.$$

Here is the normal completion needed to invoke the matrix-lift proposition. In the frame (n, e_2, e_3), with $dF(n) = 1$ and $dF(e_2) = dF(e_3) = 0$, write

$$d^2F = \begin{pmatrix} r & s^{\mathsf{T}} \\ s & H \end{pmatrix}, \qquad a = TH^{-1}s, \qquad \widehat{T} = \begin{pmatrix} 0 & 0 \\ a & T \end{pmatrix}.$$

The identity $T^{\mathsf{T}}H + HT = 0$ implies

$$dF \circ \widehat{T} = 0, \qquad \widehat{T}^{\mathsf{T}} d^2F + d^2F\,\widehat{T} = 0.$$

Indeed, TH^{-1} is skew-symmetric, so the normal–normal entry is $2s^{\mathsf{T}}TH^{-1}s = 0$, and the mixed and tangent blocks vanish by the same displayed identity. Thus the parameter correction has zero Taylor two-jet and belongs to the third-difference span. The matrix $\widehat{T}$ has entries in the incidence function field and is therefore a rational linear combination of the regular constant-matrix fields in Proposition 2.20. In the dependence contradiction this combination and the third-difference fields give a residual action inducing T on x, y; if every regular field preserved (P), this rational combination would preserve it as well. Lemma 2.18 shows that the mixed normal–tangent terms add a binary cubic multiple of ∂_W; this is immaterial after Lemma 2.2 has removed the W-dependence.

The simultaneous definition and rational choice of these two compact operators over one dominant incidence function field will be established in Lemma 2.26.

2.4. **Logarithmic case.** Let $C_a = \{f(z_1, z_2, a) = 0\} \subset \mathbb{P}^2$ and introduce the cyclic coordinate z_3 with

$$z_3^d + f(z_1, z_2, a) = 0.$$

Writing $z_3' = z_3\xi_3^{(1)}$ and $z_3'' = z_3(\xi_3^{(2)} + (\xi_3^{(1)})^2)$ gives the standard logarithmic vertical two-jet equations. For constant vectors $\tau, \beta \in \mathbb{C}^2$ set

$$L_\beta(Y - z) = \beta \cdot (Y - z), \qquad g_\tau(Y) = \tau \cdot df(Y),$$

and prescribe

$$\delta\xi^{(1)} = 0, \qquad \delta\xi^{(2)} = \tau L_\beta(\xi^{(1)})^2.$$

The parameter polynomial

$$Q^{\log}_{0,\tau,\beta}(Y; z, a) = -\frac{1}{2} g_\tau(Y, a) L_\beta(Y - z)^2$$

has zero value and gradient and has Hessian $-df_z(\tau)L_\beta^2$. The same top-degree $\mathfrak{m}_z^3$ correction used above reduces its degree to at most d without altering its two-jet.

Write $G = z_3^d + f_a(z_1, z_2)$, let $\mathscr{D}$ denote total differentiation in the source variable, and set $E_j^{\log} = \mathscr{D}^j G$ for $0 \leqslant j \leqslant 2$.

Lemma 2.22 (Full coefficient-line and cyclic-root gauge)**.** *Let $E_a = \sum_\alpha a_\alpha \partial_{a_\alpha}$ and*

$$\mathscr{E}_{\log} = E_a + \frac{1}{d} z_3 \partial_{z_3}.$$

For every regular function c on a coefficient/base chart, the full second prolongation $\operatorname{pr}^{(2)}(c\mathscr{E}_{\log})$ preserves the ideal $(E_0^{\log}, E_1^{\log}, E_2^{\log})$ and the boundary $z_3 = 0$. More precisely,

$$\operatorname{pr}^{(2)}(c\mathscr{E}_{\log})\big(\mathscr{D}^j G\big) = \mathscr{D}^j(cG), \qquad 0 \leqslant j \leqslant 2. \tag{2.10}$$

Thus

$$\begin{aligned} E_0^{\log} &\longmapsto cE_0^{\log}, \\ E_1^{\log} &\longmapsto cE_1^{\log} + (\mathscr{D}c)E_0^{\log}, \\ E_2^{\log} &\longmapsto cE_2^{\log} + 2(\mathscr{D}c)E_1^{\log} + (\mathscr{D}^2c)E_0^{\log}. \end{aligned}$$

In logarithmic-normal coordinates $\chi = \mathscr{D}\log z_3$ *and* $\psi = \mathscr{D}^2 \log z_3$*, its root part is*

$$\delta \log z_3 = \frac{c}{d}, \qquad \delta\chi = \frac{1}{d}\mathscr{D}c, \qquad \delta\psi = \frac{1}{d}\mathscr{D}^2 c.$$

Proof. Homogeneity gives $\mathscr{E}_{\log}(G) = G$. The defining commutation rule for full prolongation yields (2.10), and Leibniz' rule gives the triangular formulas. The root component is divisible by z_3, so it preserves the boundary. The last three identities follow by differentiating $\delta \log z_3 = c/d$. In particular, the derivatives of c cannot be omitted when c depends on the base point. □

Proposition 2.23 (Projective normalisation before gauge correction)**.** *Let* V *be a homogeneous cone field tangent to the logarithmic two-jet incidence, with coefficient component* $q = (q_\alpha)$*. On* $a_\rho \neq 0$ *put*

$$c_\rho = \frac{q_\rho}{a_\rho}, \qquad V^{(\rho)} = V - \mathrm{pr}^{(2)}(c_\rho \mathscr{E}_{\log}).$$

Then $V^{(\rho)}$ *has zero* a_ρ*-component, and the representatives* $V^{(\rho)}$ *on different coefficient charts differ by a full gauge field from Lemma 2.22. Hence they define one field on the projective coefficient space, without adjoining a root of* $-f$*.*

More generally, suppose that tangent cone fields V_i*, multiplied by regular functions* h_i*, have one common coefficient homogeneity and satisfy*

$$\sum_i h_i q_i = 0 \tag{2.11}$$

as a coefficient vector. Then $\sum_i h_i V_i$ *is already a homogeneous horizontal representative. If the* V_i *used for the cancellation are third-difference fields, they have no jet component and therefore do not change the residual action on the intrinsic logarithmic jet variables. No separate Euler or root term survives.*

Proof. The coefficient component of $\mathscr{E}_{\log}$ is a. On an overlap,

$$V^{(\rho)} - V^{(\sigma)} = \mathrm{pr}^{(2)}\big((c_\sigma - c_\rho)\mathscr{E}_{\log}\big),$$

which proves the first assertion by Lemma 2.22. For the second, tangent fields form an $\mathcal{O}$-module and (2.11) makes the coefficient component identically zero. The quotient from homogeneous cone fields to the projective tangent sheaf is $\mathcal{O}$-linear. Finally, a parameter-only third-difference field acts trivially on every jet variable before passage to that quotient. □

The compact analogue of Proposition 2.23 is obtained by replacing $\mathscr{E}_{\log}$ with the coefficient Euler field E_a. Since $E_a(F_a) = F_a$, the same prolongation and horizontal cone-combination proof applies, with no cyclic-root coordinate.

Proposition 2.24 (Logarithmic quadratic acceleration)**.** *After the coefficient-Euler and cyclic-root gauge correction, the preceding field descends to the projective parameter space, preserves the logarithmic boundary, is reparametrization equivariant, and has base twist at most* $\mathcal{O}_{\mathbb{P}^2}(3)$*.*

Proof. Denote the three logarithmic jet equations by $E_j^{\log} = 0$, $0 \leqslant j \leqslant 2$. The zero value and gradient of the parameter polynomial give $V(E_0^{\log}) = V(E_1^{\log}) = 0$, while its Hessian cancels exactly the acceleration term in $V(E_2^{\log})$. Thus tangency on the coefficient cone is an identity.

On a coefficient chart $a_\rho \neq 0$, apply Proposition 2.23 with $c_\rho = q_\rho^{\log}/a_\rho$. Lemma 2.22, including the $\mathscr{D}c_\rho$ and $\mathscr{D}^2 c_\rho$ terms, proves tangency and compatibility on coefficient-chart overlaps. It also proves preservation of $z_3 = 0$. This is projective descent of the full prolonged field, not merely invariance under the deck group.

For the base-chart check write $\widehat{z}_3 = gz_3$ and let $\chi = \xi_3^{(1)}$, $\psi = \xi_3^{(2)}$, $h = \log g$. Then

$$\widehat{\chi} = \chi + dh(\xi^{(1)}), \qquad \widehat{\psi} = \psi + dh(\xi^{(2)}) + d^2h(\xi^{(1)}, \xi^{(1)}).$$

The acceleration fixes the base point and first jet, so $\delta\widehat{\psi} = dh(\delta\xi^{(2)})$. The transformed acceleration has pole order at most 2, and dh contributes at most one further projective factor; hence the logarithmic-frame term has order at most 3. If $\rho = g^d$, the equations themselves transform triangularly:

$$\begin{aligned}
\widehat{E}_0^{\log} &= \rho E_0^{\log},\\
\widehat{E}_1^{\log} &= \rho E_1^{\log} + d\rho(\xi^{(1)})E_0^{\log},\\
\widehat{E}_2^{\log} &= \rho E_2^{\log} + 2d\rho(\xi^{(1)})E_1^{\log} + \big(d\rho(\xi^{(2)}) + d^2\rho(\xi^{(1)},\xi^{(1)})\big)E_0^{\log}.
\end{aligned}$$

Thus the local fields glue. To check the second Semple boundary explicitly, work on the logarithmic interior and eliminate the logarithmic-normal jet. For tangent first jets (u, v) one has

$$\delta u = \delta v = 0, \qquad \delta(\xi_1^{(2)}, \xi_2^{(2)}) = \tau L_\beta(u, v)^2.$$

Consequently

$$\delta W_{12} = (u\tau_2 - v\tau_1)L_\beta(u, v)^2 =: S_3(u, v),$$

a binary cubic. On $u \neq 0$ put

$$p_S = \frac{v}{u}, \qquad q_S = \frac{W_{12}}{u^3}, \qquad r_S = q_S^{-1}.$$

Lemma 2.19, with $T = 0$, gives

$$\delta q_S = S_3(1, p_S), \qquad \delta r_S = -S_3(1, p_S)r_S^2,$$

in the two second-stage coordinates. Both are polynomial at the relevant boundary, proving the asserted extension. □

On the incidence chart $p = f_1(z) \neq 0$, write $q = f_2(z)$ and take

$$\tau = (q/p, -1), \qquad \beta = (0, 1).$$

This is a function-field linear combination of two regular fields of the same twist, and $df_z(\tau) = 0$. Hence its parameter two-jet is removable by the third-difference fields. After the common-twist normalisation of Lemma 2.5, form the cancelling combination on the coefficient cone. Its coefficient vector is identically zero, so Proposition 2.23 leaves the raw acceleration action on the logarithmic jets; no cyclic-normal gauge term remains. On the interior $f \neq 0$, eliminate the logarithmic normal jet and put

$$u = \xi_1^{(1)}, \quad v = \xi_2^{(1)}, \quad X = \xi_1^{(2)}, \quad Y = \xi_2^{(2)}, \quad W_{12} = uY - vX.$$

The third-difference cancellation changes no jet component. Therefore

$$\delta u = \delta v = 0, \qquad \delta X = \frac{q}{p}v^2, \qquad \delta Y = -v^2,$$

and hence

$$\delta W_{12} = u\delta Y - v\delta X = -\left(u + \frac{q}{p}v\right)v^2.$$

Thus the residual derivation on the entire unlocalised invariant ring is

$$\delta_{\mathrm{acc}}^{\log} = -\left(u + \frac{q}{p}v\right)v^2\partial_{W_{12}}, \tag{2.12}$$

whose cubic coefficient is nonzero.

We next recall Rousseau's logarithmic matrix-lift package [21, Lemma 5]. The following uniform completion records the value, gradient, Hessian and chart-gluing interfaces needed below; it is a reformulation of that classical package rather than a new matrix-lift construction.

Proposition 2.25 (Rousseau's logarithmic matrix-lift package)**.** *Let*

$$A = \begin{pmatrix} A_T & 0\\ b & 0\end{pmatrix}, \qquad A_T \in M_2(\mathbb{C}), \quad b \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}),$$

and put

$$\begin{aligned}
B_A(Y, z) = A_T(Y - z), &\qquad \ell_A(Y, z) = b(Y - z),\\
G_{A,0}^{\log}(Y; z, a) = -df_Y\big(B_A(Y, z)\big) &+ df(Y, a)\ell_A(Y, z),
\end{aligned}$$

where the scalar d is the cyclic-cover degree. After cancelling the top Y-degree by a polynomial in $\mathfrak{m}_z^3$, the expansion of $G_{A,0}^{\log}$ gives a parameter correction of degree at most d. Together with $\delta\xi^{(r)} = A\xi^{(r)}$, it defines, after the same coefficient-Euler/root gauge as above, a regular logarithmic slanted field of base twist $\mathcal{O}_{\mathbb{P}^2}(3)$ which extends holomorphically across the second logarithmic Semple boundary.

Proof. Before the top-degree correction, prescribe the infinitesimal ambient motion

$$\delta Y = B_A, \qquad \delta \log z_3 = \ell_A, \qquad \delta f = G_{A,0}^{\log}.$$

It satisfies the identity

$$\delta(z_3^d + f) = d\ell_A(z_3^d + f).$$

Since $B_A(z,z) = \ell_A(z,z) = 0$, the first two prolongations at the base point act by A on both logarithmic jet levels. Differentiating the last identity twice therefore gives tangency to all three logarithmic jet equations.

Only the term $df(Y)\ell_A(Y,z)$ can have top degree $d+1$. For every top monomial $h_\alpha Y^\alpha$, choose $\gamma \leqslant \alpha$ with $|\gamma| = 3$ and subtract

$$h_\alpha (Y-z)^\gamma Y^{\alpha-\gamma}.$$

This lies in $\mathfrak{m}_z^3$, so it changes none of the value, gradient or Hessian data, but it reduces the Y-degree to at most d. The resulting parameter coefficients are linear in a and have z-degree at most 3.

It remains to check the logarithmic frame. On a base overlap write $\widehat{z} = \phi(z)$, $\widehat{z}_3 = g(z)z_3$, $h = \log g$, and use χ, ψ for the first and second logarithmic-normal jets. Direct variation gives

$$\begin{aligned}
\delta\widehat{\xi}_T^{(1)} &= D\phi\, A_T\xi_T^{(1)}, \\
\delta\widehat{\xi}_T^{(2)} &= D\phi\, A_T\xi_T^{(2)} + 2D^2\phi(\xi_T^{(1)}, A_T\xi_T^{(1)}), \\
\delta\widehat{\chi} &= b\xi_T^{(1)} + dh(A_T\xi_T^{(1)}), \\
\delta\widehat{\psi} &= b\xi_T^{(2)} + dh(A_T\xi_T^{(2)}) + 2d^2h(\xi_T^{(1)}, A_T\xi_T^{(1)}).
\end{aligned}$$

The first and second derivatives of a projective transition have pole orders at most 2 and 3; the displayed logarithmic-frame terms have the same bound. Thus $\mathcal{O}(3)$ clears both jet and parameter components. The full-gauge calculation in Lemma 2.22 and Proposition 2.23 proves descent on parameter-chart overlaps, and linearity on both jet levels proves reparametrization equivariance.

On the logarithmic interior, eliminate the logarithmic-normal jet and use a rational tangent frame. The induced actions have the form

$$\delta(u,v) = T(u,v), \qquad \delta(X,Y) = T(X,Y) + B\big((u,v),(u,v)\big).$$

Lemma 2.18 gives the exact formula

$$\delta W_{12} = (\operatorname{tr} T)W_{12} + S_3(u,v).$$

Lemma 2.19 then writes the field as an affine-linear polynomial in the finite second-stage coordinate and as a linear-plus- quadratic polynomial in its inverse. This gives the required holomorphic extension over the logarithmic Semple boundary and completes the proof. $\square$

For the second field, select a direction in Rousseau's matrix-lift package and put

$$F = f, \qquad \mathcal{H} = F\,\mathrm{Hess}(f) - df \otimes df.$$

On $F \det \mathcal{H} \neq 0$ take

$$T = \operatorname{adj}(\mathcal{H})J.$$

Then $\operatorname{tr} T = 0$ and $\det T = \det \mathcal{H} \neq 0$. If $\mathcal{H} = \left(\begin{smallmatrix} -A & -C \\ -C & B \end{smallmatrix}\right)$, the logarithmic-normal completion has entries

$$N_1 = \frac{-pC + qA}{dF}, \qquad N_2 = \frac{pB + qC}{dF}.$$

For clarity, write $p = f_1$, $q = f_2$, $r = f_{11}$, $s = f_{12}$, $t = f_{22}$ and $T = \left(\begin{smallmatrix} h & \ell \\ k & -h \end{smallmatrix}\right)$. Vanishing of the parameter Taylor coefficients through order two is equivalent to

$$\begin{aligned}
dpN_1 - rh - sk &= 0, & dqN_2 - s\ell + th &= 0, \\
d(qN_1 + pN_2) - r\ell - tk &= 0, & dFN_1 - ph - qk &= 0, \\
dFN_2 - p\ell + qh &= 0.
\end{aligned}$$

The last two equations determine the displayed N_1, N_2; substituting them in the first three gives precisely

$$T^{\mathsf{T}}\mathcal{H} + \mathcal{H}T = 0.$$

Thus the above formula is a complete allowed logarithmic matrix lift, not only a tangent-plane value. Taking $A_T = T$ and $b = (N_1, N_2)$ makes it a rational linear combination, over the incidence function field, of the regular constant-matrix fields in Proposition 2.25. The five identities make its parameter polynomial eligible for the same common-twist, pre-gauge third-difference cancellation. Proposition 2.23 shows that the resulting horizontal cone representative induces the semisimple action of T on u, v and, by Lemma 2.18, acts by $\delta W_{12} = R_3(u, v)$. Its simultaneous definition with the logarithmic acceleration, rationally over one dominant incidence function field, will be established in Lemma 2.26.

The rational coefficients q/p, $1/F$ and $1/\det\mathcal{H}$ do not increase the pole order of the field ultimately used for differentiation. They occur only in a function-field linear combination in the dependence contradiction: if every regular $\mathcal{O}(3)$ field preserved (P), then so would this combination. The contradiction below therefore implies that at least one regular member of Rousseau's classical package gives a derivative cutting (P) properly.

2.5. **The simultaneous-operator incidence open.** The compact and logarithmic constructions above are initially made on rational incidence charts. The following lemma verifies, in either family, that the two required operators coexist on a dominant incidence open and that their moving choices and common-twist corrections are all defined over one function field.

Lemma 2.26 (Dominance of the simultaneous-operator open)**.** *Let $d \geqslant 2$. In each of the compact and one-component logarithmic universal families, there is a dense parameter open such that each fibre over it contains a point where the acceleration field and the semisimple low-coefficient field are simultaneously defined. On the corresponding dominant incidence open, both fields, together with the third-difference corrections putting them in the common twist, can be chosen rationally over its function field. Each rational choice lies in the function-field span of a fixed finite basis of regular constant-direction fields, and the third-difference package is finite.*

Proof. In each case let U_d denote the irreducible open in the relevant projective space of degree-d equations which parametrizes smooth members. The constant compact acceleration tensors C, compact matrices A, logarithmic acceleration directions, and logarithmic matrices all range in finite-dimensional complex vector spaces. Fix bases once and for all. Likewise, for fixed d there are only finitely many pairs (β, γ) with $|\gamma| = 3$ and $|\beta| + 3 \leqslant d$, so the third-difference fields form a finite package.

In the compact case consider the universal incidence

$$\mathfrak{X}_d = \{(z, a) \in \mathbb{P}^3 \times U_d : F_a(z) = 0\}.$$

It is irreducible and dominates U_d: before restriction to U_d, its projection to $\mathbb{P}^3$ is the projective bundle whose fibre consists of the degree-d equations vanishing at the chosen point. On the affine chart $F_1 \neq 0$, let H be the restriction of d^2F to the tangent plane. The simultaneous domain required in the compact construction is

$$\Omega_d^{\mathrm{c}} = \{(z, a) \in \mathfrak{X}_d : F_1(z, a)\det H(z, a) \neq 0\}.$$

This open is nonempty. Indeed, at the origin the local two-jet

$$F = z_1 + z_2^2 + z_3^2 + O(|z|^3)$$

has $F_1 \neq 0$ and $\det H \neq 0$. For $d = 2$ it is realised at $[1 : 0 : 0 : 0]$ by the smooth quadric

$$Z_0Z_1 + Z_2^2 + Z_3^2 = 0.$$

For $d \geqslant 3$, prescribe the same two-jet at that point and choose the remaining degree-d coefficients generally. The variable linear system is base-point-free away from the prescribed point, while the prescribed first derivative prevents singularity at that point; Bertini's theorem therefore gives a smooth completion. Thus Ω_d^{c} is a nonempty open of $\mathfrak{X}_d$.

In the logarithmic case let

$$\mathfrak{C}_d = \{(z, a) \in \mathbb{P}^2 \times U_d : f_a(z) = 0\}, \qquad \mathfrak{L}_d = (\mathbb{P}^2 \times U_d) \setminus \mathfrak{C}_d.$$

The space $\mathfrak{L}_d$ is irreducible and dominates U_d. On the chart $p = f_1 \neq 0$, set

$$\mathcal{H} = f\,\mathrm{Hess}(f) - df \otimes df$$

and consider

$$\Omega_d^{\log} = \{(z,a) \in \mathfrak{L}_d : p(z,a)\det \mathcal{H}(z,a) \neq 0\}.$$

It too is nonempty: the local two-jet

$$f = 1 + z_1 + z_1^2 + z_2^2 + O(|z|^3)$$

has $f \neq 0$, $p = 1$ and $\det \mathcal{H} = 2$ at the origin. For $d = 2$ it is realised by the smooth conic

$$Z_0^2 + Z_0 Z_1 + Z_1^2 + Z_2^2 = 0,$$

and for $d \geqslant 3$ the same prescribed-jet and Bertini argument gives a smooth completion. Hence $\Omega_d^{\log}$ is a nonempty open of $\mathfrak{L}_d$.

Because each incidence space is irreducible, its nonempty simultaneous open contains its generic point. The incidence projection is dominant, so the constructible image of that open contains the generic point of U_d and therefore contains a dense open of U_d. After shrinking to this dense parameter open, the simultaneous locus meets every remaining fibre.

It remains to verify that "simultaneous" here includes compatible rational choices, rather than merely pointwise existence. In the compact chart $F_1 \neq 0$, use the rational frame

$$e_2 = \partial_{z_2} - \frac{F_2}{F_1}\partial_{z_1}, \qquad e_3 = \partial_{z_3} - \frac{F_3}{F_1}\partial_{z_1}, \qquad n = \frac{1}{F_1}\partial_{z_1}.$$

Define $C(e_2, e_2) = e_3$ and let C vanish on the other symmetric pairs in the frame (n, e_2, e_3). Its coefficients are rational on Ω_d^{c}, and Proposition 2.21 then gives the nonzero acceleration shear (2.9) rationally there. Since that construction is linear in C, this moving choice is a rational linear combination of the regular constant-C acceleration fields. For the semisimple direction put

$$T = \mathrm{adj}(H)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Writing

$$d^2F = \begin{pmatrix} r & s^{\mathsf{T}} \\ s & H \end{pmatrix} \quad \text{in } (n, e_2, e_3), \qquad a = TH^{-1}s, \qquad \widehat{T} = \begin{pmatrix} 0 & 0 \\ a & T \end{pmatrix},$$

one has

$$dF \circ \widehat{T} = 0, \qquad \widehat{T}^{\mathsf{T}} d^2F + d^2F\,\widehat{T} = 0.$$

Indeed, $T^{\mathsf{T}}H + HT = 0$; consequently TH^{-1} is skew-symmetric, which kills the normal–normal block, and the same identity kills the mixed and tangent blocks. Thus $\widehat{T}$ is a rational choice in the compact matrix-lift package and hence a rational linear combination of the regular constant-matrix fields of Proposition 2.20; its parameter correction has vanishing Taylor two-jet.

On $\Omega_d^{\log}$, writing $q = f_2$, the acceleration choice

$$\tau = (q/p, -1), \qquad \beta = (0, 1)$$

is rational and satisfies $df(\tau) = 0$, so Proposition 2.24 supplies the nonzero logarithmic acceleration shear (2.12). For fixed β the construction is linear in τ, so this too is a rational linear combination of regular constant-direction fields. The same adjugate prescription

$$T = \mathrm{adj}(\mathcal{H})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is rational and invertible on this open. The normal entries N_1, N_2 displayed immediately before the present lemma complete it to an allowed zero-third-column matrix; the Taylor identities verified there show that it is a rational linear combination of the regular fields in Proposition 2.25.

Finally, write K_Ω for the function field of the simultaneous incidence open under consideration. In both cases consider the finite linear map

$$\mathrm{ev}_3 \colon K_\Omega^{N_3} \longrightarrow \mathfrak{m}_z^3 \cap K_\Omega[Y]_{\leqslant d}, \qquad (c_i) \longmapsto \sum_i c_i \Phi_i.$$

The spanning calculation preceding Lemma 2.5 says that this map is surjective. Choose monomial bases in source and target. Its generic rank is the dimension of the target, so some maximal minor is nonzero in K_Ω. Restricting once to the nonvanishing locus of that single minor gives a rational right inverse

$$\mathfrak{m}_z^3 \cap K_\Omega[Y]_{\leqslant d} \longrightarrow K_\Omega^{N_3}.$$

The same right inverse is applied to the parameter remainders of both selected operators. It makes all cancellation coefficients in Lemma 2.5 rational on that one open; no second, operator-dependent function field is introduced. When $d = 2$ the displayed target is zero, so no correction or minor inversion is needed. Since every moving tensor or matrix was expressed above in the K_Ω-span of the fixed finite bases, both selected operators and all their corrections are defined over the function field of one and the same dominant incidence open. □

2.6. **All-weight low-pole differentiation.** The preceding subsections have separated the four inputs used here: the two-operator contradiction (Proposition 2.4), the relative-factor and stationary-saturation interfaces, the common-twist lemma (Lemma 2.5), and the simultaneous incidence open (Lemma 2.26). The theorem below is where these inputs are assembled. Its conclusion is deliberately narrower than global frame generation: one regular member of a fixed finite package cuts the chosen prime divisor properly. The only numerical restriction is $t \geqslant 4$, which keeps the differentiated twist $-(t-3)$ negative.

Theorem 2.27 ($\mathcal{O}(3)$ differentiation theorem)**.** *In either the compact surface case or the one-component logarithmic curve case, let P_a be the regular-jet polynomial associated with a holomorphic family of invariant two-jet sections of weighted degree $m \geqslant 3$ and base twist $-t$. Assume that their stationary-saturated zero divisors are irreducible and reduced and that, over the relevant incidence function field K, the residual equation P generates a prime ideal in the whole unlocalised invariant ring $K[u, v, W]$. Assume $t \geqslant 4$. After shrinking to a dense parameter open, there is a regular slanted vector field V of base pole order at most 3 such that*

$$P_a \nmid V(P_a).$$

Here a regular field means an actual local section obtained from the fixed finite-dimensional constant-direction packages after the common ℓ/ℓ^2 twist; the conclusion is not merely the existence of a rational tangent-vector value. Because P_a is an equivariant polynomial on the universal vertical jet space and V is invariant, the natural derivative $V(P_a) = \mathcal{L}_V P_a$ is a genuine global invariant two-jet differential of base twist $-(t-3)$ on the fibre. It is not merely a normal derivative defined on the divisor $(P_a = 0)$. Consequently the zero divisors of P_a and $V(P_a)$ have no common component along the stationary-saturated residual divisor, so these equations are two independent negatively twisted invariant two-jet equations there. In the logarithmic case the stationary divisor is removed by (2.6); the pulled-back logarithmic boundary is kept separate and is not hidden by a localisation in u, v, W.

Proof. Choose the finite bases fixed in the proof of Lemma 2.26. After multiplication by ℓ for the parameter-untwisted fields and by ℓ^2 for the raw third-difference fields, denote the resulting finite list of regular common-twist sections by

$$\mathscr{B} = \{B_1, \ldots, B_N\}.$$

Assume for contradiction that every regular available derivative contains the residual divisor. In particular this holds for each B_i. Lemma 2.16 gives

$$B_i(P) \in (P), \qquad 1 \leqslant i \leqslant N.$$

Ideal preservation is closed under K-linear combinations. Hence (P) is stable under the entire K-span of $\mathscr{B}$.

By Lemma 2.26, after shrinking once we work over one dominant incidence open with function field K. The moving acceleration, the semisimple matrix, and the output of the single third-difference right inverse all lie in the K-span of $\mathscr{B}$. Lemma 2.5 puts their parameter components into the same bundle. The cancelling combination is formed on the coefficient cone as in Proposition 2.23 and its compact analogue; hence it is horizontal and no chartwise gauge term remains. Equations (2.9) and (2.12) give

$$\delta_{\text{acc}} = S_3(u, v)\partial_W, \qquad S_3 \neq 0.$$

The residual matrix action has the form

$$\delta_{\mathrm{ss}} = \mathcal{D}_T + R_3(u,v)\partial_W, \qquad T \in \mathfrak{sl}_2(K), \quad \det T \neq 0;$$

the cubic term allows for the mixed normal–tangent contribution. Thus all hypotheses of Proposition 2.4 hold, which is impossible.

It follows that at least one member B_i of the fixed finite regular list satisfies $B_i(P) \notin (P)$. Because (P) is prime, this is exactly the assertion that its derivative cuts the residual divisor properly. No denominator-bearing moving combination is used as the final field.

Every B_i has base twist $\mathcal{O}(3)$ and common parameter twist $\mathcal{O}(1)$. After fixing the parameter, the latter is a one-dimensional scalar factor, whereas differentiation changes the base twist by

$$-t \longmapsto -t + 3 = -(t-3).$$

This is negative precisely when $t \geqslant 4$.

Finally, the class of $B_i(P)$ in the generic quotient $K[u,v,W]/(P)$ is nonzero. Clearing its finitely many coefficient denominators and shrinking the base preserves this nonvanishing, so the same regular field works on a dense Zariski open for the fixed family. The different parameter quantifiers in the compact and logarithmic applications are recorded separately in Corollary 2.31. □

The theorem starts with a factor already spread in a family. In applications one may instead begin with a prime component on a single fibre. The next proposition is the relative-extension bridge between these two formulations.

Proposition 2.28 (Fibrewise factor alternative)**.** *Let S be an integral parameter space and $\pi\colon \mathcal{Y} \to S$ the relative compact or logarithmic second Semple stage. Assume that S has been restricted to the dense parameter open furnished by Lemma 2.26. Fix $m \geqslant 3$, $t \geqslant 4$, and a point $s \in S$.*

Let θ_s be a section on $\mathcal{Y}_s$ and assume that it admits a regular relative extension on a neighbourhood of s (for example, by cohomology and base change). After removal of its full stationary multiplicity, suppose that its divisor has an irreducible prime Cartier component Z_s occurring with multiplicity one. Then either the projection of Z_s to the underlying surface is proper, or the following holds: for every regular relative extension Θ of θ_s, one regular field B in the fixed finite common-twist $\mathcal{O}(3)$ package satisfies

$$Z_s \not\subset Z\big(B(\Theta)|_{\mathcal{Y}_s}\big)\,.$$

In particular, when Θ is a spread family and P_a denotes its regular-jet polynomial representative with reduced prime residual divisor, this conclusion is

$$P_s \nmid B(P_a)|_{a=s}\,.$$

Proof. The proper-projection alternative is immediate, so assume that Z_s dominates the underlying surface. By hypothesis, fix any regular relative extension Θ on a Zariski neighbourhood of s.

Work at the generic point of Z_s. After stationary saturation, trivialisation of the relevant line bundle, and the choice of a rational tangent frame, its equation is an irreducible weighted-homogeneous polynomial

$$P \in R := K[u,v,W],$$

and (P) is prime in the whole unlocalised invariant ring. Since Z_s occurs with multiplicity one, the restriction of Θ to the fibre has the form

$$\theta_s = PQ, \qquad Q \in R_{(P)}^{\times}.$$

Let $\mathscr{B} = \{B_1, \dots, B_N\}$ be the fixed finite list of regular common-twist fields used in the proof of Theorem 2.27. Suppose, for contradiction, that every derivative contains Z_s. Then

$$B_i(\Theta)|_{\mathcal{Y}_s} \in (P)R_{(P)}, \qquad 1 \leqslant i \leqslant N.$$

The same containment holds for every K-linear combination of the B_i.

Form the acceleration and semisimple combinations on the coefficient cone, including their third-difference corrections, exactly as in Proposition 2.23 and its compact analogue. Their coefficient components cancel identically. Consequently their restrictions to the fibre are honest derivations of R of the form

$$\delta_{\mathrm{acc}} = S_3(u,v)\partial_W, \qquad \delta_{\mathrm{ss}} = \mathcal{D}_T + R_3(u,v)\partial_W,$$

where $S_3 \neq 0$, $T \in \mathfrak{sl}_2(K)$, and $\det T \neq 0$.

Because these two combinations are horizontal, no derivative of the chosen relative extension remains after restriction to the fibre. Hence, for either $\delta = \delta_{\mathrm{acc}}$ or $\delta = \delta_{\mathrm{ss}}$,

$$\delta(\theta_s) = \delta(PQ) \equiv Q\,\delta(P) \pmod{(P)}.$$

The assumed containment of the corresponding combination and the fact that Q is a unit in $R_{(P)}$ give

$$\delta(P) \in (P)R_{(P)}.$$

Since $\delta(P) \in R$ and (P) is prime,

$$(P)R_{(P)} \cap R = (P),$$

so both δ_{acc} and δ_{ss} preserve (P) in the whole ring R. This contradicts Proposition 2.4.

Thus at least one regular member B_i cuts Z_s properly. The argument uses only horizontal combinations at its decisive step and therefore works for every chosen relative extension of θ_s. $\square$

We now apply this fibrewise alternative to the logarithmic and compact relative-factor packages constructed above. The pole cost is the same in both settings, but the logarithmic statement must also keep the boundary and the stationary divisor separate.

Corollary 2.29 (Logarithmic relative pole-three differentiation)**.** *Let $\bar\eta$, $\eta_1 = (s_\Gamma^{\log})^{b_0}\bar\eta$, $\mathscr{Z}^{\log} = Z(\bar\eta)$ be supplied by Proposition 2.14, and let $P^{\log}$ be their regular-jet polynomial representative defined above. Then either the first alternative of Lemma 2.15 already forces algebraic degeneracy, or the generic polynomial of the saturated residual ideal $P_L^{\log}$ generates a prime ideal in the whole unlocalised ring $L[u,v,W]$. In the latter case, if $t \geqslant 4$, then after shrinking U there is a regular logarithmic slanted vector field V of base pole order at most 3 such that*

$$P_a^{\log} \nmid V(P_a^{\log})$$

over the simultaneous logarithmic interior. Hence the common zero set of the two equations has codimension at least two along the residual divisor there. Every remaining component is supported over the proper algebraic complement of that interior (including the pulled-back base boundary); an entire curve trapped in this complement is already algebraically degenerate. The twists are $-t$ and $-(t-3)$, respectively.

Proof. The proper-projection branch is the first alternative of Lemma 2.15. In the dominant branch, that lemma identifies the saturated generic residual ideal with the prime ideal $(P_L^{\log}) \subset L[u,v,W]$. It also upgrades divisorial dependence along the relative residual divisor for every regular logarithmic field to the unlocalised prime-ideal preservation identity (2.7). Lemma 2.26 puts the acceleration, the semisimple field and their third-difference corrections over this same field L. The coefficient-line Euler and cyclic-root gauge checks in Propositions 2.24 and 2.25 show that these operators descend to L itself; no cyclic field extension adjoining a root of $-f$ is made, so the prime ideal is not altered by a hidden scalar extension. Lemma 2.5 places the acceleration, semisimple and third-difference terms in the single bundle

$$T_{J_2^v} \otimes \mathcal{O}_{\mathbb{P}^2}(3) \boxtimes \mathcal{O}_{\mathbb{P}(V_d^{(2)})}(1).$$

Thus the hypotheses of Proposition 2.28 hold on each fibre under consideration, and it gives the regular field cutting the residual divisor properly. The base boundary does not create another factor: if the residual divisor projects into it, one is in the proper-projection branch, whereas in the dominant branch the argument takes place on $f \neq 0$. Likewise, the equations $p\det\mathcal{H} = 0$ defining the complement of the simultaneous chart cut out a proper algebraic subset of $\mathscr{Q}_U$; a lifted entire curve contained above this subset has algebraically degenerate base projection. The twist calculation is the final calculation in the proof of the theorem. $\square$

Corollary 2.30 (Compact relative pole-three differentiation)**.** *Let $\bar\omega$, $\omega_1 = s_\Gamma^{b_0}\bar\omega$ and $\mathscr{Z} = Z(\bar\omega)$ be supplied by Proposition 2.12, and let P be their regular-jet polynomial representative defined above. Then either the first alternative of Corollary 2.13 already forces algebraic degeneracy, or the stationary-saturated equations P_a form a holomorphic family with irreducible and reduced residual zero divisors, and their generic equation P_K generates a prime ideal in the whole unlocalised ring*

$K[x,y,W]$. *In the latter case, if* $t \geqslant 4$, *then after shrinking* U *there is a regular compact slanted vector field* V *of base pole order at most* 3 *such that*

$$P_a \nmid V(P_a).$$

The two equations have twists $-t$ *and* $-(t-3)$, *respectively.*

Proof. The first alternative is precisely the algebraic-degeneracy branch of Corollary 2.13. In the dominant alternative, the same corollary identifies the generic defining ideal with the prime ideal $(P_K) \subset K[x,y,W]$, while Proposition 2.12 supplies the required regular family and its irreducible and reduced nonstationary divisors. Thus Proposition 2.28 applies to the chosen fibrewise factor and gives the asserted regular field. The twist calculation is the final calculation in the proof of Theorem 2.27. □

Corollary 2.31 (Parameter quantifiers)**.** *The compact fibrewise alternative holds simultaneously for all numerical factor types on a very general set of surface parameters. For the degree-*11 *logarithmic application, the factor required by the proof can be chosen on one dense Zariski-open set of curve parameters.*

Proof. For the compact statement, generic freeness and cohomology and base change give a dense open for each triple (m,t,b_0), applied to the residual line bundle $\mathscr{L}_{m,t}(-b_0\Gamma_2)$. There are only countably many such triples. Intersecting these opens with the simultaneous-operator open gives a very general set, and Proposition 2.28 applies to every stationary-saturated factor on every fibre in that set.

For the logarithmic statement, write K for the function field of the smooth degree-11 parameter space, choose the initial Riemann–Roch section over K, and spread it over a dense open. Its residual divisor has finitely many geometric prime components. After a finite normalisation

$$\nu\colon \widetilde{U} \longrightarrow U$$

if necessary, choose a maximal-slope component. Since the ground field has characteristic zero, shrink once so that ν is finite étale and the corresponding factor is defined over the function field of $\widetilde{U}$. Write $\widetilde{K} = \mathbb{C}(\widetilde{U})$. Let b_0 be the full stationary multiplicity of this factor on the generic fibre and divide there to obtain a residual generator

$$0 \neq \bar{\eta}_{\widetilde{K}} \in H^0\left(\mathscr{P}^{\log}_{2,\widetilde{K}}, \mathscr{L}^{\log}_{m,t}(-b_0\Gamma_2^{\log})|_{\mathscr{P}^{\log}_{2,\widetilde{K}}}\right).$$

After shrinking $\widetilde{U}$, spread $\bar{\eta}_{\widetilde{K}}$ as a regular residual section $\bar{\eta}$, require $\bar{\eta}_a|_{\Gamma^{\log}_{a,2}} \neq 0$ on every fibre, and reconstruct $\eta_1 = (s^{\log}_{\Gamma})^{b_0}\bar{\eta}$. Put $\mathscr{Z}^{\log} = Z(\bar{\eta})$. The restriction condition implies that $\bar{\eta}_a \neq 0$ on every fibre, so Lemma 2.9 makes $\mathscr{Z}^{\log} \to \widetilde{U}$ projective and flat. Its generic fibre is the selected geometrically integral prime divisor; after one more shrinking, openness of geometric integrality makes every fibre geometrically integral. Pull back the simultaneous-operator and base-change packages to $\widetilde{U}$; finite étaleness identifies the relative parameter directions, so Proposition 2.28 applies there to the resulting stationary-saturated family.

The good locus upstairs is dense and constructible. Its image under the finite dominant map contains a dense open $U^\circ \subset U$. Since $\mathbb{C}$ is algebraically closed, every complex point of U° has a complex point above it; the selected factor and the cutting field on that lifted fibre are sections on the same curve C_a. Thus this existential fibrewise conclusion descends to U°, so the factor-selection conclusion is general. If the relevant finite full-rank assertions are successfully verified, upper semicontinuity also supplies the finitely many Key Vanishing opens; their intersection with U° still contains a dense Zariski open for the conditional endpoint argument. This is why the conditional logarithmic endpoint is stated for general parameters, whereas the compact endpoint is stated for very general parameters. □

This completes the geometric differentiation input. The next two sections address only the finite low-twist systems left by the theorem.

3. Logarithmic symmetry reduction

This section gives the exact reduction underlying the logarithmic clause of Lemma 1.3. Following the three-step strategy of [13, §3], we encode a hypothetical negatively twisted logarithmic two-jet differential, impose the involution eigencondition, and obtain a finite integral linear system. The matrix and its proof-to-certificate interface are derived here. The required full-column-rank calculations are currently undergoing exact computer-assisted verification. Completion of those calculations, generation and public release of the corresponding certificates, and independent reproduction all remain pending.

The exposition follows the construction of that matrix. We first fix and check a smooth symmetric fibre, then write the invariant two-jet normal form and its degree bounds. The chart transition converts the involution into coefficientwise covariance equations. Finally, holomorphicity at infinity and covariance are assembled into one exact system. The normal form and every matrix equation are derived in this section; what remains is the ongoing exact computer-assisted full-rank verification and the subsequent certificate process described below.

3.1. **Local description of invariant logarithmic two-jet differentials.** Use homogeneous coordinates $[T : X : Y]$ on $\mathbb{P}^2$ and affine coordinates (x, y) on the chart $T = 1$. Let $\mathcal{C} \subset \mathbb{P}^2$ be a smooth curve of degree $d = 11$. For the symmetry reduction we choose

$$R(x, y) = x^{11} + y^{11} + 1 + x^5 + y^5,$$

which is invariant under exchanging x and y. We shall use the following exact chart check.

Lemma 3.1 (The logarithmic special fibre)**.** *The projective curve defined by*

$$\widehat{R} = X^{11} + Y^{11} + T^{11} + T^6X^5 + T^6Y^5$$

is smooth. On the affine chart $T = 1$, the restrictions of R_x and R_y to $\mathcal{C}$ have no common zero.

Proof. Put $h(s) = s^{11} + s^5$. If $h'(s) = 0$, then either $s = 0$, or $s^6 = -5/11$ and

$$|h(s)| = \frac{6}{11}\left(\frac{5}{11}\right)^{5/6}.$$

Since

$$2\frac{6}{11}\left(\frac{5}{11}\right)^{5/6} < 1,$$

as follows after taking sixth powers from $12^6 5^5 < 11^{11}$, the equations $R = R_x = R_y = 0$ have no affine solution. At $T = 0$ the equation is $X^{11} + Y^{11} = 0$, while its X- and Y-derivatives cannot vanish simultaneously at a projective point. These two observations prove both assertions. □

On the open set $\{R_y \neq 0\}$, the fibres of the bundle of invariant logarithmic 2-jet differentials are generated by x', $(\log R)' = R'/R$, and the logarithmic Wronskian

$$\Delta'''_{xR} := \begin{vmatrix} x' & \frac{R'}{R} \\ x'' & \frac{R''}{R} - \frac{R'^2}{R^2} \end{vmatrix} = x'\frac{R''}{R} - x'\frac{R'^2}{R^2} - x''\frac{R'}{R},$$

where

$$R' = x'R_x + y'R_y, \qquad R'' = x''R_x + y''R_y + x'^2R_{xx} + 2x'y'R_{xy} + y'^2R_{yy}.$$

According to [13, Proposition 3.1], any global section ω of $E_{2,m}T^*_{\mathbb{P}^2}(\log \mathcal{C})$ can be written on $\{R_y \neq 0\}$ as

$$\omega_{xR} = \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{j=0}^{m-3k} \frac{F_{j,k}}{R_y^{m-2k}} (x')^{m-3k-j} \left(\frac{R'}{R}\right)^j \left(\Delta'''_{xR}\right)^k, \tag{3.1}$$

with polynomials $F_{j,k} \in \mathbb{C}[x, y]$.

3.2. **Degree bounds from vanishing along the ample divisor.** A section of $E_{2,m}T^*_{\mathbb{P}^2}(\log\mathcal{C})\otimes\mathcal{O}(-t)$, with $t\geqslant 1$, is represented by a section vanishing to order at least t along $\{X=0\}$. Thus $F_{j,k}=x^tK_{j,k}$. The transition to the chart $\{X\neq 0\}$, computed in [13, §3.2], gives

$$\deg K_{j,k}\leqslant (m-2k)(d-1)-m+j+k+\mathsf{Twist},$$

where $\mathsf{Twist}=-t$. Each $K_{j,k}$ therefore has finitely many unknown coefficients.

3.3. **The eigenvector condition from symmetry.** The polynomial R is invariant under the involution σ that exchanges x and y:

$$\sigma(x,y)=(y,x),\qquad \sigma^*R=R.$$

This involution lifts to an action on jet coordinates:

$$\sigma^*(x')=y',\quad \sigma^*(y')=x',\quad \sigma^*(x'')=y'',\quad \sigma^*(y'')=x''.$$

Consequently, the logarithmic Wronskian transforms as $\sigma^*(\Delta'''_{xR})=\Delta'''_{yR}$, where Δ'''_{yR} denotes the Wronskian with x and y exchanged.

Suppose that the negatively twisted space is nonzero. Choose an eigensection

$$0\neq s\in H^0\big(\mathbb{P}^2,E_{2,m}T^*_{\mathbb{P}^2}(\log\mathcal{C})\otimes\mathcal{O}(-t)\big)\,,\qquad \sigma^*s=\varepsilon s,\quad \varepsilon\in\{\pm 1\}.$$

Inject it into the untwisted bundle by $\Omega_x:=X^ts$. On $T=1$ the coefficients in (3.1) are therefore $F_{j,k}=x^tK_{j,k}$. Since $\sigma^*X=Y$, the injected section is covariant, not invariant:

$$x^t\sigma^*\Omega_x=\varepsilon y^t\Omega_x. \tag{3.2}$$

We impose this identity after writing both sides on the chart $\{R_x\neq 0\}$. In the transition formulas below, ω denotes Ω_x.

General transformation to the $\{R_x\neq 0\}$ chart. On the one hand, by applying σ^* to the local expression (3.1) on $\{R_y\neq 0\}$ we obtain

$$\sigma^*\omega_{xR}=\sum_{k=0}^{\lfloor m/3\rfloor}\sum_{j=0}^{m-3k}\frac{\sigma^*(F_{j,k})}{R_x^{m-2k}}\,(y')^{m-3k-j}\left(\frac{R'}{R}\right)^j(\Delta'''_{yR})^k. \tag{3.3}$$

On the other hand, to apply the covariance identity (3.2), we must rewrite ω_{xR} on the chart $\{R_x\neq 0\}$. The next proposition gives the required form.

Proposition 3.2. *When we change from the chart $\{R_y\neq 0\}$ to $\{R_x\neq 0\}$, the local expression (3.1) transforms into*

$$\omega_{yR}:=\sum_{\substack{0\leqslant q\leqslant\lfloor m/3\rfloor\\ 0\leqslant p\leqslant m-3q}}\frac{\overline{F}_{p,q}}{R_x^{m-2q}R_y^p}\left(\frac{R'}{R}\right)^p(y')^{m-3q-p}\,(\Delta'''_{yR})^q, \tag{3.4}$$

where each $\overline{F}_{p,q}\in\mathbb{C}[x,y]$ is a polynomial obtained from the $F_{j,k}$ via the transformation rules given in the proof.

Proof. As in the proof of [13, Proposition 3.1], the total exponent of R_x in the denominator multiplying $\left(\frac{R'}{R}\right)^p(y')^{m-3q-p}\left(\Delta'''_{yR}\right)^q$ is at most $m-2q$. We therefore focus on the exponent of R_y.

The transformation formulas for the 1-jet and the logarithmic Wronskian are:

$$\begin{aligned}
x'&=-\frac{R_y}{R_x}y'+\frac{R}{R_x}\frac{R'}{R},\\
\Delta'''_{xR}&=-\frac{R_y}{R_x}\,\Delta'''_{yR}+\frac{R}{R_x^3}(R_{xx}R-R_x^2)\left(\frac{R'}{R}\right)^3+\frac{2R}{R_x^3}(R_{xy}R_x-R_yR_{xx})\left(\frac{R'}{R}\right)^2y'\\
&\qquad\qquad+\frac{1}{R_x^3}(R_y^2R_{xx}-2R_xR_yR_{xy}+R_x^2R_{yy})\left(\frac{R'}{R}\right)(y')^2.
\end{aligned}$$

Substituting these into (3.1), the term $\left(\frac{R'}{R}\right)^p(y')^{m-3q-p}\left(\Delta'''_{yR}\right)^q$ arises from expanding

$$\frac{F_{j,k}}{R_y^{m-2k}}\,(x')^{m-3k-j}\left(\frac{R'}{R}\right)^j(\Delta'''_{xR})^k\quad\text{with } k\geqslant q \text{ and } j\leqslant p.$$

In this expansion, $\big(\Delta'''_{xR}\big)^k$ contributes a factor $\big(\Delta'''_{yR}\big)^q$ together with q powers of $-R_y/R_x$ from the leading term. The remaining $k-q$ factors of Δ'''_{xR} contribute additional powers of R'/R and y'. More precisely, suppose that $\big(\Delta'''_{xR}\big)^k$ contributes η extra powers of R'/R, where $k-q \leqslant \eta \leqslant 3(k-q)$. The factor $(x')^{m-3k-j}$ then contributes $p-j-\eta$ powers of R'/R and $m-3k-j-(p-j-\eta)$ powers of $-\frac{R_y}{R_x}y'$. Consequently, the total exponent of R_y in the denominator of the coefficient multiplying $\big(\frac{R'}{R}\big)^p\,(y')^{m-3q-p}\,\big(\Delta'''_{yR}\big)^q$ is

$$m-2k-q-\big(m-3k-j-(p-j-\eta)\big)=p+k-q-\eta\leqslant p.$$

The upper bound p is attained by expanding the single term $\frac{F_{p,q}}{R_y^{m-2q}}\,(x')^{m-3q-p}\big(\frac{R'}{R}\big)^p\big(\Delta'''_{xR}\big)^q$.

Lemma 3.1 supplies the required overlap of the two derivative charts. The calculation above puts every coefficient over the common denominator R_y^p; any cancellation can only lower its actual pole order. This proves the representation (3.4). □

The transition is linear in the coefficient numerators. Hence $F_{j,k}=x^tK_{j,k}$ implies $\overline{F}_{j,k}=x^t\overline{K}_{j,k}$, while $\sigma^*F_{j,k}=y^t\sigma^*K_{j,k}$. Comparing (3.3) and (3.4) through (3.2) first gives

$$x^tR_y^j\sigma^*F_{j,k}=\varepsilon y^t\overline{F}_{j,k}.$$

Cancelling x^ty^t in $\mathbb{C}[x,y]$ yields the computational equations

$$R_y^j\sigma^*K_{j,k}=\varepsilon\overline{K}_{j,k},\quad 0\leqslant k\leqslant\lfloor m/3\rfloor,\quad 0\leqslant j\leqslant m-3k. \tag{3.5}$$

3.4. **Certificate protocol and verification status.** For each admissible pair (m,t) in the logarithmic clause of Lemma 1.3, the degree bounds leave finitely many unknown coefficients in the polynomials $K_{j,k}$. We impose two sets of constraints:

(1) **Eigenvector equations from symmetry:**

$$R_y^j\,\sigma^*K_{j,k}=\varepsilon\,\overline{K}_{j,k},$$

where $\overline{K}_{j,k}$ is obtained from the linear transformation in Proposition 3.2 and $\varepsilon=\pm1$.

(2) **Holomorphicity at infinity:** the equations expressing regular extension across the line $T=0$.

Holomorphicity condition. We compactify $\mathbb{C}^2$ to $\mathbb{P}^2$ with homogeneous coordinates $[T:X:Y]$, where $T=0$ corresponds to the line at infinity. Following [13, Subsection 3.2], we homogenise the polynomials $F_{j,k}=x^tK_{j,k}\in\mathbb{C}[x,y]$ to

$$\widehat{F}_{j,k}=X^t\widehat{K}_{j,k}\in\mathbb{C}[T,X,Y],$$

where each $\widehat{K}_{j,k}(T,X,Y)$ is homogeneous of degree

$$\deg\widehat{K}_{j,k}=(m-2k)(d-1)-m+j+k-t,$$

for $k=0,\dots,\lfloor m/3\rfloor$ and $j=0,\dots,m-3k$.

The requirement that the jet differential be holomorphic across $T=0$ is equivalent to the divisibility condition

$$T^{m-3q-p}\;\Big|\;\mathbf{HEq}_{p,q}(\widehat{K}_{j,k},T,X)\quad\text{in }\mathbb{C}[T,X,Y],$$

for every $q=0,\dots,\lfloor m/3\rfloor$ and $p=0,\dots,m-3q$. Here $\mathbf{HEq}_{p,q}$ is obtained by expanding the transformed expression and collecting powers of T. Using $(-1)^{-3k}=(-1)^k$, it takes the form

$$\begin{aligned}\mathbf{HEq}_{p,q}=\sum_{k=q}^{\lfloor m/3\rfloor}\ \sum_{r=\max(0,\,p+q-k)}^{\min(p,\,m-3k)}\ \sum_{j=r}^{m-3k}\\ (-1)^k\,2^{p-r}\,d^{\,j+k-p-q}\binom{j}{r}\frac{k!}{(p-r)!\,q!\,(k+r-p-q)!}\\ \cdot\,X^{\,m-q-j-2k}\,T^{\,k-q}\,\widehat{R}_Y^{2(k-q)}\,\widehat{K}_{j,k},\end{aligned}$$

where $\widehat{R}_Y=\partial\widehat{R}/\partial Y$ in homogeneous coordinates, and $\widehat{R}(T,X,Y)=T^dR(X/T,Y/T)$ is the homogenization of R.

Instead of checking divisibility directly, we truncate each $\mathbf{HEq}_{p,q}$ by discarding all terms where the exponent of T is at least $m-3q-p$. More precisely, we define

$$\mathbf{HEq}_{p,q}^{(\mathrm{trunc})} := \mathbf{HEq}_{p,q} \pmod{T^{m-3q-p}},$$

i.e., the polynomial obtained by keeping only terms with T-degree $< m-3q-p$. The original divisibility holds exactly when $\mathbf{HEq}_{p,q}^{(\mathrm{trunc})} = 0$. This is a linear system in the coefficients of the $K_{j,k}$.

Certificate protocol and verification status. For the symmetric curve $R = x^{11} + y^{11} + 1 + x^5 + y^5$, the equations above define an integral sparse matrix for each pair in the logarithmic clause of Lemma 1.3 and each sign $\varepsilon \in \{\pm 1\}$. A reproducible certificate must record the block dimensions, modulus, pivots, and deterministic hash after denominators are cleared. If exact elimination exhibits full column rank modulo one good prime, the matrix has full column rank over $\mathbb{Q}$, and hence over $\mathbb{C}$. The matrix and certificate interface have therefore been derived, but the required full-column-rank calculations are currently undergoing exact computer-assisted verification. Completion of those calculations, generation and public release of the corresponding certificates, and independent reproduction all remain pending. Conditional on successful completion of this verification, every coefficient $K_{j,k}$ vanishes, so

$$H^0\big(\mathbb{P}^2,\ E_{2,m}T^*_{\mathbb{P}^2}(\log \mathcal{C}) \otimes \mathcal{O}(-t)\big) = 0$$

for the symmetric curve $\mathcal{C}$.

3.5. **Extension to general curves.** Conditional on successful completion of the exact computer-assisted full-rank verification, the preceding special-fibre vanishing holds, and upper semicontinuity in the universal smooth degree-11 family [10] then makes it a Zariski-open condition. Under the same condition, the finite intersection of these opens gives the logarithmic clause of Lemma 1.3 for a general, not necessarily symmetric, curve.

4. Compact symmetry reduction

The reduction follows the same symmetry method in degree 15. Put $a = 7$ and consider the symmetric surface

$$R(x,y,z) = x^d + y^d + z^d + 1 + y^a + z^a. \tag{4.1}$$

The local description of invariant 2-jet differentials on surfaces, the degree bounds from vanishing at infinity, and the eigenvector condition from the involution exchanging y and z define an exact finite linear system. The order of presentation mirrors the logarithmic section: special-fibre geometry, local normal form, degree bounds, covariance, holomorphicity, and the certificate interface. The compact differences are the hypersurface quotient normal form and the need to reduce every equation modulo R; the matrix and its certificate interface are derived here, while the required full-column-rank calculations are currently undergoing exact computer-assisted verification.

4.1. **Local description of invariant two-jet differentials.** Use homogeneous coordinates $[T : X : Y : Z]$ on $\mathbb{P}^3$ and affine coordinates (x,y,z) on the chart $T = 1$. Let $X \subset \mathbb{P}^3$ be defined by

$$R(x,y,z) = x^d + y^d + z^d + 1 + y^a + z^a,$$

which is invariant under exchanging y and z.

Lemma 4.1 (The compact special fibres)**.** *For $(d,a) = (15,7)$, the projective homogenization*

$$\widehat{R} = X^d + Y^d + Z^d + T^d + T^{d-a}(Y^a + Z^a)$$

defines a smooth surface. On the affine chart, the supports of the restrictions of R_x, R_y, R_z have pairwise zero-dimensional intersections on X. Moreover,

$$\mathrm{ht}(R, y, R_z) = 3 \quad \textit{in } \mathbb{C}[x,y,z].$$

Proof. Write $h_{d,a}(s) = s^d + s^a$. At an affine singular point one must have $x = 0$ and $h'_{d,a}(y) = h'_{d,a}(z) = 0$. For $(d,a) = (15,7)$, every nonzero critical point satisfies $s^8 = -7/15$, and hence

$$|h_{15,7}(s)| = \frac{8}{15}\left(\frac{7}{15}\right)^{7/8}, \qquad 2|h_{15,7}(s)| < 1.$$

The last inequality is equivalent to $16^8 7^7 < 15^{15}$. Thus $1 + h_{15,7}(y) + h_{15,7}(z)$ cannot vanish. At $T = 0$ the equation is the Fermat surface $X^d + Y^d + Z^d = 0$, whose three displayed partial derivatives never vanish simultaneously. This proves smoothness.

Each of R_x, R_y, R_z depends on only one affine coordinate and has finitely many zeros in that coordinate. Two such equations therefore leave only finitely many points after imposing $R = 0$, proving the support assertion. Finally, after setting $y = 0$, the equation $R_z = 0$ leaves finitely many values of z, and $R = 0$ then leaves finitely many values of x. Hence $V(R, y, R_z)$ is zero-dimensional, which is the stated height equality. ☐

On the open set $\{R_z \neq 0\}$, the fibres of the bundle of invariant 2-jet differentials are generated by x', y', and the Wronskian

$$\Delta'''_{yx} := \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix} = y'\,x'' - y''\,x'.$$

According to [13, Proposition 4.1], any global section ω of $E_{2,m}T^*_X$ can be written on $\{R_z \neq 0\}$ as

$$J_{yx} = \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{j=0}^{m-3k} \frac{F_{j,k}}{R_z^{m-2k}} (y')^j (x')^{m-3k-j} (\Delta'''_{yx})^k, \tag{4.2}$$

with polynomials $F_{j,k} \in \mathbb{C}[x,y,z]$, viewed as elements of the quotient ring $\mathfrak{R} = \mathbb{C}[x,y,z]/(R)$ (hence as regular functions on $X \cap \{T \neq 0\}$).

To determine a proper reduced representative of an element in $\mathfrak{R}$, we employ an algorithmic reduction based on the relation $x^d = -1 - y^d - z^d - y^a - z^a$. Every element of $\mathfrak{R}$ can be uniquely represented by a polynomial in $\mathbb{C}[x,y,z]$ with $\deg_x < d$. Indeed, for any monomial x^k with $k \geqslant d$, writing $k = dq + r$ where $0 \leqslant r < d$ via Euclidean division, we may replace

$$x^k = (x^d)^q \cdot x^r = (-1 - y^d - z^d - y^a - z^a)^q \cdot x^r,$$

which involves only powers of x with exponent $r < d$. Applying this substitution to each monomial yields a reduced representative $\operatorname{red}(F) \in \mathbb{C}[x,y,z]$ satisfying $\deg_x \operatorname{red}(F) < d$ and $\operatorname{red}(F) \equiv F \pmod{R}$.

The uniqueness of this reduced form follows from the observation that if two polynomials F_1, F_2 with $\deg_x < d$ satisfy $F_1 \equiv F_2 \pmod{R}$, then their difference is a multiple of R. Any nonzero multiple of R must have x-degree at least d, since R itself contains the monomial x^d with coefficient 1 and no other term can cancel it after multiplication. Hence $F_1 - F_2 = 0$, establishing uniqueness.

Therefore, we may assume without loss of generality the following:

Convention 4.2. All polynomials $F_{j,k}$ are assumed to have x-degree strictly less than d; equivalently, every monomial appearing in them satisfies $\deg_x < d$.

4.2. **Degree bounds from vanishing along the ample divisor.** We require the jet differential to be a section of $E_{2,m}T^*_X \otimes \mathcal{O}_X(-t)$ with $t \geqslant 1$. It may be represented by a section vanishing to order at least t along the hyperplane section $\{Y = 0\}$.

Lemma 4.3. *The rational function $\frac{F_{j,k}}{R_z^{m-2k}}$ vanishes along the divisor $X \cap \{y = 0\}$ with multiplicity at least t if and only if $F_{j,k} \equiv y^t K_{j,k} \pmod{R}$ for some polynomial $K_{j,k} \in \mathbb{C}[x,y,z]$ satisfying $\deg_x K_{j,k} < d$.*

Proof. Put $A = \mathbb{C}[x,y,z]/(R)$ and let $D = \operatorname{div}_X(y)$. Because X is smooth, A is normal. The height assertion in Lemma 4.1 says that R_z is nonzero at the generic point of every component of D; hence

$$\operatorname{ord}_E(R_z) = 0 \qquad \text{for every prime component } E \subset D.$$

It follows that $F_{j,k}/R_z^{m-2k}$ vanishes along D to order at least t exactly when

$$\operatorname{ord}_E(F_{j,k}) \geqslant t \operatorname{ord}_E(y) \qquad \text{for every } E \subset D.$$

Under these inequalities, $F_{j,k}/y^t$ has nonnegative valuation at every height-one prime of A: along D this is the displayed inequality, and away from D the element y is a unit. Normality gives

$$A = \bigcap_{\operatorname{ht}\mathfrak{p}=1} A_{\mathfrak{p}} \subset \operatorname{Frac}(A),$$

so $F_{j,k}/y^t$ belongs to A. Thus $F_{j,k} = y^t K_{j,k}$ in A. The converse is immediate from $\operatorname{ord}_E(R_z) = 0$. Finally choose the unique representative of $K_{j,k}$ with $\deg_x K_{j,k} < d$ furnished by Convention 4.2. ☐

The transition to the chart $\{X \neq 0\}$ gives the degree bounds for $F_{j,k} = y^t K_{j,k}$; see [13, §4.4]. Explicitly,

$$\deg K_{j,k} \leqslant (m-2k)(d-1) - m + \mathsf{Twist} \quad \text{and} \quad \deg_x K_{j,k} < d,$$

where $\mathsf{Twist} = -t$. These bounds leave only finitely many coefficients. We use $d = 15$ for the compact clause of Lemma 1.3 in the symmetric family (4.1).

4.3. **The eigenvector condition from symmetry.** The polynomial R is symmetric under the involution σ that exchanges y and z:

$$\sigma(y,z) = (z,y), \qquad \sigma^* R = R.$$

This involution lifts to an action on the jet coordinates by

$$\sigma^*(x') = x', \quad \sigma^*(y') = z', \quad \sigma^*(x'') = x'', \quad \sigma^*(y'') = z'',$$

and on the Wronskian we have $\sigma^*(\Delta'''_{yx}) = \Delta'''_{zx}$.

Suppose that the negatively twisted space is nonzero. Choose an eigensection

$$0 \neq s \in H^0(X, E_{2,m} T_X^* \otimes \mathcal{O}_X(-t)), \qquad \sigma^* s = \varepsilon s, \quad \varepsilon \in \{\pm 1\}.$$

Inject it into the untwisted bundle by $\Omega_y := Y^t s$. On $T = 1$, the coefficients in (4.2) are $F_{j,k} = y^t K_{j,k}$. Since $\sigma^* Y = Z$, the correct identity for the injected section is

$$y^t \sigma^* \Omega_y = \varepsilon z^t \Omega_y. \tag{4.3}$$

We impose this covariance after writing both sides on $\{R_y \neq 0\}$. In the transition formulas below, ω denotes Ω_y.

First, by applying σ^* directly to the local expression (4.2) on the chart $\{R_y \neq 0\}$ we obtain an immediate pullback formula:

$$\sigma^* \omega = \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{j=0}^{m-3k} \frac{\sigma^*(F_{j,k})}{R_y^{m-2k}} (z')^j (x')^{m-3k-j} \left(\Delta'''_{zx}\right)^k. \tag{4.4}$$

Second, we must rewrite the original ω on the same chart.

Proposition 4.4. *Let X be either compact special fibre of Lemma 4.1, and let ω be given on $\{R_z \neq 0\}$ by (4.2). After transition to $\{R_y \neq 0\}$, the same differential form admits the local expression*

$$\omega = \sum_{\substack{0 \leqslant q \leqslant \lfloor m/3 \rfloor \\ 0 \leqslant p \leqslant m-3q}} \frac{\overline{F}_{p,q}}{R_y^{m-2q} R_z^p} (z')^{m-3q-p} (x')^p \left(\Delta'''_{zx}\right)^q, \tag{4.5}$$

where each $\overline{F}_{p,q}$ is a polynomial in $\mathbb{C}[x,y,z]$, regarded as an element of the quotient ring $\mathfrak{R} = \mathbb{C}[x,y,z]/(R)$. The coefficients $\overline{F}_{p,q}$ are explicit linear combinations of the original $F_{j,k}$, with coefficients that are polynomials in the partial derivatives of R.

Proof. As in [13, Proposition 4.1], the maximal admissible exponent of R_y in the coefficients multiplying $(x')^p (z')^{m-3q-p} \left(\Delta'''_{zx}\right)^q$ is $m-2q$. We only need to consider the maximal admissible exponent of R_z in the coefficients.

Following a similar computation in [13, Subsection 3.1], the 1-jet and the Wronskian transform as:

$$y' = -\frac{R_x}{R_y}\,x' - \frac{R_z}{R_y}\,z',$$

$$\begin{aligned}\Delta'''_{yx} &= -\frac{R_z}{R_y}\,\Delta'''_{zx}\\ &\quad + z'z'x'\left(\frac{R_{zz}}{R_y} - 2\,\frac{R_z}{R_y}\,\frac{R_{yz}}{R_y} + \frac{R_z}{R_y}\,\frac{R_z}{R_y}\,\frac{R_{yy}}{R_y}\right)\\ &\quad + z'x'x'\left(2\,\frac{R_{xz}}{R_y} - 2\,\frac{R_z}{R_y}\,\frac{R_{xy}}{R_y} - 2\,\frac{R_x}{R_y}\,\frac{R_{yz}}{R_y} + 2\,\frac{R_x}{R_y}\,\frac{R_z}{R_y}\,\frac{R_{yy}}{R_y}\right)\\ &\quad + x'x'x'\left(\frac{R_{xx}}{R_y} - 2\,\frac{R_x}{R_y}\,\frac{R_{xy}}{R_y} + \frac{R_x}{R_y}\,\frac{R_x}{R_y}\,\frac{R_{yy}}{R_y}\right).\end{aligned}$$

Substituting these into (4.2), we see that the term $(x')^p\,(z')^{m-3q-p}\,\big(\Delta'''_{zx}\big)^q$ arises from expanding terms of the form

$$\frac{F_{j,k}}{R_z^{m-2k}}\,(x')^{m-3k-j}(y')^j\big(\Delta'''_{yx}\big)^k \quad \text{with } k \geqslant q \text{ and } m-3k-j \leqslant p.$$

When expanding such a term, $\big(\Delta'''_{yx}\big)^k$ contributes a factor of $\big(\Delta'''_{zx}\big)^q$ together with q powers of $-R_z/R_y$ from the leading term. We may assume that $\big(\Delta'''_{yx}\big)^k$ also contributes an additional η powers of x' from the remaining cubic terms, where η satisfies $k-q \leqslant \eta \leqslant 3(k-q)$ and $m-3k-j+\eta \leqslant p$. The factor $(y')^j$ contributes $p-(m-3k-j+\eta)$ powers of x' and $j-(p-(m-3k-j+\eta))$ powers of $-\frac{R_z}{R_y}z'$. Hence, the total exponent of R_z in the denominator of the coefficient multiplying $(x')^p\,(z')^{m-3q-p}\,\big(\Delta'''_{zx}\big)^q$ satisfies

$$m-2k-q-\big(j-(p-(m-3k-j+\eta))\big) = p+k-q-\eta \leqslant p.$$

Thus every coefficient can be written over the common denominator R_z^p. The pairwise support statement in Lemma 4.1 ensures that the derivative charts have no hidden common divisorial component; any coefficientwise cancellation only lowers the actual pole order. This proves the claimed expression. □

The transition is linear in the coefficient numerators. Hence $F_{j,k} = y^t K_{j,k}$ implies $\overline{F}_{p,q} = y^t \overline{K}_{p,q}$, while $\sigma^* F_{j,k} = z^t \sigma^* K_{j,k}$. Comparing (4.4) and (4.5) through (4.3) first gives, with $p = m-3k-j$,

$$y^t R_z^p \sigma^* F_{j,k} = \varepsilon z^t \overline{F}_{p,k} \quad \text{in } \mathbb{C}[x,y,z]/(R).$$

The quotient is a domain by Lemma 4.1; cancelling $y^t z^t$ gives the fundamental computational system

$$R_z^{m-3k-j}\sigma^* K_{j,k} - \varepsilon \overline{K}_{m-3k-j,k} \equiv 0 \pmod{R}. \tag{4.6}$$

These equations hold for $0 \leqslant k \leqslant \lfloor m/3 \rfloor$ and $0 \leqslant j \leqslant m-3k$.

4.4. **Certificate protocol and verification status.** For each admissible pair (m,t) in the compact clause of Lemma 1.3, the degree bounds in Subsection 4.2 leave finitely many unknown coefficients. The complete obstruction system has two parts:

(1) **Covariance equations from symmetry:** the equations derived in Section 4.3, which encode the intrinsic eigensection condition after the twist injection;
(2) **Holomorphicity at infinity:** the equations expressing regular extension of ω across $\{T = 0\}$.

We treat each part in turn and then describe the resulting exact linear system and its certificate protocol.

Symmetry constraints. According to (4.6), for each $0 \leqslant k \leqslant \lfloor m/3 \rfloor$ and $0 \leqslant j \leqslant m-3k$ we impose

$$\mathbf{Eq}_{j,k} := R_z^{m-3k-j}\sigma^* K_{j,k} - \varepsilon \overline{K}_{m-3k-j,k} \equiv 0 \pmod{R},$$

where the polynomials $\overline{K}_{p,q}$ are induced by the linear transformation in Proposition 4.4.

Holomorphicity at infinity. It remains to check that the jet differential (4.2) extends holomorphically along the divisor $\{T = 0\}$ at infinity. Following [13, Section 4], we homogenise the local data.

Let $d = \deg R$ be the total degree of the defining polynomial. For every index pair (j, k) we factor the local coefficient as

$$F_{j,k} = y^t\, K_{j,k}, \qquad K_{j,k} \in \mathbb{C}[x, y, z],$$

and lift each $K_{j,k}$ to a homogeneous polynomial $\widehat{K}_{j,k} \in \mathbb{C}[T, X, Y, Z]$ of degree

$$\deg \widehat{K}_{j,k} = (m - 2k)(d - 1) - m - t,$$

with the additional requirement $\deg_X \widehat{K}_{j,k} < d$. The corresponding homogeneous lift of $F_{j,k}$ is then

$$\widehat{F}_{j,k} = Y^t\, \widehat{K}_{j,k}.$$

This factorisation separates the fixed pole order prescribed by the geometry from the remaining degrees of freedom carried by the polynomials $\widehat{K}_{j,k}$.

As shown in [13, Subsection 3.2], the holomorphicity conditions translate into the following divisibility requirements in the homogeneous quotient ring $\mathbb{C}[T, X, Y, Z]/(\widehat{R})$:

$$T^{\,m-3k-\ell} \;\Big|\; \mathbf{HEq}_{\ell,k}(\widehat{K}_{j,k}, X, Y) \quad \text{in } \mathbb{C}[T, X, Y, Z]/(\widehat{R}), \tag{4.7}$$

for every $k = 0, \dots, \lfloor m/3 \rfloor$ and $\ell = 0, \dots, m - 3k$, where the auxiliary polynomials $\mathbf{HEq}_{\ell,k}$ are defined by

$$\mathbf{HEq}_{\ell,k}(\widehat{K}_{j,k}, X, Y) \;=\; \sum_{j=\ell}^{m-3k} \widehat{K}_{j,k} \binom{j}{\ell} X^{m-3k-j}\, Y^{\,j-\ell}. \tag{4.8}$$

Algebraic reduction. The two systems use the same quotient relation at different stages. The symmetry equation $\mathbf{Eq}_{j,k}$ is first reduced in the affine quotient $\mathbb{C}[x, y, z]/(R)$ by $x^d = -1 - y^d - z^d - y^a - z^a$; its coefficient identities may then be homogenised to a common degree. The infinity equation $\mathbf{HEq}_{\ell,k}$ is homogeneous from the outset and is reduced in $\mathbb{C}[T, X, Y, Z]/(\widehat{R})$ by

$$X^d = -\big(Y^d + Z^d + T^d + T^{d-\lfloor d/2 \rfloor} Y^{\lfloor d/2 \rfloor} + T^{d-\lfloor d/2 \rfloor} Z^{\lfloor d/2 \rfloor}\big),$$

until its representative has X-degree $< d$. Both procedures are the homogeneous and affine forms of the unique normal form fixed in Convention 4.2; they are kept distinct in the implementation.

Expanding the reduced affine symmetry identities gives the first homogeneous linear system in the coefficients of the $K_{j,k}$.

The divisibility condition (4.7) is equivalent to the vanishing of the truncated reduced polynomial

$$\mathbf{HEq}^{(\mathrm{trunc})}_{\ell,k} \;:=\; \mathrm{red}(\mathbf{HEq}_{\ell,k}) \pmod{T^{\,m-3k-\ell}}, \tag{4.9}$$

obtained by discarding all terms of total T-degree at least $m-3k-\ell$ from the reduced representative. This converts the geometric divisibility constraints into a second explicit homogeneous linear system in the coefficients of the polynomials $\widehat{K}_{j,k}$.

Certificate protocol and verification status. The combined system is homogeneous. For every degree-15 target in the compact clause of Lemma 1.3 and each sign $\varepsilon \in \{\pm 1\}$, it defines an integral sparse matrix after denominators are cleared. A reproducible certificate must record the row and column counts, pivot data, modulus, and deterministic block hashes. Full column rank modulo one good prime implies full column rank over $\mathbb{Q}$, hence over $\mathbb{C}$. The matrix and certificate interface have therefore been derived, but the required full-column-rank calculations are currently undergoing exact computer-assisted verification. Completion of those calculations, generation and public release of the corresponding certificates, and independent reproduction all remain pending.

Conditional on successful completion of this verification, in every admissible case (m, t) the only solution is the trivial one: all coefficients of the polynomials $K_{j,k}$ vanish. Consequently,

$$H^0\big(X,\ E_{2,m} T^*_X \otimes \mathcal{O}_X(-t)\big) = 0$$

for the symmetric surface $X = \{R = 0\}$ defined by $R = x^d + y^d + z^d + 1 + y^{\lfloor d/2 \rfloor} + z^{\lfloor d/2 \rfloor}$.

4.5. **Extension to general surfaces.** Conditional on successful completion of the exact computer-assisted full-rank verification, the preceding special-fibre vanishing holds, and upper semicontinuity in the universal smooth family [10] then makes it a Zariski-open condition. Under the same condition, intersecting the finitely many opens gives the compact clause of Lemma 1.3 for general, not necessarily symmetric, surfaces.

5. Conditional derivation of the hyperbolicity theorems

This section records the geometric and analytic deductions that would follow from successful exact computer-assisted verification of the pending full-rank assertions underlying Lemma 1.3; it introduces no additional computational input. Under that verification hypothesis, the compact argument combines the slope trichotomy with the two classical foliation results below, while the logarithmic argument keeps the same trichotomy and tracks its Nevanlinna coefficient in each branch.

We first isolate the intrinsic evaluation needed in both arguments. This also records exactly where the coefficient 3 in the logarithmic estimate enters.

Lemma 5.1 (Tautological evaluation of a restricted section)**.** *Let Y be a smooth projective surface, let $D = \sum_i D_i \subset Y$ be a simple-normal-crossing divisor (possibly empty), and let*

$$\pi_{2,0}\colon Y_2(\log D) \longrightarrow Y$$

be the logarithmic second Semple tower, with second vertical divisor Γ_2. Let A be ample, let $Z \subset Y_2(\log D)$ be a reduced Cartier divisor, and suppose that

$$\sigma \in H^0\Big(Z,\, \big(\mathcal{O}_{Y_2(\log D)}(3p)(-2p\Gamma_2) \otimes \pi_{2,0}^* A^{-t}\big)\big|_Z\Big), \qquad p,t \in \mathbb{Z}_{>0}.$$

Let $f\colon \mathbb{C} \to Y$ be a nonconstant entire curve not contained in D, and assume that its regular logarithmic second lift $f_{[2]}$ is contained in Z. On $\mathbb{C} \setminus f^{-1}(D)$, regard $df_{[1]}\colon T_{\mathbb{C}} \to f_{[1]}^* V_1$ as a differential in the directed structure. Wherever it is nonzero, its image is the tautological line $f_{[2]}^*\mathcal{O}(-1)$. It therefore defines, by meromorphic continuation across the remaining discrete set, the intrinsic tautological derivative

$$\mathcal{D}_D f_{[1]} \in H^0_{\mathrm{mer}}\Big(\mathbb{C}, K_{\mathbb{C}} \otimes f_{[2]}^*\mathcal{O}(-1)\Big).$$

The canonical inclusion

$$\iota_p\colon \mathcal{O}(3p)(-2p\Gamma_2) \hookrightarrow \mathcal{O}(3p)$$

and this tautological derivative define, without choosing an ambient lift of σ, a global meromorphic section

$$s_{\sigma,f} := \Big\langle f_{[2]}^*(\iota_p\sigma), (\mathcal{D}_D f_{[1]})^{\otimes 3p} \Big\rangle \in H^0_{\mathrm{mer}}\Big(\mathbb{C}, K_{\mathbb{C}}^{3p} \otimes f^* A^{-t}\Big). \tag{5.1}$$

It is holomorphic away from $f^{-1}(D)$. At a zero of f' outside $f^{-1}(D)$, the evaluation has only a removable singularity or a zero; at a zero lying over D, no pole occurs beyond the logarithmic boundary poles counted in the following estimate:

$$\mathrm{div}_\infty(s_{\sigma,f}) \leqslant 3p \sum_i (f^* D_i)_{\mathrm{red}}. \tag{5.2}$$

If $s_{\sigma,f} \not\equiv 0$, then

$$t T_{f,A}(r) \leqslant 3p \sum_i N_f^{[1]}(r, D_i) + O\big(\log^+ T_{f,A}(r) + \log r\big) \quad \|. \tag{5.3}$$

If $D = \varnothing$, the compact fundamental-vanishing argument instead forces $s_{\sigma,f} \equiv 0$.

Proof. Pairing its $3p$-th tensor power with the pullback of $\iota_p\sigma$ gives (5.1). This is an intrinsic pairing of dual tautological lines, so it is independent of Semple coordinates and of every local representative of σ.

Equip the line bundles with smooth Hermitian metrics. Since Z is projective, the norm of $\iota_p\sigma$ is bounded on Z; hence

$$\|s_{\sigma,f}\| \leqslant C\,\|\mathcal{D}_D f_{[1]}\|^{3p}$$

for a uniform constant C. In logarithmic base coordinates, the horizontal components of the derivative of the regular first lift are (f_j'/f_j) in the boundary directions. Its vertical projective-coordinate components are holomorphic. Thus $\mathcal{D}_D f_{[1]}$ has at most a simple pole along each $(f^*D_i)_{\mathrm{red}}$; when $D=\varnothing$, it is holomorphic. At a critical point outside $f^{-1}(D)$, regularity of the Semple lift gives only a removable singularity or a zero. At a critical point over D, a logarithmic component may still have a simple pole, but it is already counted by the reduced boundary pullback above; no additional Semple-type pole is introduced. This proves (5.2).

When $D=\varnothing$, the bounded coefficient $\iota_p\sigma$ and the negative twist A^{-t} put this intrinsic evaluation exactly in the standard Ahlfors–Schwarz proof of the fundamental vanishing theorem for invariant jets; see [4]. That proof is local along the regular Semple lift and therefore does not require extending σ away from Z. It gives $s_{\sigma,f}\equiv 0$.

Choose a finite logarithmic coordinate cover and, above it, finitely many relatively compact Semple charts. A partition of unity is used only to estimate the norm of the already-defined global section. On each chart the components of $\mathcal{D}_D f_{[1]}$ are logarithmic derivatives of local coordinate functions. A projective Semple coordinate u is a rational function on $Y_1(\log D)$ and, in a logarithmic frame, a ratio of first logarithmic derivatives. Consequently

$$T_{u\circ f_{[1]}}(r) = O\big(T_{f,A}(r)+\log r\big).$$

For a partition function χ supported in a relatively compact Semple chart, choose $a\notin\overline{u(\operatorname{supp}\chi)}$. On this support, $u-a$ is bounded above and away from zero, and

$$(u\circ f_{[1]})' = (u\circ f_{[1]}-a)\frac{(u\circ f_{[1]})'}{u\circ f_{[1]}-a}.$$

The logarithmic-derivative lemma therefore controls every vertical component by

$$O\big(\log^+ T_{f,A}(r)+\log r\big).$$

Summing these estimates and the analogous horizontal estimates over the finite cover gives the proximity term. Finally trivialize $K_{\mathbb{C}}^{3p}$ by $(dz)^{3p}$, give it the flat metric, and apply Poincaré–Lelong and Jensen's formula. This proves (5.3), following [15, Thm. 3.1, proof of (3.3)–(3.6)] with the Semple-coordinate adaptation made explicit. Notice that the direct-image identification for jet differentials is not invoked for a section defined only on Z. □

5.1. **Compact case.** Two classical results close the compact argument once two independent jet equations have been produced.

Theorem 5.2 (Clemens [2] and Xu [26]). *A very general surface $X\subset\mathbb{P}^3$ of degree $d\geqslant 5$ contains neither rational nor elliptic curves.*

Theorem 5.3 (McQuillan [16]). *Every parabolic leaf of an algebraic multi-foliation on a surface of general type is algebraically degenerate.*

The remaining mechanism is a three-way slope dichotomy: the Demailly–El Goul zero-locus method handles small slope, pole-three differentiation handles twist at least four, and successful verification of the compact clause of the finite Key Vanishing Lemma would exclude the narrow strip between them.

Theorem 5.4 (The compact endpoint). *Assume that the ongoing exact computer-assisted full-rank verification successfully establishes the compact clause of Lemma 1.3. Then every very general surface $X\subset\mathbb{P}^3$ of degree* 15 *is Kobayashi hyperbolic.*

Proof. Assume the verification hypothesis in the theorem. Together with upper semicontinuity, that hypothesis supplies the finite Key Vanishing open described in Section 4.5. Choose X in its intersection with the parameter locus supplied by Corollary 2.31 and the very-general locus supplied by Theorem 5.2. Let $f\colon \mathbb{C} \to X$ be an entire curve and write $f_{[1]}$ and $f_{[2]}$ for its first two Semple lifts. The existence, irreducible-factor, and relative-extension steps are those of [13, Sections 5–6]; Proposition 2.12 supplies the prime-divisor interface needed for differentiation. It remains to close the numerical gap for an irreducible factor of weight m and twist $-t$. By Lemma 2.11, $m \geqslant 3$. Set

$$\rho_{15} = \frac{17}{66}.$$

The positive integer t falls into exactly one of the following branches. If $t/m < \rho_{15}$, the Demailly–El Goul zero-locus argument applies; compare [5, Proposition 3.4] and [13, Proposition 6.1]. If $t/m \geqslant \rho_{15}$ and $t \geqslant 4$, Corollary 2.30 produces an independent derivative with negative twist $-(t-3)$.

It remains only to exclude $t/m \geqslant \rho_{15}$ with $1 \leqslant t \leqslant 3$. The inequality $17m \leqslant 66t$ and integrality give

$$t = 1 \Longrightarrow 3 \leqslant m \leqslant 3, \qquad t = 2 \Longrightarrow 3 \leqslant m \leqslant 7, \qquad t = 3 \Longrightarrow 3 \leqslant m \leqslant 11.$$

For $t = 1$ the compact clause of Lemma 1.3 covers the only weight. For $t = 2$, its twist-one vanishings cover the lower weights by twist monotonicity and its twist-two clause covers $m = 7$. For $t = 3$, the same argument leaves only the twist-three range, also covered by that lemma. Thus the residual branch is empty. In either of the first two branches, the two negatively twisted equations vanish on $f_{[2]}$ and cut properly. In the differentiation branch they have no common divisorial component. In the zero-locus branch, apply Lemma 5.1 with $D = \varnothing$ to the second section restricted to the irreducible threefold Z. Its canonical evaluation vanishes by the compact clause of the lemma. Since the tautological derivative is nonzero away from a discrete set, the intrinsic pairing (5.1) implies that the restricted section vanishes along $f_{[2]}$. Since it does not vanish identically on Z, it cuts a divisor there. Hence $f_{[2]}(\mathbb{C})$ is contained in an algebraic subset

$$W \subset X_2, \qquad \dim W \leqslant 2.$$

Since $\dim X_1 = 3$, the closed set

$$Y = \overline{\pi_{2,1}(W)} \subsetneq X_1$$

is proper and contains $f_{[1]}(\mathbb{C})$. Thus the first lift is algebraically degenerate.

If $\pi_{1,0}(Y)$ is proper in X, then f is already algebraically degenerate. Otherwise an irreducible component of Y dominates X; over a dense open of X, its finite collection of tangent directions defines the standard algebraic multi-foliation associated with the two-jet zero locus, and f is one of its parabolic leaves. This is the Demailly–El Goul reduction used in [5, Proposition 3.4] and [13, Sections 5–6]. Theorem 5.3 therefore makes f algebraically degenerate in X. The normalization of an algebraic curve supporting a nonconstant entire curve has genus at most one, whereas Theorem 5.2 excludes rational and elliptic curves on the chosen very general surface. Hence f is constant. □

The theorem above is therefore a conditional implication: successful exact verification of the compact full-rank assertions would yield the new degree-15 endpoint and the compact assertion of Theorem 1.1. The public version of [13] covers $d \geqslant 17$, and the public first version of the present symmetry paper covers $d \geqslant 16$ [3]. Those earlier proofs are not reproduced or used in the degree-15 argument.

5.2. **Logarithmic case.** At degree 11, algebraic degeneracy alone is not enough for the conditional Second Main Theorem. The next proposition supplies the quantitative small-slope estimate; under the verification hypothesis in Theorem 1.2, pole-three differentiation and the pending logarithmic Key Vanishing clause handle the complementary large-slope range.

Proposition 5.5 (Quantitative restricted zero-locus package)**.** *Let $C \subset \mathbb{P}^2$ be a smooth curve of degree 11, and write*

$$\pi_{2,1}\colon X_2^{\log} \longrightarrow X_1^{\log}, \qquad \pi_{2,0}\colon X_2^{\log} \longrightarrow \mathbb{P}^2.$$

Let Z be the reduced nonstationary component cut out by an irreducible logarithmic two-jet differential of weight m and twist $-t$. Assume that Z dominates $X_1^{\log}$. With

$$u_1 = \pi_{2,1}^* \mathcal{O}_{X_1^{\log}}(1), \qquad u_2 = \mathcal{O}_{X_2^{\log}}(1), \qquad h = c_1(\mathcal{O}_{\mathbb{P}^2}(1)),$$

assume, equivalently as in the standard irreducible-factor decomposition, that

$$Z \sim b_1 u_1 + b_2 u_2 - t\pi_{2,0}^* h, \qquad b_1 \geqslant 2b_2 > 0, \qquad b_1 + b_2 = m.$$

Put $\rho = t/m$. If $0 < \rho < 13/96$, if $\tau \in \mathbb{Q}_{>0}$, and if

$$0 < \tau < \tau_1(\rho), \tag{5.4}$$

then

$$Q_{11}(\rho,\tau) > 0, \qquad Q_{11}(\rho,\tau) := 3\tau^2 + (9\rho - 96)\tau + 13 - 96\rho,$$

and, for every sufficiently large and divisible ℓ, the restriction to Z of

$$\Big(\mathcal{O}_{X_2^{\log}}(2,1) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-\tau)\Big)^{\ell}, \qquad \mathcal{O}_{X_2^{\log}}(2,1) := \mathcal{O}_{X_2^{\log}}(3) \otimes \mathcal{O}(-2\Gamma_2^{\log}),$$

has a nonzero section. If $f : \mathbb{C} \to \mathbb{P}^2$ is an entire curve whose image is not contained in C, whose second logarithmic lift $f_{[2]}$ is contained in Z, and if the canonical evaluation of such a section, as defined in Lemma 5.1, is nonzero, then

$$T_f(r) \leqslant \frac{3}{\tau} N_f^{[1]}(r, C) + o\big(T_f(r)\big) \ \|. \tag{5.5}$$

Proof. Put

$$L_\tau = \mathcal{O}_{X_2^{\log}}(2,1) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-\tau).$$

Under the displayed divisor-class hypotheses, the restricted Riemann–Roch calculation of [13, Section 6] and [12, Proposition 6.2] applies. For a degree-11 plane curve,

$$\bar{c}_1 = -8h, \qquad \bar{c}_1^2 = 64, \qquad \bar{c}_2 = 91.$$

Substitution in the intersection formula gives

$$L_\tau^3 \cdot Z = m\left\{3\tau^2 + (9\rho - 96)\tau + 13 - 96\rho\right\} = mQ_{11}(\rho,\tau).$$

The higher-direct-image vanishings controlling the error term are obtained by the method of [12, Proposition 6.1]. Consequently $Q_{11}(\rho,\tau) > 0$ implies that $L_\tau|_Z$ is big.

The smaller positive root of $Q_{11}(\rho,\tau) = 0$ is

$$\tau_1(\rho) = 16 - \frac{3}{2}\rho - \frac{\sqrt{3}}{6}\sqrt{27\rho^2 - 192\rho + 3020}.$$

Since $Q_{11}(\rho, 0) > 0$ for $0 < \rho < 13/96$, the positivity interval issuing from $\tau = 0$ is precisely $0 < \tau < \tau_1(\rho)$. This proves the algebraic assertion.

We now justify the coefficient in (5.5). The Wronskian has the standard double vertical factor, but the residual factor is a section of $\mathcal{O}(2,1)$, not a nowhere-vanishing frame. Apply Lemma 5.1 with $p = \ell$, $t = \ell\tau$, $A = \mathcal{O}_{\mathbb{P}^2}(1)$, and $D = C$. Its canonical evaluation has pole divisor at most $3\ell(f^*C)_{\mathrm{red}}$, and (5.3), divided by $\ell\tau$, gives (5.5) with coefficient $3/\tau$. This is the intrinsic form of the local method used in [12, Proposition 6.3 and Theorem 6.4]; the point here is that the residual zeros of the Wronskian must be retained. Only the divisor and Chern numbers change in the present application. □

We shall also use the following quantitative logarithmic form of McQuillan's theorem; see [16] and the precise formulation in [12, Theorem 1.1].

Theorem 5.6 (Quantitative McQuillan estimate)**.** *Let $C \subset \mathbb{P}^2$ be a normal-crossing divisor of total degree $d \geqslant 4$, and let $f \colon \mathbb{C} \to \mathbb{P}^2$ be algebraically nondegenerate. If the first logarithmic lift $f_{[1]}$ is algebraically degenerate, then*

$$(d-3)T_f(r) \leqslant N_f^{[1]}(r, C) + o\big(T_f(r)\big) \ \|.$$

Proof of Theorem 1.2. Assume the successful-verification hypothesis stated in the theorem. Under this hypothesis there are four numerical outputs to compare. The initial jet differential gives a coefficient below 30 when it does not vanish on the curve. In the residual divisor, the restricted zero-locus estimate gives a coefficient below 1422 at small slope; pole-three differentiation gives at most 29 at large slope; and algebraic degeneracy gives 1/8. We verify these branches in turn.

The initial factor. Corollary 2.31 supplies a dense open of curve parameters. Choose C in this open. Logarithmic Riemann–Roch gives an initial section of weight M and twist $-T$ whose ratio T/M may be chosen arbitrarily close from below to

$$\rho_{\mathrm{RR}} = \frac{64-\sqrt{4018}}{18} > \frac{1}{30}.$$

The maximal-slope irreducible-factor reduction of [13] then supplies a factor of weight m and twist $-t$ satisfying

$$\frac{t}{m} \geqslant \frac{T}{M} > \frac{1}{30}.$$

If the pullback of this factor by f is nonzero, the jet-differential estimate of [15, Theorem 3.1] gives

$$\frac{m}{t} \leqslant \frac{M}{T} < 30.$$

We may therefore suppose that the second lift of f lies in the residual component Z. The stationary component is removed by Lemma 2.15.

Let $\pi_{2,1}\colon X_2^{\log} \to X_1^{\log}$. If $\pi_{2,1}(Z)$ is a proper algebraic subset of $X_1^{\log}$, then $f_{[1]}$ is algebraically degenerate. Since f is algebraically nondegenerate, Theorem 5.6 immediately gives the stronger coefficient 1/8. We may therefore assume that Z dominates $X_1^{\log}$. The standard divisor-class decomposition of the nonstationary irreducible factor then gives

$$Z \sim b_1u_1 + b_2u_2 - t\pi_{2,0}^{*}h, \qquad b_1 \geqslant 2b_2 > 0, \qquad b_1 + b_2 = m,$$

so all geometric hypotheses of Proposition 5.5 are satisfied.

Small slope. Set $\rho_0 = 2/15$. First assume $t/m \leqslant \rho_0$. Since $\rho_0 < 13/96$, Proposition 5.5 applies. For fixed $\tau < 96/9$, the polynomial $Q_{11}(\rho,\tau)$ decreases with ρ, so it is enough to test ρ_0. The smaller positive root of $Q_{11}(\rho_0,\tau) = 0$ is

$$\tau_* = \frac{237-7\sqrt{1146}}{15}, \qquad \frac{3}{\tau_*} = 711 + 21\sqrt{1146} < 1422. \tag{5.6}$$

The strict inequality in (5.6) leaves a rational τ with $3/1422 < \tau < \tau_*$. Then $Q_{11}(t/m,\tau) > 0$ and $3/\tau < 1422$, so (5.5) gives the asserted coefficient unless the canonical evaluation of the restricted section vanishes. In the latter case the tautological derivative is generically nonzero, so (5.1) implies that the section itself vanishes along $f_{[2]}$. Its zero divisor therefore cuts the irreducible threefold Z properly. The resulting locus in $X_2^{\log}$ has dimension at most two, so its image in the threefold $X_1^{\log}$ is proper and contains $f_{[1]}(\mathbb{C})$. Theorem 5.6 then gives the stronger coefficient $1/(11-3) = 1/8$.

Large slope. Now assume $t/m > 2/15$. If $t \leqslant 3$, then integrality gives $m \leqslant 7$ for $t = 1$, $m \leqslant 14$ for $t = 2$, and $m \leqslant 22$ for $t = 3$. The logarithmic clause of Lemma 1.3, together with twist monotonicity, excludes these three ranges without any gap. Hence $t \geqslant 4$. Corollary 2.29 produces an independent logarithmic two-jet differential of twist $-(t-3)$. If its pullback is nonzero, [15, Theorem 3.1] gives coefficient

$$\frac{m}{t-3} \leqslant 29;$$

indeed, $m < 15t/2$ gives $m \leqslant 29$ when $t = 4$, while for $t \geqslant 5$ the quotient is $< 15t/(2(t-3)) \leqslant 75/4$. If its pullback vanishes, the two independent jet equations cut a locus of dimension at most two in $X_2^{\log}$; its image in $X_1^{\log}$ is therefore proper, and Theorem 5.6 again gives 1/8. Thus every branch is bounded by

$$\max\left\{30, 29, \frac{1}{8}, 711 + 21\sqrt{1146}\right\} < 1422,$$

which proves the stated Second Main Theorem under its successful-verification hypothesis.

From the estimate to hyperbolicity. Still under the same verification hypothesis, for an entire curve omitting C the counting function is zero; the estimate excludes the algebraically nondegenerate case. The standard algebraically degenerate case is treated as in [8, 21, 13], and yields constancy. Hence the complement is hyperbolic. The result for degrees $d \geqslant 12$ is already contained in [13]; thus successful exact verification of the pending logarithmic full-rank assertions would yield the logarithmic part of Theorem 1.1 at degree 11. □

6. Outlook

Degrees 15 and 11 are the positivity endpoints of Demailly's classical invariant two-jet Riemann–Roch calculation. Successful verification of the pending finite full-rank assertions would therefore complete the present existence–vanishing–differentiation mechanism at its natural numerical boundary. At present, the argument reduces these endpoints to an ongoing exact computation: Riemann–Roch supplies sections, pole-three geometry removes the infinite high-twist range, and the residual strip is encoded by the Key Vanishing systems. This conditional reduction does not prove that every conceivable two-jet argument must stop there. A further decrease would require new positivity, a usable three-jet theory, or a different geometric input.

The broader methodological point is the division of labour between geometry and algebra. Effective exponent and denominator bounds make a global vanishing problem finite; low-pole differentiation should then remove as many targets as possible before elimination; symmetry finally adds exact equations without enlarging the unknown space. In this sense, the complex-analytic hyperbolicity problem exposes a concrete problem in hard exact algebra, while the geometry determines which part of that algebra must actually be done.

A forthcoming manuscript of Hou–Merker–Xie develops the corresponding finite-reduction programme [14]. For fixed jet order, weight, and twist, effective bounds turn the relevant negatively twisted global Green–Griffiths H^0 space, in both compact and logarithmic settings, into the kernel of an exact finite system derived from chart transitions, regularity, and divisibility. In the sense delimited in the Introduction, this concerns the full global section space rather than only generators of the invariant jet algebra, and offers a theoretical platform for jet order at least three.

The resulting systems quickly exceed present resources. Massively parallel and, potentially, quantum-assisted exact linear algebra may eventually make some of them accessible and thereby provide an algorithmic route towards the Green–Griffiths and Kobayashi conjectures. This is a prospective programme, not a proved quantum advantage and not an input to the present theorems. Exact certificates, completeness of the transition and divisibility systems, independence of the resulting sections, control of their common base locus, and geometric interpretation would all still be required.

For the finite endpoint systems used in this manuscript, exact full-rank verification is currently in progress. Generation and public release of the modular certificates, followed by independent reproduction, remain pending. Accordingly, the degree-15 and degree-11 endpoint statements remain conditional.

Acknowledgement. This project began when Lei Hou and Song-Yan Xie participated in a 2024 research-group program at the Tianyuan Mathematics Research Center in Kunming, Yunnan, held in conjunction with the conference "Progress in Several Complex Variables and Complex Geometry". During the conference, Professor Yum-Tong Siu suggested to Song-Yan Xie the direct-image argument underlying the relative-family passage. We thank him and the Center for its excellent research environment. We thank Joël Merker for valuable discussions on computational algebra, notably concerning the limitations of Maple's modular arithmetic for linear systems of the size encountered here.

Author contributions. Lei Hou and Song-Yan Xie developed the theoretical framework, including the symmetry method and the pole-three argument, and prepared the manuscript. Tao Cui and Jianjun Liu designed the exact C++ implementation and its parallel-computing architecture, and are carrying out the computational verification of the finite Key Vanishing Lemma systems. All authors have checked the interface between the theoretical reduction and the exact computations and accept responsibility for the manuscript.

AI-assisted preparation. Generative artificial-intelligence tools were used only to assist with language polishing and presentation. All original mathematical ideas and arguments developed in this paper, including the symmetry method and the pole-three strategy, are due to the authors. The authors have independently checked the theoretical arguments and the proof-to-certificate interface. The finite full-rank verification underlying the Key Vanishing Lemmas remains in progress and is not asserted to be complete in this manuscript. All authors accept full responsibility for the current manuscript.

Funding. S.-Y. Xie acknowledges partial support from the National Key R&D Program of China under Grants No. 2023YFA1010500 and No. 2021YFA1003100, from the National Natural Science Foundation of China under Grants No. 12288201 and No. 12471081, and from the Xiaomi Young Talents Program.

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