OKA GEOMETRY OF COMPLEX SIX-SPHERES

YUN-HENG DU, BIN GUO, YUTA KUSAKABE, AND SONG-YAN XIE

ABSTRACT. Alpöge constructed a complex structure on S^6 through a torus fibration with exactly three special fibres. We prove that every member of the associated admissible period family is Oka and that the family projection over each relatively compact parameter disc is an Oka map. The Oka property survives the deletion or simultaneous blow-up of any finite set of distinct points in any member. Blowing up a single point therefore produces an Oka complex manifold diffeomorphic, but not biholomorphic, to $\mathbf{P}^3$. The proofs combine explicit cyclic coverings with analytic extension and the invariance of the Oka property under holomorphic coverings.

CONTENTS

1. Introduction 1
2. Period data and relative compactification 3
2.1. Period data and the smooth family 3
2.2. Local fillings 6
2.3. Global compactification and smooth type 17
3. Cyclic coverings and Oka maps 19
4. Point modifications and band complements 23
Appendix A. Gluing and choice-independence 25
Appendix B. The source complex structure on the six-sphere 27
Acknowledgements 29
AI Use Disclosure 29
References 29

1. INTRODUCTION

The Oka principle begins with a flexibility question: when can a continuous map from a Stein space be deformed to a holomorphic map? Informally, a complex manifold is *Oka* if these deformations can be made with approximation, interpolation, and parameters [10, 9]. Gromov proved, using holomorphic sprays, that elliptic manifolds are Oka [14]. For a holomorphic submersion, an *Oka map* is the relative version: liftings over a fixed map to the base enjoy the same flexibility, and the submersion is a Serre fibration [18].

Kusakabe's analytic-Zariski localization theorem [17] assembles the Oka property from an analytic-Zariski open cover. His Oka-map criterion extends this strategy to holomorphic submersions [18]. These tools are well suited to a fibration whose general fibre is homogeneous while its special fibres are singular or multiple.

Alpöge constructed a torus-fibred complex threefold diffeomorphic to S^6 [1]. Its general fibre is a complex two-torus, and the fibration has exactly three special values on $\mathbf{P}^1$: two multiple fibres with bielliptic reductions and one irreducible nonnormal cusp fibre, whose normalization is a degree-six del Pezzo surface. Any ordered triple of distinct values on $\mathbf{P}^1$ can be moved to $0, 1, \infty$ by an automorphism; these coordinates are used throughout the paper. The orders 3 and 4 and the nature of the cusp remain essential to the construction. We use Alpöge's construction and smooth-type theorem, including its complex structure on S^6; Appendix B records the particular source ingredients used here and their locations.

The smooth fibres and $\mathbf{P}^1$ are Oka, so it is natural to ask whether the total space is Oka. The base of the smooth fibration, $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, is hyperbolic; recovering flexibility therefore depends on the three special fillings. The proof passes to cyclic covers containing explicit two-row conic bands, obtained by retaining two adjacent toric rows in the cusp model. Their algebraic charts provide Oka open subsets. The analytic

2020 *Mathematics Subject Classification.* Primary 32Q56.

Key words and phrases. Oka manifolds, Oka maps, complex six-sphere, torus fibrations, point blow-ups, deformations, cyclic coverings.

extension theorem [2, Theorem 1.9] extends the Oka property from such a nonempty analytic-Zariski open subset to its ambient manifold.

An additive period direction determines the admissible half-plane $\mathcal{A} \subset \mathbf{C}$ in Proposition 2.2. Over each relatively compact parameter disc $\Delta \Subset \mathcal{A}$ the fixed filling data give a holomorphic family $\pi_\Delta : \mathfrak{X}_\Delta \to \Delta$ of compact threefolds X_a. Here U is the uniformizing upper half-plane of the base and τ, μ, β_0 are the holomorphic period functions on U constructed in Subsection 2.1. The marked matrix is

$$\Omega_a(z) = \begin{pmatrix} 6\mu(z) & \tau(z) & 1 & 0 \\ \beta_0(z) + a & \mu(z) & 0 & 1 \end{pmatrix}.$$

The construction fixes the monodromy, finite logarithmic twists and untranslated cusp gluing as a varies. Its proper submersion and smooth-type comparison are proved in Section 2. Write $f_a : X_a \to \mathbf{P}^1$ for the torus fibration and $S_{1,a}, S_{2,a}$ for the two finite reductions. The special fibres are

$$f_a^*(0) = 3S_{1,a}, \qquad f_a^*(1) = 4S_{2,a}, \qquad f_a^{-1}(\infty) = W_a. \tag{1.1}$$

Theorem 1.1. *For every disc $\Delta \Subset \mathcal{A}$, the projection $\pi_\Delta : \mathfrak{X}_\Delta \to \Delta$ is an Oka map. For every $a \in \mathcal{A}$, the manifold X_a and both $X_a \setminus S_{1,a}$ and $X_a \setminus S_{2,a}$ are Oka.*

Deleting either finite multiple fibre leaves one multiple fibre, which is removed by a finite root base change. A cyclic period-lattice cover then contains a two-row conic band over an elliptic surface. Removing a section from that surface gives an explicit Danielewski model. Over a two-dimensional algebraic torus in this model, the principal conic open is a holomorphically trivial $\mathbf{C}^*$-bundle, hence isomorphic to $(\mathbf{C}^*)^3$. Analytic extension gives the Oka property of the band and the cyclic cover. Classical invariance under holomorphic coverings [9, Proposition 2.8] gives the finite-fibre complement, and a final analytic extension gives X_a.

The family assertion requires more than fibrewise Oka. The chosen period-lattice kernel has periods independent of a, so its covering family over a disc is a product with one fixed Oka manifold. The parametric Oka property of the product projection passes through this covering by Kusakabe's invariance theorem [18, Corollary 5.6]. Localization then gives the Oka property of π_Δ. The cyclic-cover construction and relative argument appear in Section 3.

Chen [6] independently established the Oka property for the admissible members of Alpöge's period family.

Analytic extension also applies to the torus-fibred complex six-spheres of Liu–Rollenske [20] and Viaclovsky [23]. In each construction, deleting the unique multiple fibre leaves a proper fibration over $\mathbf{C}$ with a holomorphic section and connected complex two-tori as its smooth general fibres. The torus-fibration theorem [3, Theorem 7.1] makes this complement Oka. Analytic extension [2, Theorem 1.9] then gives the Oka property of the compact total space.

Removing or blowing up points tests the flexibility of this geometry. For a finite centre, its complete lift to a cyclic cover meets each two-row band in two parts: finitely many full deck orbits in the common core and finitely many toroidal points.

Theorem 1.2. *For every $a \in \mathcal{A}$ and every finite set $P \subset X_a$ of distinct points, both $X_a \setminus P$ and the simultaneous blow-up* $\mathrm{Bl}_P X_a$ *are Oka.*

On one principal band chart, a proper monomial character makes the logarithmic lift of this complete centre tame. Its complement is Oka. Analytic extension fills the rest of the band and the cyclic cover. Invariance under holomorphic coverings, followed by analytic extension, gives $X_a \setminus P$. The exceptional-divisor complement in $\mathrm{Bl}_P X_a$ is precisely this punctured manifold, so one further analytic extension proves the blow-up result. The same principal-open argument yields the band curve complements in Section 4.

For a single point p, the Oka manifold $\mathrm{Bl}_p X_a$ is diffeomorphic to $\mathbf{P}^3$. Its complex structure differs from the standard projective one: the first Chern class of its tangent bundle has divisibility 2, whereas that of the standard $\mathbf{P}^3$ has divisibility 4. The topology and Chern-class calculation are spelled out in Corollary 4.6.

The band geometry also supplies curve complements. If any chosen subcollection of the canonical exceptional curves is deleted from a two-row band, the result remains Oka, even together with the mixed finite/deck point deletions used above. The complement of the complete residual-axis curve configuration is Oka as well (Theorems 4.4 and 4.5). These are band statements.

The Oka–1 property is the corresponding Runge approximation and Weierstrass jet-interpolation property for maps from open Riemann surfaces [11, Definition 1.1]. For an arbitrary smooth compact curve $C \subset X_a$, the blow-up theorem of Forstnerič–Lárusson gives $\mathrm{Bl}_C X_a$ this property (Corollary 4.7); the local $(-1,-1)$ conifold model has an algebraically elliptic blow-up (Proposition 4.8).

The paper is organized as follows. Section 2 constructs the period family and its fillings. Section 3 develops the cyclic bands, identifies their algebraic charts, and proves the absolute and relative Oka statements. Section 4 uses these charts for finite-point deletions, simultaneous point blow-ups, and band curve complements. Appendix A contains the Hausdorff gluing and choice-independence verifications, while Appendix B records the ingredients of Alpöge's complex-six-sphere construction that enter this paper.

2. Period data and relative compactification

2.1. **Period data and the smooth family.** Throughout, all complex spaces are reduced unless otherwise stated.

A discrete action on a Hausdorff space is called properly discontinuous if, for every pair of compact subsets, only finitely many group elements carry the first to meet the second.

Let

$$B = \mathbf{P}^1, \qquad p_1 = 0, \quad p_2 = 1, \quad p_0 = \infty, \qquad B^\circ = B \setminus \{p_0, p_1, p_2\}.$$

We use Alpöge's oriented $(3, 4, \infty)$ uniformization [1, Proposition 2.11(i)–(iii)]: a copy U of the upper half-plane carries a properly discontinuous Fuchsian action of

$$\Gamma = \langle g_1, g_2 \mid g_1^3 = g_2^4 = 1 \rangle,$$

with fixed points $z_1, z_2 \in U$ of g_1, g_2, respectively, such that $U/\Gamma \simeq B \setminus \{p_0\}$, with finite orbifold orders 3 and 4 at p_1, p_2. The element $g_0 = (g_1 g_2)^{-1}$ is a primitive parabolic generator of the cusp stabilizer. On

$$U^\circ = U \setminus (\Gamma z_1 \cup \Gamma z_2)$$

the action is free, and the orbit map

$$\varpi : U^\circ \longrightarrow B^\circ \tag{2.1}$$

is a holomorphic covering with deck group Γ.

Let

$$\Lambda = \mathbf{Z}\langle \widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta} \rangle.$$

Write $\Lambda_{\mathbf{R}} = \Lambda \otimes_{\mathbf{Z}} \mathbf{R}$. The homological monodromy matrices in this ordered basis are

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ -6 & -1 & -1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -6 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{pmatrix}. \tag{2.2}$$

Direct multiplication gives $A_1^3 = A_2^4 = I$. The displayed presentation of Γ therefore yields a representation $A : \Gamma \to \mathrm{GL}(\Lambda)$ by $A(g_j) = A_j$; its representation law is $A(gh) = A(g)A(h)$. The first row of each matrix in (2.2) is $(1, 0, 0, 0)$. Write $\gamma : \Lambda \to \mathbf{Z}$ for this invariant first-coordinate functional.

Proposition 2.1. *There are holomorphic functions*

$$\tau, \mu, \beta_0 : U \longrightarrow \mathbf{C}, \qquad \Im\tau > 0,$$

and holomorphic matrices $R_g : U \to \mathrm{GL}(2, \mathbf{C})$ satisfying

$$R_{gh}(z) = R_g(hz) R_h(z), \tag{2.3}$$

$$\Omega_a(gz) = R_g(z) \Omega_a(z) A(g)^{-1} \tag{2.4}$$

for every $g, h \in \Gamma$ and every $a \in \mathbf{C}$, where

$$\Omega_a(z) = \begin{pmatrix} 6\mu(z) & \tau(z) & 1 & 0 \\ \beta_0(z) + a & \mu(z) & 0 & 1 \end{pmatrix}. \tag{2.5}$$

Let t_c be the pullback to a sufficiently small lifted cusp neighbourhood of Alpöge's local coordinate at p_0. There is a holomorphic unit u_1 at $t_c = 0$ such that

$$\mathrm{e}^{2\pi \mathrm{i} \tau} = t_c u_1(t_c). \tag{2.6}$$

After choosing a holomorphic logarithm of u_1, set

$$h(t_c) = \frac{1}{2\pi \mathrm{i}} \log u_1(t_c), \qquad s = \tau - h(t_c). \tag{2.7}$$

Then $\mathrm{e}^{2\pi \mathrm{i} s} = t_c$, and both μ and $\beta_0 + \tau$ extend holomorphically to $t_c = 0$.

Set

$$\mathscr{D}_a(z) = \Im(\beta_0(z) + a) - \frac{6(\Im\mu(z))^2}{\Im\tau(z)}. \tag{2.8}$$

The function $\mathscr{D}_0$ is Γ-invariant, descends to $B \setminus \{p_0\}$, and tends to $-\infty$ as p_0 is approached. Its descended function is continuous at the two finite orbifold points; together with the cusp limit, this makes $\mathscr{D}_0$ bounded above.

Proof. In the ordered basis $(\widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta})$ of (2.2), Alpöge [1, Theorem 3.4(i)–(v)] constructs τ, μ and a solution β_0, unique up to an additive constant, with the equivariance for $a = 0$. Parts (i)–(iii) give (2.6)–(2.7) and the stated extensions of μ and $\beta_0 + \tau$. In the present generator convention the transformation laws are

$$\tau(g_1 z) = \frac{\tau(z) - 1}{\tau(z)}, \qquad \tau(g_2 z) = -\frac{1}{\tau(z)}, \qquad \tau(g_0 z) = \tau(z) - 1, \tag{2.9}$$

$$\mu(g_1 z) = \frac{1 - \mu(z)}{\tau(z)}, \qquad \mu(g_2 z) = 1 + \frac{\mu(z)}{\tau(z)}. \tag{2.10}$$

With $\varphi_1 = 2 - 6(1-\mu)^2/\tau$ and $\varphi_2 = -3 - 6\mu^2/\tau$, the remaining laws are

$$\beta_0(g_j z) = \beta_0(z) + \varphi_j(z), \qquad j = 1, 2. \tag{2.11}$$

The same theorem gives $R_{g_0} = I_2$. With $e_2 = (0,1)^t$, the fixed-column identity $A(g)\widehat{\delta} = \widehat{\delta}$ and the equivariance at $a = 0$ give

$$R_g(z) e_2 = R_g(z)\Omega_0(z)\widehat{\delta} = \Omega_0(gz)A(g)\widehat{\delta} = e_2.$$

Since $\Omega_a = \Omega_0 + a e_2 \gamma$ and $\gamma A(g)^{-1} = \gamma$, it follows that

$$R_g(z)\Omega_a(z)A(g)^{-1} = \Omega_0(gz) + a e_2\gamma = \Omega_a(gz)$$

for every $a \in \mathbf{C}$, with the same cocycle R_g. Together with the representation law for A, this gives the orbifold descent action. The real-lattice condition is imposed in Proposition 2.2. Finally, Theorem 3.4(iv) gives the invariance, continuity at the finite orbifold points, and cusp limit of $\mathscr{D}_0$. These properties yield its upper bound on $B \setminus \{p_0\}$. □

Proposition 2.2. *Let*

$$M_0 = \sup_{z \in U} \mathscr{D}_0(z) < \infty.$$

Then

$$\mathcal{A} := \{a \in \mathbf{C} : \Im a < -M_0\} \tag{2.12}$$

is a nonempty open half-plane, and for every $a \in \mathcal{A}$ the four columns of $\Omega_a(z)$ are linearly independent over $\mathbf{R}$ for all $z \in U$.

Proof. Equation (2.8) gives

$$\mathscr{D}_a(z) = \mathscr{D}_0(z) + \Im a.$$

If $\Im a < -M_0$, then $\mathscr{D}_a(z) < 0$ for every z.

Write

$$Z_a = \begin{pmatrix} 6\mu & \tau \\ \beta_0 + a & \mu \end{pmatrix}, \qquad \Omega_a = [Z_a \mid I_2].$$

A direct real block-determinant calculation gives

$$\det_{\mathbf{R}} \Omega_a = -\det(\Im Z_a) = \Im\tau\, \mathscr{D}_a. \tag{2.13}$$

Here $\Im\tau > 0$ and $\mathscr{D}_a < 0$, so the determinant is nonzero. □

We call a parameter $a \in \mathcal{A}$ *admissible*.

The period variation has one invariant lattice direction. The invariant functional γ has kernel

$$\Lambda_0 = \ker\gamma = \mathbf{Z}\langle \widehat{u}, \widehat{w}, \widehat{\delta}\rangle.$$

In particular, $\gamma A(g) = \gamma$ for every $g \in \Gamma$. For any two admissible parameters,

$$\Omega_a - \Omega_{a_*} = (a - a_*)e_2\gamma, \qquad \Omega_a\widehat{\delta} = e_2, \qquad R_g e_2 = e_2. \tag{2.14}$$

Translation by $u e_2$ therefore induces a compatible action

$$\Theta_{\exp(2\pi i u)}[z, \zeta_1, \zeta_2] = [z, \zeta_1, \zeta_2 + u]$$

on the quotients by Λ_0. Its extension across the finite and cusp fillings will be checked from their formulas.

The equivariant period family determines the smooth core over the punctured base. Fix a disc $\Delta \Subset \mathcal{A}$; the relative construction uses parameters $a \in \Delta$.

Here U is the orbifold universal cover, whereas $U^\circ \to B^\circ$ is the ordinary covering (2.1), with deck group Γ. The space U° is not simply connected: a small loop around a deleted orbit point has nonzero winding number about that point. The lattice quotient below uses this marked covering and does not require its total space to be simply connected.

Let Λ act on $U^\circ \times \mathbf{C}^2 \times \Delta$ by

$$\lambda \cdot (z, \zeta, a) = (z, \zeta + \Omega_a(z)\lambda, a). \tag{2.15}$$

Lemma 2.3. *The action* (2.15) *is free and properly discontinuous. Its quotient*

$$\widetilde{\mathcal{J}}_\Delta = (U^\circ \times \mathbf{C}^2 \times \Delta)/\Lambda$$

is a complex manifold, and the projection

$$\widetilde{p} : \widetilde{\mathcal{J}}_\Delta \longrightarrow U^\circ \times \Delta$$

is a proper holomorphic submersion with fibres complex two-tori.

Proof. Freeness follows from the injectivity of the real-linear map $\Omega_a(z) : \Lambda_{\mathbf{R}} \to \mathbf{C}^2$, proved in Proposition 2.2.

Let K_1, K_2 be compact subsets of $U^\circ \times \mathbf{C}^2 \times \Delta$. Their projections to $U^\circ \times \Delta$ lie in a compact set K. The smallest singular value of the real-linear map $\Omega_a(z)$ is a positive continuous function on K, hence has a positive lower bound c_K. If $\lambda K_1 \cap K_2 \neq \varnothing$, then for suitable points with the same (z, a) the difference of their ζ-coordinates is $\Omega_a(z)\lambda$. The compactness of K_1, K_2 bounds the norm of that difference by a constant C_K. Hence

$$c_K \|\lambda\| \le \|\Omega_a(z)\lambda\| \le C_K.$$

Only finitely many lattice vectors satisfy this inequality. Thus the action is properly discontinuous, and Lee's holomorphic quotient theorem [19, Theorem 1.16] makes the quotient a complex manifold.

For properness, set

$$\Phi : (U^\circ \times \Delta) \times (\Lambda_{\mathbf{R}}/\Lambda) \longrightarrow \widetilde{\mathcal{J}}_\Delta, \qquad ((z, a), [x]) \longmapsto [z, \Omega_a(z)x, a].$$

The map is well defined and bijective. Its inverse is continuous because the inverse real-linear matrices $\Omega_a(z)^{-1}$ depend continuously on (z, a). Thus Φ is a homeomorphism over $U^\circ \times \Delta$. Since $\Lambda_{\mathbf{R}}/\Lambda$ is compact, the projection is proper. It is a holomorphic submersion because locally it is the quotient of the coordinate projection $U^\circ \times \mathbf{C}^2 \times \Delta \to U^\circ \times \Delta$ by translations. □

We next descend the lattice quotient by the deck group.

Equation (2.4) permits the action

$$g \cdot [z, \zeta, a] = [gz, R_g(z)\zeta, a] \tag{2.16}$$

on $\widetilde{\mathcal{J}}_\Delta$. It is well defined because

$$R_g(z)\Omega_a(z)\lambda = \Omega_a(gz)A(g)\lambda.$$

The group law follows from (2.3).

Proposition 2.4. *The action* (2.16) *is free and properly discontinuous, and the quotient*

$$\mathcal{J}_\Delta = \widetilde{\mathcal{J}}_\Delta/\Gamma$$

is a connected complex manifold. For every $a \in \Delta$, the parameter slice $(\mathrm{pr}_\Delta \circ p^\circ)^{-1}(a)$ is connected. The induced map

$$p^\circ : \mathcal{J}_\Delta \longrightarrow B^\circ \times \Delta$$

is a proper holomorphic submersion, and each fibre over a point of $B^\circ \times \{a\}$ is a complex two-torus with local period matrix Ω_a.

Proof. The action of Γ on U° is free and properly discontinuous. If an element fixes a point of $\widetilde{\mathcal{J}}_\Delta$, it fixes its base point in U°, hence is the identity. For compact subsets of $\widetilde{\mathcal{J}}_\Delta$, their projections to U° are compact; only finitely many group elements can make the projected compact sets meet. Thus the action is properly discontinuous. Lee's holomorphic quotient theorem [19, Theorem 1.16] makes the quotient a complex manifold. The homeomorphism in Lemma 2.3 identifies $\widetilde{\mathcal{J}}_\Delta$ topologically with $(U^\circ \times \Delta) \times (\Lambda_{\mathbf{R}}/\Lambda)$; all three

factors are connected. Hence $\widetilde{\mathcal{J}}_\Delta$ is connected, and its continuous surjective quotient by Γ is connected. If $a \in \Delta$ is fixed, the same homeomorphism identifies the parameter slice upstairs with $U^\circ \times (\Lambda_{\mathbf{R}}/\Lambda)$, which is connected; its quotient by Γ is exactly $(\mathrm{pr}_\Delta \circ p^\circ)^{-1}(a)$ and is therefore connected.

Let $V \subset B^\circ$ be a simply connected relatively compact coordinate disc. Choose one connected lift $\widetilde{V} \subset U^\circ$. Every Γ-orbit over V meets $\widetilde{V}$ in exactly one point, because V is simply connected and $U^\circ \to B^\circ$ is a covering. Hence the restriction over $V \times \Delta$ is isomorphic to the lattice family over $\widetilde{V} \times \Delta$, so the restriction of p° is proper by Lemma 2.3. Properness is local on the target for maps between Hausdorff spaces, and submersivity follows in the same local chart. □

2.2. **Local fillings.** The smooth family has three missing fibres, two at the finite orbifold points and one at the cusp. Alpöge [1, Theorem 5.4] constructs the fixed-parameter logarithmic fillings at the finite points. We apply the same affine actions to the family of period matrices Ω_a and keep the parameter explicit in the quotient and gluing maps.

For $j = 1, 2$, set $m_1 = 3$ and $m_2 = 4$. The marked base has the following local uniformization data at p_j:

(F1)
$$\mathrm{Stab}_\Gamma(z_j) = \langle g_j \rangle, \qquad |\langle g_j \rangle| = m_j. \tag{2.17}$$

(F2) There is a g_j-invariant disc $\widetilde{D}_j \subset U$ centred at z_j such that
$$g\widetilde{D}_j \cap \widetilde{D}_j = \varnothing \qquad \text{for every } g \in \Gamma \setminus \langle g_j \rangle. \tag{2.18}$$
The two discs are chosen so that their images in B have disjoint closures.

(F3) There is a holomorphic coordinate s_j on $\widetilde{D}_j$ satisfying
$$s_j(g_j z) = \mathrm{e}^{-2\pi \mathrm{i}/m_j} s_j(z). \tag{2.19}$$

The stabilizer assertion follows from the triangle uniformization above, and proper discontinuity gives the discs in (F2). The Cayley coordinate $s_j(z) = (z - z_j)/(z - \overline{z}_j)$ conjugates g_j to the rotation in (2.19) under the chosen inverse-vertex convention. Let
$$\widetilde{D}_j^* = \widetilde{D}_j \setminus \{z_j\}.$$
By (2.18), the quotient map identifies
$$\widetilde{D}_j^*/\langle g_j \rangle \xrightarrow{\cong} D_j^* := \varpi(\widetilde{D}_j^*) \subset B^\circ. \tag{2.20}$$
Let D_j be the image of $\widetilde{D}_j$ in the coarse quotient $U/\Gamma \cong B \setminus \{p_0\}$. Then $D_j = D_j^* \cup \{p_j\}$, and the invariant coordinate $t_j = s_j^{m_j}$ identifies $\widetilde{D}_j/\langle g_j \rangle$ biholomorphically with D_j.

The twist vectors are
$$v_1 = \widehat{\gamma} + 2\widehat{u} - 4\widehat{w}, \qquad v_2 = -\widehat{\gamma} - 3\widehat{u} + 3\widehat{w}. \tag{2.21}$$
Direct multiplication using (2.2) gives
$$A_1 v_1 = v_1, \qquad A_2 v_2 = v_2. \tag{2.22}$$
The invariant first-coordinate functional introduced above satisfies
$$\gamma(\widehat{\gamma}) = 1, \qquad \gamma(\widehat{u}) = \gamma(\widehat{w}) = \gamma(\widehat{\delta}) = 0.$$
Then
$$\gamma \circ A_j = \gamma, \qquad \gamma(v_1) = 1, \qquad \gamma(v_2) = -1. \tag{2.23}$$

Fix $j \in \{1, 2\}$. Let Λ act on $\widetilde{D}_j \times \mathbf{C}^2 \times \Delta$ by
$$\lambda \cdot (z, \zeta, a) = (z, \zeta + \Omega_a(z)\lambda, a),$$
and set
$$\mathcal{P}_{j,\Delta} = (\widetilde{D}_j \times \mathbf{C}^2 \times \Delta)/\Lambda.$$
The argument of Lemma 2.3 applies on $\widetilde{D}_j \times \Delta$: the action is free and properly discontinuous, and Lee's holomorphic quotient theorem [19, Theorem 1.16] makes $\mathcal{P}_{j,\Delta}$ a complex manifold.

The map
$$(\widetilde{D}_j \times \Delta) \times (\Lambda_{\mathbf{R}}/\Lambda) \longrightarrow \mathcal{P}_{j,\Delta}, \qquad ((z, a), [x]) \longmapsto [z, \Omega_a(z)x, a],$$
is a homeomorphism over $\widetilde{D}_j \times \Delta$. Since $\Lambda_{\mathbf{R}}/\Lambda$ is compact, the projection
$$\mathcal{P}_{j,\Delta} \longrightarrow \widetilde{D}_j \times \Delta$$

is proper. In a local quotient chart it is induced by the coordinate projection from $\widetilde{D}_j \times \mathbf{C}^2 \times \Delta$, so its differential is surjective and it is a holomorphic submersion.

The finite local data induce the following affine action.

Set

$$\widetilde{g}_j[z,\zeta,a] = \left[g_j z, R_{g_j}(z)\zeta + \Omega_a(g_j z)\frac{v_j}{m_j}, a\right]. \tag{2.24}$$

This formula is jointly holomorphic in (z,ζ,a). It is well defined on the lattice quotient: replacing ζ by $\zeta + \Omega_a(z)\lambda$ changes the target by

$$R_{g_j}(z)\Omega_a(z)\lambda = \Omega_a(g_j z)A_j\lambda,$$

which is an integral target period. The formula has a holomorphic inverse because its m_j-th power is the identity, as proved next.

Lemma 2.5. *The biholomorphism $\widetilde{g}_j$ has order exactly m_j, and the action of $\langle\widetilde{g}_j\rangle$ on $\mathcal{P}_{j,\Delta}$ is free.*

Proof. Use the real flat coordinate $x \in \Lambda_{\mathbf{R}}/\Lambda$ determined by $\zeta = \Omega_a(z)x$. Equation (2.4) transforms (2.24) into

$$(z,x,a) \longmapsto \left(g_j z, A_j x + \frac{v_j}{m_j}, a\right). \tag{2.25}$$

By (2.22), $A_j^r v_j = v_j$ for every r. Therefore the k-th power has fibre translation

$$\frac{1}{m_j}\sum_{r=0}^{k-1} A_j^r v_j = \frac{k v_j}{m_j}.$$

For $k = m_j$, the base rotation is the identity, $A_j^{m_j} = I$, and translation by $v_j \in \Lambda$ is the identity on the torus. Thus $\widetilde{g}_j^{m_j} = 1$. For $1 \le k < m_j$, the base rotation is nontrivial, so the k-th power cannot be the identity; the order is exactly m_j.

A fixed point of a nonidentity power must lie over z_j, because the base rotation has no other fixed point in $\widetilde{D}_j$. At z_j, the fixed point equation in the real torus is

$$(I - A_j^k)x \equiv \frac{k v_j}{m_j} \pmod{\Lambda}. \tag{2.26}$$

Apply the integral functional γ. Since $\gamma \circ A_j = \gamma$, the left side has γ-value zero modulo $\mathbf{Z}$. The right side has value

$$\frac{k}{3} \quad (j=1), \qquad -\frac{k}{4} \quad (j=2),$$

by (2.23). These numbers are not integers for $1 \le k < m_j$. Hence (2.26) has no solution. □

Proposition 2.6. *The quotient*

$$\mathcal{N}_{j,\Delta} = \mathcal{P}_{j,\Delta}/\langle\widetilde{g}_j\rangle$$

is a connected complex manifold, and every parameter slice of $\mathcal{N}_{j,\Delta} \to \Delta$ is connected. It has a proper holomorphic map

$$F_j : \mathcal{N}_{j,\Delta} \longrightarrow D_j \times \Delta,$$

and its projection to Δ is a holomorphic submersion. The fibre over $p_j \times \{a\}$ has multiplicity m_j, and its reduced structure is a compact smooth complex surface.

Proof. The group is finite and acts freely, so Lee's holomorphic quotient theorem [19, Theorem 1.16] applies. The flat-coordinate homeomorphism above identifies $\mathcal{P}_{j,\Delta}$ with the product of the connected space $\widetilde{D}_j \times \Delta$ and the connected real torus $\Lambda_{\mathbf{R}}/\Lambda$. Thus $\mathcal{P}_{j,\Delta}$ is connected, and its continuous surjective finite quotient $\mathcal{N}_{j,\Delta}$ is connected. After fixing a, the same product description gives the connected space $\widetilde{D}_j \times (\Lambda_{\mathbf{R}}/\Lambda)$ upstairs; its finite quotient is the parameter slice of $\mathcal{N}_{j,\Delta}$ and is connected. Let

$$\rho_j : \widetilde{D}_j \times \Delta \longrightarrow D_j \times \Delta, \qquad (z,a) \longmapsto (s_j(z)^{m_j}, a).$$

The composite

$$\mathcal{P}_{j,\Delta} \longrightarrow \widetilde{D}_j \times \Delta \xrightarrow{\rho_j} D_j \times \Delta$$

is invariant under $\widetilde{g}_j$, because $s_j(g_j z)^{m_j} = s_j(z)^{m_j}$. It therefore descends to the holomorphic map F_j in the statement.

The map ρ_j is finite and hence proper. If $K \subset D_j \times \Delta$ is compact, its inverse image in $\mathcal{P}_{j,\Delta}$ is compact because the torus family is proper. The image of this compact set in the finite quotient is exactly $F_j^{-1}(K)$. Thus F_j is proper.

The parameter projection upstairs is a submersion and is invariant under the finite group. Since the quotient map is locally biholomorphic, the descended parameter projection is again a submersion.

Finally, the coarse base coordinate is $t_j = s_j^{m_j}$. Its pullback upstairs vanishes to order m_j along the central torus. The quotient map of the total space is locally biholomorphic because the action is free. Upstairs, the reduced central divisor is the smooth torus family $s_j = 0$; its quotient by the free finite action is therefore smooth and is the reduction of $F_j^{-1}(p_j \times \Delta)$. Consequently this Cartier divisor has multiplicity m_j along its smooth reduction. □

On a simply connected sector of $\widetilde{D}_j^*$, choose a branch of $\log s_j$ and set

$$\sigma_{j,a}(z) = \frac{\log s_j(z)}{2\pi \mathrm{i}}\, \Omega_a(z) v_j. \tag{2.27}$$

The fibre translation is

$$h_j[z, \zeta, a] = [z, \zeta + \sigma_{j,a}(z), a].$$

If the branch of the logarithm changes by $2\pi \mathrm{i} n$, then $\sigma_{j,a}$ changes by $n\Omega_a(z)v_j$, an integral period. Hence the translation on the torus quotient is independent of the branch.

Lemma 2.7. *On $\mathcal{P}_{j,\Delta}|_{\widetilde{D}_j^* \times \Delta}$ one has*

$$h_j \widetilde{g}_j h_j^{-1} = \widetilde{g}_j^0, \qquad \widetilde{g}_j^0[z, \zeta, a] = [g_j z, R_{g_j}(z)\zeta, a].$$

Consequently there is a jointly holomorphic biholomorphism

$$G_j : \mathcal{N}_{j,\Delta}|_{D_j^* \times \Delta} \xrightarrow{\cong} \mathcal{J}_\Delta|_{D_j^* \times \Delta}. \tag{2.28}$$

Proof. From (2.19), on a chosen sector one has

$$\log s_j(g_j z) = \log s_j(z) - \frac{2\pi \mathrm{i}}{m_j}.$$

Using (2.4) and $A_j v_j = v_j$, we get

$$R_{g_j}(z)\Omega_a(z)v_j = \Omega_a(g_j z) A_j v_j = \Omega_a(g_j z) v_j.$$

Therefore

$$\begin{aligned} \sigma_{j,a}(g_j z) &= \frac{\log s_j(g_j z)}{2\pi \mathrm{i}}\, \Omega_a(g_j z) v_j \\ &= R_{g_j}(z)\sigma_{j,a}(z) - \Omega_a(g_j z)\frac{v_j}{m_j}. \end{aligned}$$

Substituting this identity into the definitions of h_j and $\widetilde{g}_j$ gives the displayed conjugation formula term by term.

We now identify the untwisted quotient with the actual restriction of the global core. Let

$$\widetilde{\mathcal{J}}_{j,\Delta}^* = \widetilde{\mathcal{J}}_\Delta\big|_{\widetilde{D}_j^* \times \Delta}.$$

The subgroup $\langle g_j \rangle$ acts on this space by the untwisted maps $\widetilde{g}_j^0$. If $g\widetilde{\mathcal{J}}_{j,\Delta}^* \cap \widetilde{\mathcal{J}}_{j,\Delta}^* \neq \varnothing$, then the projections to U° give $g\widetilde{D}_j^* \cap \widetilde{D}_j^* \neq \varnothing$. Equation (2.18) therefore implies $g \in \langle g_j \rangle$, so $\widetilde{\mathcal{J}}_{j,\Delta}^*$ is precisely invariant under $\langle g_j \rangle$. The orbit maps are open, and Lee's local quotient charts [19, Theorem 1.16] give a biholomorphism

$$\widetilde{\mathcal{J}}_{j,\Delta}^* / \langle g_j \rangle \xrightarrow{\cong} \mathcal{J}_\Delta|_{D_j^* \times \Delta}.$$

The affine quotient of the same pulled-back family is $\mathcal{N}_{j,\Delta}|_{D_j^* \times \Delta}$. Hence the conjugation descends to (2.28). Every formula is holomorphic in (z, a), and branch changes alter the translation by an integral period, so the descended map is jointly holomorphic and single-valued. □

The two finite-point fillings are now identified with the smooth core on their punctured discs. At the remaining point, the cusp quotient requires a Hausdorff countable-fan model and a proper lattice action that remains controlled as $q \to 0$. It must also agree with the smooth core over the punctured cusp disc.

We choose a simply connected cusp neighbourhood $\mathcal{H}_0 \subset U^\circ$ and a biholomorphic coordinate

$$s : \mathcal{H}_0 \xrightarrow{\cong} \{w \in \mathbf{C} : \Im w > R_0\}$$

for some real number R_0, satisfying

$$s(g_0 z) = s(z) - 1.$$

The neighbourhood and coordinate can be chosen to obey

$$\mathrm{Stab}_\Gamma(\mathcal{H}_0) = \langle g_0 \rangle, \tag{2.29}$$

$$g\mathcal{H}_0 \cap \mathcal{H}_0 = \varnothing \qquad (g \in \Gamma \setminus \langle g_0 \rangle). \tag{2.30}$$

Indeed, a sufficiently small horodisc at the parabolic fixed point of g_0 is precisely invariant under $\langle g_0 \rangle$ by the discrete triangle-group action fixed above. The function $s = \tau - h(t_c)$ constructed in Proposition 2.1 satisfies $\mathrm{e}^{2\pi i s} = t_c$. On a smaller cusp disc, h is bounded, so a high enough half-plane in the s coordinate lies inside that precisely invariant horodisc. Its inverse image is $\mathcal{H}_0$, proving (2.29) and (2.30). Here the first equality means that the setwise stabilizer of the cusp neighbourhood is the infinite cyclic group generated by the primitive parabolic element g_0. After increasing R_0 and replacing $\mathcal{H}_0$ by the inverse image of the corresponding higher half-plane if necessary, set

$$q = \mathrm{e}^{2\pi i s}, \qquad \varepsilon_0 = \mathrm{e}^{-2\pi R_0}.$$

The function q identifies the quotient $\mathcal{H}_0/\langle g_0 \rangle$ biholomorphically with the punctured disc

$$D_0^* = \{q \in \mathbf{C} : 0 < |q| < \varepsilon_0\}. \tag{2.31}$$

Equation (2.30) makes $\mathcal{H}_0$ precisely invariant under $\langle g_0 \rangle$, so the global orbit map embeds this quotient biholomorphically in B°; its image is identified with D_0^*. Let

$$D_0 = D_0^* \cup \{p_0\} \cong \{|q| < \varepsilon_0\}.$$

The three discs D_0, D_1, D_2 are chosen with pairwise disjoint closures.

In the splitting

$$\Lambda_{\mathrm{tor}} = \mathbf{Z}\langle \widehat{w}, \widehat{\delta} \rangle, \qquad \overline{\Lambda} = \Lambda/\Lambda_{\mathrm{tor}} = \mathbf{Z}\langle \widehat{\gamma}, \widehat{u} \rangle,$$

write $\Omega_a = [Z_a \mid I_2]$. The nonconstant 2×2 period block has the form

$$Z_a = sB_0 + C_a(q), \qquad B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C_a(q) = C_0(q) + aE_{21}, \tag{2.32}$$

where C_0 is holomorphic on a disc D_{ε_0} about $q = 0$ and E_{21} has a single 1 in row 2, column 1. To verify the decomposition, let $b = \beta_0 + \tau$. The proof of Proposition 2.1 shows that b and μ are g_0-invariant and holomorphic in $q = t_c$ at 0, and that $\tau = s + h(q)$. Hence the displayed period matrix gives

$$C_0(q) = \begin{pmatrix} 6\mu(q) & h(q) \\ b(q) - h(q) & \mu(q) \end{pmatrix},$$

which is holomorphic at $q = 0$. The same proposition computes $R_{g_0} = I_2$, and the additive parameter contributes aE_{21}. Since $q(g_0 z) = q(z)$ and $s(g_0 z) = s(z) - 1$, one has

$$Z_a(g_0 z) = Z_a(z) - B_0. \tag{2.33}$$

These identities will be used in the punctured identification.

We recall the toric notation used for the periodic fan.

For a lattice N, write $M = \mathrm{Hom}(N, \mathbf{Z})$ and $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$. A *rational polyhedral cone* is a strongly convex cone generated by finitely many elements of N. A *rational fan* Σ is a collection of rational polyhedral cones such that every face of a cone in Σ belongs to Σ and the intersection of two cones in Σ is a face of each. No finiteness or ambient local-finiteness condition is included in this definition. For $\sigma \in \Sigma$, set

$$U_\sigma = \mathrm{Spec}\mathbf{C}[\sigma^\vee \cap M]$$

and use the same notation for its complex analytification.

For finite subfans we use the standard face-chart embeddings, separated gluing, orbit–cone correspondence, and star-fan description of toric divisors from Fulton [12, Sections 1.3–1.4, 2.1, 3.1 and 3.3]. For the specific

periodic fan below, the countable gluing and its torus action are established by Alpöge [1, Lemmas 4.2–4.3]. We identify that fan with the present coordinates next.

Let $N = \mathbf{Z}^2$ with standard basis e_1, e_2, and triangulate $N_{\mathbf{R}} = \mathbf{R}^2$ by the triangles

$$P_v^+ = \operatorname{conv}\{v, v + e_1, v + e_2\},$$
$$P_v^- = \operatorname{conv}\{v + e_1, v + e_2, v + e_1 + e_2\}, \qquad v \in \mathbf{Z}^2.$$

Let $N' = N \oplus \mathbf{Z}$. For every vertex, edge, or triangle P of the triangulation, set

$$\sigma_P = \operatorname{cone}(P \times \{1\}) \subset N'_{\mathbf{R}},$$

and set

$$\Sigma = \{0\} \cup \{\sigma_P : P \text{ is a cell of the triangulation}\}.$$

Proposition 2.8. *The collection Σ is a countable rational fan. Every cone is generated by part of a basis of N'. For each vertex $v \in \mathbf{Z}^2$, the star fan of the ray $\mathbf{R}_{\geq 0}(v, 1)$ is the finite complete hexagonal fan. Consequently the countable toric space*

$$Y = Y_\Sigma$$

is a smooth Hausdorff second-countable complex threefold, and the toric divisor E_v corresponding to the ray through $(v, 1)$ is compact.

Proof. The two families of triangles above are exactly the periodic A_2-triangulation used by Alpöge [1, Lemma 4.2(i)–(iv)], under the identification of its height-one lattice with $N' = \mathbf{Z}^2 \oplus \mathbf{Z}$. In particular, that lemma proves the fan axioms, unimodularity, Hausdorff and second-countable toric gluing, and the complete hexagonal star fan of each ray through $(v, 1)$. The source's toric threefold and divisors are therefore precisely Y_Σ and E_v in the present notation. □

Let $h = (0, 0, 1) \in M' = \operatorname{Hom}(N', \mathbf{Z})$. Since h is nonnegative on every cone, the character χ^h is holomorphic on every affine toric chart and the chartwise functions glue to

$$q : Y \longrightarrow \mathbf{C}.$$

If a maximal cone is generated by the rays through $(v_0, 1), (v_1, 1), (v_2, 1)$ and m_0, m_1, m_2 is the dual basis, then $h = m_0 + m_1 + m_2$. In the affine coordinates $z_i = \chi^{m_i}$ one has

$$q = z_0 z_1 z_2. \tag{2.34}$$

The divisor $q^{-1}(0)$ is therefore reduced with simple normal crossings. The component E_v corresponding to the ray through $(v, 1)$ has the compact hexagonal star fan described in Proposition 2.8. We shall use repeatedly that E_0 is a compact smooth toric surface.

All matrix norms in the definition below and in the cusp analysis are the operator norms induced by the standard Euclidean norms on the relevant real vector spaces.

Definition 2.9. Let $\Delta \Subset \mathcal{A}$ be a relatively compact parameter disc. A number $\varepsilon > 0$ is called a *valid cusp cutoff for* Δ if there exists a number ε_1 such that

(V1)
$$0 < \varepsilon < \varepsilon_1 < \min\{1, \varepsilon_0\};$$

(V2) with
$$M_C(\Delta, \varepsilon_1) = \sup_{\substack{|q| \leq \varepsilon_1 \\ a \in \overline{\Delta}}} \left\| -2\pi \Im C_a(q) \right\|,$$
one has
$$\frac{\|B_0^{-1}\| M_C(\Delta, \varepsilon_1)}{|\log \varepsilon|} \leq \frac{1}{2}; \tag{2.35}$$

(V3) the closed subdisc $\overline{D}_\varepsilon = \{|q| \leq \varepsilon\}$, viewed inside the original cusp coordinate disc $D_{\varepsilon_0} \cup \{p_0\} \subset B$, is disjoint from $\overline{D}_1 \cup \overline{D}_2$.

A number ε_1 satisfying these conditions is called an *auxiliary cutoff* for ε.

Lemma 2.10. *For every relatively compact parameter disc $\Delta \Subset \mathcal{A}$, valid cusp cutoffs exist. If ε is valid for Δ and $0 < \varepsilon' < \varepsilon$, then ε' is valid for Δ with the same auxiliary cutoff.*

Proof. Choose any $0 < \varepsilon_1 < \min\{1, \varepsilon_0\}$. Since C_0 is holomorphic on D_{ε_0}, the function $(q, a) \mapsto C_a(q) = C_0(q) + aE_{21}$ is continuous on the compact set $\overline{D}_{\varepsilon_1} \times \overline{\Delta}$. Thus $M_C(\Delta, \varepsilon_1) < \infty$. As $\varepsilon \downarrow 0$, the denominator $|\log \varepsilon|$ tends to infinity, so condition (V2) holds for all sufficiently small positive $\varepsilon < \varepsilon_1$. The original cusp disc was chosen with closure disjoint from the two finite discs; hence the same is true for every sufficiently small closed subdisc. This proves existence.

If $0 < \varepsilon' < \varepsilon$, then $|\log \varepsilon'| > |\log \varepsilon|$, so the left side of (2.35) can only decrease. The smaller closed cusp disc is contained in the larger one, so condition (V3) is also preserved. The same ε_1 is therefore an auxiliary cutoff for ε'. □

The cusp is the only place where an infinite group acts on a non-quasi-compact toric space. We give separate proofs of joint holomorphic extension, freeness, proper discontinuity, and properness of the quotient map. The underlying fixed-parameter cusp filling, its free and proper lattice quotient, the description of its central fibre, and its punctured gluing are due to Alpöge [1, Theorem 4.5 and Propositions 4.6–4.7]. Here the matrices C_a vary holomorphically with a, and the estimates below are made uniform on the parameter disc Δ.

Fix a valid cusp cutoff ε for Δ in the sense of Definition 2.9, and choose one auxiliary cutoff ε_1. Set

$$M_C = M_C(\Delta, \varepsilon_1) = \sup_{\substack{|q| \le \varepsilon_1 \\ a \in \overline{\Delta}}} \left\| -2\pi \Im C_a(q) \right\| < \infty. \tag{2.36}$$

By condition (V2) in Definition 2.9,

$$\frac{\|B_0^{-1}\| M_C}{|\log \varepsilon|} \le \frac{1}{2}. \tag{2.37}$$

We now restrict every cusp object to this radius and write

$$D_0 = D_\varepsilon = \{q \in \mathbf{C} : |q| < \varepsilon\}, \qquad D_0^* = D_\varepsilon \setminus \{0\}, \qquad Y_\varepsilon = q^{-1}(D_\varepsilon).$$

On the uniformizing side take the smaller horodisc

$$\mathcal{H}_{0,\varepsilon} = \{z \in \mathcal{H}_0 : 0 < |q(z)| < \varepsilon\}. \tag{2.38}$$

Since $|q(z)| = \exp(-2\pi \Im s(z))$, this is a connected horodisc. It is $\langle g_0 \rangle$-invariant, and the quotient map q induces a biholomorphism

$$\mathcal{H}_{0,\varepsilon}/\langle g_0 \rangle \xrightarrow{\cong} D_0^*. \tag{2.39}$$

Moreover, if $g\mathcal{H}_{0,\varepsilon} \cap \mathcal{H}_{0,\varepsilon} \neq \varnothing$, then $g\mathcal{H}_0 \cap \mathcal{H}_0 \neq \varnothing$; by (2.30), this forces $g \in \langle g_0 \rangle$. Thus the isolated-subgroup property survives the shrinking. Condition (V3) in Definition 2.9 gives $\overline{D}_0 \cap (\overline{D}_1 \cup \overline{D}_2) = \varnothing$.

Identify $\overline{\Lambda}$ with $\mathbf{Z}^2$ in the basis $(\widehat{\gamma}, \widehat{u})$. For $\lambda \in \mathbf{Z}^2$, the lattice automorphism

$$\phi_\lambda : N' \longrightarrow N', \qquad (y, r) \longmapsto (y + rB_0\lambda, r) \tag{2.40}$$

preserves the fan Σ. Let Φ_λ be the induced toric automorphism of Y. On the dense torus $(\mathbf{C}^*)^2 \times \mathbf{C}^*$ it is

$$\Phi_\lambda(x, q) = (q^{B_0\lambda} x, q).$$

For $(q, a) \in D_\varepsilon \times \Delta$, set

$$c_{a,\lambda}(q) = \exp(2\pi \mathrm{i} C_a(q)\lambda) \in (\mathbf{C}^*)^2, \tag{2.41}$$

where exponentiation is componentwise.

Proposition 2.11. *For every $\lambda \in \mathbf{Z}^2$, the map*

$$\Psi_\lambda(y, a) = \big(c_{a,\lambda}(q(y)) \cdot \Phi_\lambda(y), a\big) \tag{2.42}$$

is a biholomorphism of $Y_\varepsilon \times \Delta$. The dependence on (y, a) is holomorphic, and

$$\Psi_\lambda \Psi_\mu = \Psi_{\lambda+\mu}.$$

On the dense torus,

$$\Psi_\lambda(x, q, a) = \big(c_{a,\lambda}(q) q^{B_0\lambda} x, q, a\big). \tag{2.43}$$

Proof. Alpöge [1, Lemma 4.2(i)] gives the holomorphic torus multiplication map $(\mathbf{C}^*)^2 \times Y \to Y$ for this fan. The map

$$(y, a) \longmapsto c_{a,\lambda}(q(y))$$

is holomorphic because $C_a(q) = C_0(q) + aE_{21}$ is jointly holomorphic in (q, a). Therefore torus translation by this point is jointly holomorphic on $Y \times \Delta$. Composing it with the toric biholomorphism Φ_λ gives (2.42).

Both the vertical torus action and Φ_λ preserve q. Hence Ψ_λ preserves $Y_\varepsilon \times \Delta$.

The shear (2.40) restricts to the identity on the vertical sublattice $N \oplus \{0\}$. Hence Φ_λ commutes with the vertical torus action. Moreover,

$$c_{a,\lambda}(q)c_{a,\mu}(q) = c_{a,\lambda+\mu}(q)$$

because $C_a(q)$ is linear in the lattice vector. The shears add: $\phi_\lambda\phi_\mu = \phi_{\lambda+\mu}$. These identities prove the group law. Taking $\mu = -\lambda$ shows that $\Psi_{-\lambda}$ is the inverse of Ψ_λ. Formula (2.43) is the corresponding expression on the dense torus. □

Thus the action extends jointly holomorphically across the toric boundary.

For $0 < |q| < \varepsilon$, set

$$R_a(q) = -2\pi\Im C_a(q), \qquad B_{q,a} = B_0 + \frac{R_a(q)}{\log|q|}. \tag{2.44}$$

Here $R_a(q)$ is a real 2×2 matrix.

Lemma 2.12. *For the radius ε chosen in (2.37), there is a constant $K > 0$ such that*

$$\|B_{q,a}\| + \|B_{q,a}^{-1}\| \le K$$

for every $0 < |q| < \varepsilon$ and every $a \in \overline{\Delta}$.

Proof. Set

$$E_{q,a} = B_0^{-1}\frac{R_a(q)}{\log|q|}.$$

By (2.36) and (2.37),

$$\|E_{q,a}\| \le \frac{\|B_0^{-1}\|M_C}{|\log\varepsilon|} \le \frac{1}{2}, \qquad B_{q,a} = B_0(I + E_{q,a}).$$

The standard Neumann estimate gives $\|(I + E_{q,a})^{-1}\| \le 2$. Consequently,

$$\|B_{q,a}\| \le \|B_0\| + \frac{M_C}{|\log\varepsilon|}, \qquad \|B_{q,a}^{-1}\| \le 2\|B_0^{-1}\|.$$

Their sum is bounded independently of q and a. □

On the dense torus use the logarithmic modulus coordinate

$$y(x,q) = \frac{\log|x|}{\log|q|} \in \mathbf{R}^2. \tag{2.45}$$

Taking absolute values in (2.43) gives

$$y(\Psi_\lambda(x,q,a)) = y(x,q) + B_{q,a}\lambda. \tag{2.46}$$

The next lemma is the point that controls mixed sequences with $q \neq 0$ and $q \to 0$.

Lemma 2.13. *Let $K \subset Y_\varepsilon$ be compact. There exists $M_K > 0$ such that*

$$\|y(x,q)\| \le M_K$$

for every $(x,q) \in K$ lying in the dense torus with $q \neq 0$.

Proof. For every point of K, choose a relatively compact open neighbourhood whose closure lies in an affine chart associated with a maximal triangle. A finite number of these smaller neighbourhoods covers K. Their intersections with K have compact closures inside the corresponding affine charts. It is therefore enough to prove the assertion on one such compact chart piece.

Write $P = \mathrm{conv}\{v_0, v_1, v_2\}$ for the corresponding maximal triangle. Its cone is generated by $(v_i, 1)$, which form a basis of N'. For the dual basis m_0, m_1, m_2, write $z_i = \chi^{m_i}$. The affine barycentric functions are

$$\ell_i(y) = \langle m_i, (y,1)\rangle.$$

They satisfy

$$\ell_i(v_j) = \delta_{ij}, \qquad \ell_0 + \ell_1 + \ell_2 = 1, \qquad y = \sum_{i=0}^{2}\ell_i(y)v_i.$$

For a dense-torus point, the character formula gives

$$\log|z_i| = \ell_i(y)\log|q|. \tag{2.47}$$

On the chosen compact chart piece, choose $R \geq 1$ such that $|z_i| \leq R$ for all i, and set

$$c = \frac{\log R}{|\log \varepsilon|}.$$

For every $0 < |q| < \varepsilon < 1$, Equation (2.47) gives

$$\ell_i(y) \geq -c, \qquad \ell_i(y) = 1 - \sum_{j \neq i} \ell_j(y) \leq 1 + 2c.$$

Consequently

$$\|y\| \leq (1 + 2c) \sum_{i=0}^{2} \|v_i\|.$$

Taking the maximum over the finite chart cover proves the lemma. □

Proposition 2.14. *The $\mathbf{Z}^2$-action (2.42) on $Y_\varepsilon \times \Delta$ is free.*

Proof. Suppose first that $q \neq 0$ and $\Psi_\lambda(x, q, a) = (x, q, a)$. Equation (2.46) gives $B_{q,a}\lambda = 0$. By Lemma 2.12, $\lambda = 0$.

Now let $q = 0$. Torus orbits in the central fibre correspond to cells P of the height-one triangulation. The toric automorphism Φ_λ sends the orbit of P to the orbit of $P + B_0\lambda$. The additional torus translation $c_{a,\lambda}(0)$ preserves each torus orbit. Therefore a fixed point would require

$$P + B_0\lambda = P.$$

Every cell P is bounded, so it is invariant under a translation only when the translation vector is zero. Since B_0 is invertible, $\lambda = 0$. □

Proposition 2.15. *The $\mathbf{Z}^2$-action on $Y_\varepsilon \times \Delta$ is properly discontinuous.*

Proof. Let $K_1, K_2 \subset Y_\varepsilon \times \Delta$ be compact. We must show that only finitely many $\lambda \in \mathbf{Z}^2$ satisfy

$$\Psi_\lambda(K_1) \cap K_2 \neq \varnothing.$$

First consider intersections represented by points with $q \neq 0$. The projections of K_1, K_2 to Y_ε are compact, so Lemma 2.13 gives constants M_1, M_2 bounding the logarithmic coordinates of the source and target points. If $p \in K_1$ and $\Psi_\lambda(p) \in K_2$, then (2.46) gives

$$\lambda = B_{q,a}^{-1}(y(\Psi_\lambda p) - y(p)).$$

By Lemma 2.12,

$$\|\lambda\| \leq K(M_1 + M_2).$$

Only finitely many integer vectors satisfy this bound. Notice that this argument remains valid for sequences $q \neq 0$ tending to 0; that is exactly the content of Lemma 2.13.

It remains to consider intersections lying in $q = 0$. For $i = 1, 2$, let

$$K_i^0 = \mathrm{pr}_{Y_\varepsilon}\big(K_i \cap (q^{-1}(0) \times \Delta)\big).$$

Each K_i^0 is compact. A finite collection of affine toric charts covers each K_i^0, and every chosen chart contains only finitely many torus orbits, corresponding to the faces of its cone. Hence there are finite sets of cells $\mathcal{S}_1, \mathcal{S}_2$ such that K_i^0 meets only the orbits associated with cells in $\mathcal{S}_i$. If an orbit meeting K_1^0 is sent to one meeting K_2^0, then

$$P + B_0\lambda = P'$$

for some $P \in \mathcal{S}_1$ and $P' \in \mathcal{S}_2$. For each pair (P, P') there is at most one λ, because B_0 is invertible. Thus only finitely many λ occur on the central fibre as well. □

Lemma 2.16. *If $K_y \subset \mathbf{R}^2$ is bounded and $0 < r < \varepsilon$, then the closure in Y of*

$$\mathcal{S}(K_y, r) = \{(x, q) \in (\mathbf{C}^*)^2 \times \mathbf{C}^* : |q| \leq r,\ y(x, q) \in K_y\}$$

is compact.

Proof. It suffices to replace K_y by $\overline{K_y}$, since $\mathcal{S}(K_y, r) \subset \mathcal{S}(\overline{K_y}, r)$. We may therefore assume that K_y is compact. It meets only finitely many triangles of the locally finite triangulation. For each $y \in K_y$, choose a triangle $P = \mathrm{conv}\{v_0, v_1, v_2\}$ containing y. In the corresponding affine chart, the barycentric functions satisfy $\ell_i(y) \geq 0$ and $\sum_i \ell_i(y) = 1$. Equation (2.47) gives

$$|z_i| = |q|^{\ell_i(y)} \leq 1.$$

Thus the part of $\mathcal{S}(K_y, r)$ associated with P lies in the closed unit polydisc of the chart $U_P \cong \mathbf{C}^3$. Only finitely many triangles are needed, so the closure is contained in a finite union of compact closed polydiscs. It is closed in that finite union and is therefore compact. □

Proposition 2.17. *Let $K \subset D_\varepsilon \times \Delta$ be compact. There exists a compact set*

$$C_K \subset (q, a)^{-1}(K) \subset Y_\varepsilon \times \Delta$$

meeting every $\mathbf{Z}^2$-orbit over K.

Proof. Choose $r < \varepsilon$ and a compact set $K_a \Subset \Delta$ such that $K \subset \overline{D}_r \times K_a$. For $0 < |q| \leq r$ and $a \in K_a$, set

$$\mathcal{P}_{q,a} = B_{q,a}([-1/2, 1/2]^2).$$

By Lemma 2.12, all these parallelograms lie in one bounded set $K_y \subset \mathbf{R}^2$.

Let (x, q, a) have $q \neq 0$. Choose $\lambda \in \mathbf{Z}^2$ such that

$$B_{q,a}^{-1} y(x, q) + \lambda \in [-1/2, 1/2]^2.$$

Then (2.46) gives

$$y(\Psi_\lambda(x, q, a)) = B_{q,a}\big(B_{q,a}^{-1} y(x, q) + \lambda\big) \in K_y.$$

Therefore the compact set

$$C^* = \overline{\mathcal{S}(K_y, r)} \times K_a$$

meets every orbit with $q \neq 0$ over the larger box $\overline{D}_r \times K_a$, by Lemma 2.16.

Now consider $q = 0$. Let a point lie in the orbit corresponding to a cell P. Choose a vertex v of P. Since $B_0 \in \mathrm{GL}(2, \mathbf{Z})$, there is a unique $\lambda \in \mathbf{Z}^2$ with $B_0\lambda = -v$. The translated cell $P + B_0\lambda$ contains 0. Hence the translated torus orbit is contained in the component E_0 of the central fibre. The torus factor $c_{a,\lambda}(0)$ preserves that orbit and therefore does not alter this conclusion. Thus the compact set $E_0 \times K_a$ meets every central-fibre orbit.

Set

$$C_{\mathrm{box}} = C^* \cup (E_0 \times K_a).$$

It is compact and meets every orbit over $\overline{D}_r \times K_a$. Because the group action preserves (q, a), the set

$$C_K = C_{\mathrm{box}} \cap (q, a)^{-1}(K)$$

is compact and still meets every orbit over K. □

Lemma 2.18. *Let*

$$D_Y = q^{-1}(0) \subset Y.$$

Then D_Y is reduced, and its normalization is the disjoint union

$$\nu_Y : \coprod_{v \in \mathbf{Z}^2} E_v \longrightarrow D_Y \tag{2.48}$$

of its smooth irreducible toric components.

Proof. In a maximal toric chart the central equation is $q = z_0 z_1 z_2$; at a point where exactly $r \in \{1, 2, 3\}$ components meet, the germ of D_Y is $\{z_0 \cdots z_{r-1} = 0\}$. Thus D_Y is reduced and its local branches are smooth coordinate hyperplanes. By the branch description and uniqueness of analytic normalization [13, Proposition 8.4.1 and Theorem 8.4.2], its local normalization is the disjoint union of these branches. On chart overlaps the branches are the restrictions of the globally defined toric components E_v, so the local normalizations glue to (2.48). □

Proposition 2.19. *Fix $a \in \Delta$ and set*

$$\widetilde{D}_Y = \coprod_{v \in \mathbf{Z}^2} E_v.$$

The action of $\mathbf{Z}^2$ on D_Y lifts to a free properly discontinuous action by biholomorphisms on $\widetilde{D}_Y$. Its analytic orbit space is biholomorphic to E_0.

Proof. Each biholomorphism $\Psi_\lambda|_{D_Y}$ sends E_v to $E_{v+B_0\lambda}$ and lifts uniquely to the normalization. The uniqueness of these lifts gives the group law. Since $B_0 \in \mathrm{GL}(2,\mathbf{Z})$, the map

$$\mathbf{Z}^2 \times E_0 \longrightarrow \widetilde{D}_Y, \qquad (\lambda, x) \longmapsto \widetilde{\Psi}_{\lambda,a}(x)$$

is a biholomorphism, with $\mathbf{Z}^2$ discrete. It identifies the lifted action with translation on the first factor. The action is therefore free and properly discontinuous, and its analytic quotient is E_0. □

Theorem 2.20. *The quotient*

$$\mathcal{N}_{0,\Delta} = (Y_\varepsilon \times \Delta)/\mathbf{Z}^2$$

is a connected Hausdorff second-countable complex manifold, and every parameter slice of $\mathcal{N}_{0,\Delta} \to \Delta$ is connected. The induced map

$$F_0 : \mathcal{N}_{0,\Delta} \longrightarrow D_\varepsilon \times \Delta$$

is proper, and the projection to Δ is a holomorphic submersion. The inverse image

$$W_\Delta := F_0^{-1}(\{0\} \times \Delta)$$

is a reduced normal-crossing divisor in $\mathcal{N}_{0,\Delta}$. For every $a \in \Delta$, the fibre

$$W_a := F_0^{-1}(0, a)$$

is a compact, reduced, irreducible, nonnormal surface with normal-crossing singularities, and its normalization is isomorphic to the compact toric surface E_0.

Proof. Freeness and proper discontinuity are Propositions 2.14 and 2.15; hence Lee's holomorphic quotient theorem [19, Theorem 1.16] makes the quotient a complex manifold. The dense torus of Y meets Y_ε in

$$(\mathbf{C}^*)^2 \times D_\varepsilon^*,$$

which is connected. This subset is dense in Y_ε, so Y_ε is connected as the closure of a connected subset. The product $Y_\varepsilon \times \Delta$ is therefore connected, and its continuous surjective quotient $\mathcal{N}_{0,\Delta}$ is connected. For fixed $a \in \Delta$, the slice upstairs is the connected space Y_ε, and its quotient by the restricted $\mathbf{Z}^2$-action is the corresponding parameter slice of $\mathcal{N}_{0,\Delta}$; hence that slice is connected as well. For a compact set $K \subset D_\varepsilon \times \Delta$, Proposition 2.17 supplies a compact set C_K meeting every orbit over K. Its image in the quotient is exactly $F_0^{-1}(K)$, which is therefore compact. Thus F_0 is proper. The parameter projection upstairs is the ordinary projection from $Y_\varepsilon \times \Delta$, hence is a submersion. Since the quotient map is locally biholomorphic, the descended map is also a submersion.

In every maximal toric chart, the equation $q = 0$ is $z_0 z_1 z_2 = 0$ by (2.34). The quotient map is locally biholomorphic, so the same equation describes

$$W_\Delta = F_0^{-1}(\{0\} \times \Delta).$$

Hence W_Δ is a reduced normal-crossing divisor in the four-dimensional manifold $\mathcal{N}_{0,\Delta}$. Fix $a \in \Delta$. Intersecting these local charts with the parameter slice $a =$ constant shows that $W_a = F_0^{-1}(0,a)$ is a reduced normal-crossing complex surface. It is compact because F_0 is proper. Set $D_Y = q^{-1}(0)$.

By Lemma 2.18, the normalization of D_Y is $\nu_Y : \widetilde{D}_Y = \coprod_v E_v \to D_Y$. The lifted action in Proposition 2.19 gives

$$\widetilde{D}_Y/\mathbf{Z}^2 \xrightarrow{\cong} E_0. \tag{2.49}$$

The equivariant map ν_Y, followed by the covering $p : D_Y \to W_a$, therefore induces a holomorphic map

$$\nu_a : E_0 \longrightarrow W_a. \tag{2.50}$$

It is surjective because every component E_v has a translate equal to E_0.

Choose a sufficiently small evenly covered open set $U \subset W_a$ and one sheet $\widetilde{U} \subset D_Y$, so that

$$p^{-1}(U) = \coprod_{\lambda \in \mathbf{Z}^2} \Psi_\lambda(\widetilde{U}).$$

Every orbit in the normalization over U has a unique representative over $\widetilde{U}$. Hence the lifted action identifies

$$\nu_a^{-1}(U) \simeq \nu_Y^{-1}(\widetilde{U}),$$

and under this identification ν_a is the local normalization map $\nu_Y^{-1}(\widetilde{U}) \to \widetilde{U} \simeq U$ from Lemma 2.18. These identifications show that ν_a is the normalization of W_a. Since E_0 is irreducible and ν_a is surjective, W_a is irreducible.

Finally, the triangulation has edges. At a point of the toric orbit corresponding to an edge, the local equation of the central fibre is $z_0 z_1 = 0$. The quotient map is locally biholomorphic, so W_a contains a nonempty double-curve stratum with the same local equation. The local normalization has two points over a general point of this stratum, and therefore ν_a is not an isomorphism. Thus W_a is nonnormal. □

On the cusp horodisc, use the adapted fibre coordinates in which Λ_{tor} is represented by the last two columns of the period matrix. Set

$$E_{\exp}(z, \zeta, a) = \big(\mathrm{e}^{2\pi \mathrm{i}\zeta_1}, \mathrm{e}^{2\pi \mathrm{i}\zeta_2}, q(z), a\big). \tag{2.51}$$

Translations by Λ_{tor} disappear under the first two exponentials. For $\lambda \in \Lambda$, let $\bar{\lambda}$ denote its image in $\overline{\Lambda} \cong \mathbf{Z}^2$. The decomposition (2.32) gives

$$E_{\exp}(z, \zeta + \Omega_a(z)\lambda, a) = \Psi_{\bar{\lambda}}(E_{\exp}(z, \zeta, a)). \tag{2.52}$$

Under the cusp deck transformation the adapted vertical coordinate is unchanged because $R_{g_0} = I_2$, while $q(g_0 z) = q(z)$. Hence $E_{\exp}(g_0 z, \zeta, a) = E_{\exp}(z, \zeta, a)$. Equation (2.33) is the corresponding compatibility of the period lattice with this deck action.

Proposition 2.21. *For the disc $D_0 = D_\varepsilon$ and the smaller horodisc $\mathcal{H}_{0,\varepsilon}$ defined in (2.38), Equation (2.51) descends to a jointly holomorphic biholomorphism*

$$G_0 : \mathcal{J}_\Delta|_{D_0^* \times \Delta} \xrightarrow{\cong} \mathcal{N}_{0,\Delta}|_{D_0^* \times \Delta}. \tag{2.53}$$

Proof. We first identify the relevant part of the global core. Let

$$\widetilde{\mathcal{J}}^*_{0,\Delta} = \widetilde{\mathcal{J}}_\Delta\big|_{\mathcal{H}_{0,\varepsilon} \times \Delta}.$$

If $g\widetilde{\mathcal{J}}^*_{0,\Delta} \cap \widetilde{\mathcal{J}}^*_{0,\Delta} \neq \varnothing$, then the projections to U° give

$$g\mathcal{H}_{0,\varepsilon} \cap \mathcal{H}_{0,\varepsilon} \neq \varnothing.$$

The inherited separation property proved immediately after (2.39) therefore implies $g \in \langle g_0 \rangle$. Thus $\widetilde{\mathcal{J}}^*_{0,\Delta}$ is precisely invariant under $\langle g_0 \rangle$. Since the orbit maps are open, Lee's local quotient charts [19, Theorem 1.16], together with (2.39), give

$$\widetilde{\mathcal{J}}^*_{0,\Delta}/\langle g_0 \rangle \xrightarrow{\cong} \mathcal{J}_\Delta|_{D_0^* \times \Delta}. \tag{2.54}$$

Thus it remains only to identify the quotient on the left with the punctured toric quotient.

First quotient $\mathbf{C}^2$ by the invariant lattice $\Lambda_{\mathrm{tor}} = \mathbf{Z}\langle \widehat{w}, \widehat{\delta} \rangle$. In the adapted coordinates its two generators are the last two columns of Ω_a, so the map

$$(\zeta_1, \zeta_2) \longmapsto (\mathrm{e}^{2\pi \mathrm{i}\zeta_1}, \mathrm{e}^{2\pi \mathrm{i}\zeta_2})$$

identifies that quotient with $(\mathbf{C}^*)^2$. The remaining quotient lattice is $\overline{\Lambda} \cong \mathbf{Z}^2$, and (2.52) says exactly that its action is (2.43). Thus $E_{\exp}$ induces a bijection after the two lattice quotients are taken.

The cusp deck transformation leaves q and the exponential fibre coordinates unchanged, so taking its quotient replaces the additive coordinate s by the punctured-disc coordinate $q = \mathrm{e}^{2\pi \mathrm{i}s}$. To construct the inverse near a point of the quotient, choose local logarithms of the two nonzero torus coordinates and a local logarithm of q. Dividing the first two logarithms by $2\pi\mathrm{i}$ gives ζ, and dividing the third by $2\pi\mathrm{i}$ gives a lift s of the base point. Changing a fibre logarithm adds a generator of Λ_{tor}; changing the logarithm of q changes s by an integer and therefore changes the chosen point of $\mathcal{H}_{0,\varepsilon}$ by a power of the deck transformation g_0. A different representative under the residual $\mathbf{Z}^2$ action changes ζ by the remaining periods, by (2.52). Hence the local inverse is independent of all choices after passing to the quotients. The local logarithms are holomorphic, so this inverse is holomorphic. Both directions are jointly holomorphic in a. Combining this identification with (2.54) proves (2.53). □

2.3. **Global compactification and smooth type.** For a fixed parameter, this gluing construction and its independence of the auxiliary choices are due to Alpöge [1, Construction 6.1, Theorem 6.2, and Proposition 6.3]. We carry out the same gluing over Δ and record the compatibility as the parameter disc and the common cusp cutoff vary.

Choose the three discs $D_0, D_1, D_2 \subset B$ with pairwise disjoint closures. The four base opens

$$B^\circ, \quad D_0, \quad D_1, \quad D_2$$

cover B, and the only nonempty overlaps involving a filling disc are $D_j^* = D_j \cap B^\circ$.

Glue the disjoint union

$$\mathcal{J}_\Delta \sqcup \mathcal{N}_{0,\Delta} \sqcup \mathcal{N}_{1,\Delta} \sqcup \mathcal{N}_{2,\Delta}$$

by G_0 from (2.53) and by G_j from (2.28). Since the filling discs are disjoint, there are no nontrivial triple overlaps among three different pieces; hence the cocycle condition is automatic once the three displayed maps are fixed.

Lemma 2.22. *The glued topological space $\mathfrak{X}_\Delta$ is Hausdorff and second-countable. The quotient charts determine a unique complex manifold structure on it.*

The filling discs are disjoint, so every gluing class is either a singleton or a two-point class joining the core to one filling. Each piece therefore embeds openly, and disjoint neighbourhoods inside one piece have disjoint open saturations. The complete verification is given in Appendix A.

The local maps glue to a holomorphic map

$$\mathfrak{f} : \mathfrak{X}_\Delta \longrightarrow B \times \Delta. \tag{2.55}$$

Theorem 2.23. *The map (2.55) is proper. The parameter projection*

$$\pi = \mathrm{pr}_\Delta \circ \mathfrak{f} : \mathfrak{X}_\Delta \longrightarrow \Delta$$

is a proper holomorphic submersion. Its fibre over a is the compactification obtained from Ω_a by the two logarithmic fillings and the cusp filling with the chosen valid cusp cutoff.

Proof. On the four members of the base open cover, the restrictions of $\mathfrak{f}$ are p°, F_0, F_1, F_2, all of which are proper by Propositions 2.4 and 2.6 and Theorem 2.20. The base $B \times \Delta$ is metrizable, and the local properness just proved shows that $\mathfrak{f}$ is proper.

Let $K \subset \Delta$ be compact. Since B is compact, $B \times K$ is compact. Hence

$$\pi^{-1}(K) = \mathfrak{f}^{-1}(B \times K)$$

is compact, so π is proper.

On the core, the parameter projection is a submersion by Proposition 2.4. On the finite fillings, it is a submersion by Proposition 2.6; this remains true at the multiple central fibres because the free quotient map is locally biholomorphic and the parameter a is unchanged by the group action. On the cusp filling, it is a submersion by Theorem 2.20; this remains true at the normal-crossing fibre because upstairs the total space is the smooth product $Y_\varepsilon \times \Delta$ and the action is free. These local parameter maps agree under the gluing biholomorphisms, so π is a holomorphic submersion everywhere.

All group actions preserve a, and all gluing maps are defined by formulas whose restriction at a fixed parameter is the corresponding fixed-parameter formula. Thus the fibre over a is the compactification made with the present parameter disc and the present cutoff. Its independence of these choices is established in Proposition 2.24. □

Proposition 2.24. *The relative compactification has the following compatibility properties.*

(i) *Fix a relatively compact parameter disc $\Delta \Subset \mathcal{A}$. If $0 < \varepsilon' < \varepsilon$ are two valid cusp cutoffs for Δ in the sense of Definition 2.9, then the two relative compactifications are canonically biholomorphic over $B \times \Delta$. The biholomorphism is the identity on the core and on the two finite fillings.*

(ii) *Let $\Delta_1, \Delta_2 \Subset \mathcal{A}$ be parameter discs, with any valid cusp cutoffs in the sense of Definition 2.9 chosen in the two constructions. Over $\Delta_1 \cap \Delta_2$, their relative compactifications are canonically biholomorphic. These identifications satisfy the cocycle identity on triple intersections.*

Consequently, for every $a \in \mathcal{A}$, the fibre together with its distinguished map to B is independent, up to the canonical biholomorphisms over B constructed above, of the parameter disc and of the cusp cutoff.

Nested valid cutoffs give invariant open embeddings of cusp models, and the punctured identifications are restrictions of the same exponential formula. Two parameter discs are compared locally using one smaller common cutoff; the restriction identities make these comparisons independent of that cutoff and give the cocycle law. The complete verification is given in Appendix A.

Definition 2.25. For $a \in \mathcal{A}$, let (X_a, f_a) be the fibre pair obtained from any relatively compact disc $\Delta \Subset \mathcal{A}$ containing a and any valid cusp cutoff, whose existence follows from Lemma 2.10. Proposition 2.24 identifies all such choices by canonical biholomorphisms over B satisfying the cocycle law. We use (X_a, f_a) for any fixed representative.

Every X_a is connected: the core and each of the three connected fillings meet along a nonempty punctured annulus. Moreover, X_a is compact and $f_a(X_a)$ contains the dense open set B°; hence $f_a : X_a \to B$ is surjective.

Corollary 2.26. *For any two admissible parameters $a, a' \in \mathcal{A}$, the fixed-parameter compactifications X_a and $X_{a'}$ are diffeomorphic.*

Proof. Choose a disc $\Delta \Subset \mathcal{A}$ containing a and a'. The projection of the resulting relative compactification is a proper holomorphic submersion by Theorem 2.23; hence its fibres over a and a' are diffeomorphic by Ehresmann's theorem [7, pp. 154–155]. The comparison maps of Proposition 2.24 and Definition 2.25 identify these fibres with the fixed representatives X_a and $X_{a'}$. □

Lemma 2.27. *Let $(X^{\mathrm{A}}, f^{\mathrm{A}})$ be the compact pair of Alpöge* [1, Construction 6.1]*, with its specified twists $v_1 = \widehat{\gamma} + 2\widehat{u} - 4\widehat{w}$ and $v_2 = -\widehat{\gamma} - 3\widehat{u} + 3\widehat{w}$, and untranslated cusp gluing. For its chosen admissible period function β^{A}, there is a unique constant $a_{\mathrm{A}} \in \mathcal{A}$ for which $\beta^{\mathrm{A}} = \beta_0 + a_{\mathrm{A}}$. For this parameter there is an isomorphism of pairs*

$$(X^{\mathrm{A}}, f^{\mathrm{A}}) \cong (X_{a_{\mathrm{A}}}, f_{a_{\mathrm{A}}}) \quad \text{over } B.$$

Proof. Use the same oriented $(3, 4, \infty)$ triangle and the same affine normalization of its quotient: the order-three, order-four, and cusp points are $0, 1, \infty$, respectively. This identifies the marked coverings and their generators g_1, g_2, g_0. The integral matrices (2.2) are Alpöge's dual monodromy matrices in the ordered basis $(\widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta})$. His τ and μ have the generator laws (2.9) and (2.10), and the same cusp normalization as ours. The uniqueness assertions in [1, Theorem 3.4(i),(ii)] identify them. His β^{A} and our β_0 satisfy (2.11), and both $\beta + \tau$ extend across the cusp. Their difference descends to a holomorphic function on $B = \mathbf{P}^1$, hence is a constant a_{A}:

$$\beta^{\mathrm{A}} = \beta_0 + a_{\mathrm{A}}, \qquad \Pi^{\mathrm{A}} = \Omega_{a_{\mathrm{A}}}.$$

This also fixes the vertical matrices R_g, since they are determined by the same period matrix and integral monodromy.

Alpöge's admissibility condition is $\mathscr{D}_{a_{\mathrm{A}}}(z) < 0$ for every $z \in U$. By Proposition 2.1, this invariant function is continuous on $B \setminus \{p_0\}$ and tends to $-\infty$ at p_0; it therefore attains a strictly negative maximum. Since $\mathscr{D}_{a_{\mathrm{A}}} = \mathscr{D}_0 + \Im a_{\mathrm{A}}$, we obtain $M_0 + \Im a_{\mathrm{A}} < 0$, or $a_{\mathrm{A}} \in \mathcal{A}$.

The four local pieces now agree at a_{A}. On the smooth core they are quotients of the same marked period family by the same deck action. At p_j, the vectors in (2.21) equal the two source vectors; hence the finite actions (2.24) and the logarithmic translations (2.27) agree with Alpöge's finite fillings and gluing maps. At the cusp, both constructions use the triangles $\{v, v + e_1, v + e_2\}$ and $\{v + e_1, v + e_2, v + e_1 + e_2\}$ for $v \in \mathbf{Z}^2$, the same $B_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and the same decomposition $Z_{a_{\mathrm{A}}} = sB_0 + C_{a_{\mathrm{A}}}(q)$. Thus the toric lattice action (2.42) and the exponential gluing (2.51) agree. In particular, neither construction inserts a fibre translation at the cusp. In the discrete notation of [1, Section 6.4], these facts say $(\ell_0, \ell_1, \ell_2) = (0, 1, -1)$, with the last two values given by (2.23).

Take the finite linearizing discs and the cusp cutoff to be common smaller choices in the two constructions. On these choices, the identity on each of the four pieces respects every gluing map, hence descends to a biholomorphism of pairs over B. Restriction to smaller discs removes only punctured collars already covered by the core. Changing a finite logarithm branch changes its fibre translation by an integral period; changing the cusp logarithm changes C by an integral multiple of B_0, whose exponential action is unchanged. Thus the comparison is independent of these choices, consistently with [1, Proposition 6.3]. The intrinsic representative $X_{a_{\mathrm{A}}}$ is independent of its parameter disc and cusp cutoff by Proposition 2.24 and Definition 2.25, giving the stated isomorphism. □

Corollary 2.28. *Every compact threefold X_a in the admissible family is diffeomorphic to S^6.*

Proof. By Lemma 2.27, X_{a_Δ} is Alpöge's threefold, which is diffeomorphic to S^6 by [1, Theorem 8.1]. The smooth type is constant throughout $\mathcal{A}$ by Corollary 2.26. □

3. Cyclic coverings and Oka maps

An open subset is called analytic-Zariski open if its complement in the stated ambient manifold is a closed analytic subset.

For the family in Theorem 2.23, set

$$\mathcal{S}_j = (\mathfrak{f}_\Delta^{-1}(\{p_j\} \times \Delta))_{\mathrm{red}}, \qquad \mathcal{U}_j = \mathfrak{X}_\Delta \setminus \mathcal{S}_j.$$

Here $(\cdot)_{\mathrm{red}}$ denotes the underlying reduced complex space. The two complements form an analytic-Zariski cover of $\mathfrak{X}_\Delta$. The normalized base maps are

$$\rho_4(b) = \frac{1}{1-b^4} : \mathbf{C}_b \longrightarrow \mathbf{P}^1 \setminus \{0\}, \qquad \rho_3(b) = \frac{b^3}{b^3-1} : \mathbf{C}_b \longrightarrow \mathbf{P}^1 \setminus \{1\}.$$

For $(m, j) = (4, 1)$ or $(3, 2)$, let $\mathcal{M}_m$ be the normalization of the following fibre product in the category of reduced complex spaces:

$$\mathcal{U}_j \times_{(\mathbf{P}^1 \setminus \{p_j\}) \times \Delta} (\mathbf{C}_b \times \Delta).$$

There are maps

$$r_m : \mathcal{M}_m \longrightarrow \mathcal{U}_j, \qquad h_m : \mathcal{M}_m \longrightarrow \mathbf{C}_b \times \Delta. \tag{3.1}$$

The root base change removes the remaining multiple fibre. The local model from Section 2 is the quotient of $t = s^m$ by a free affine cyclic action of order m, so r_m is a finite unramified holomorphic covering of degree m and $\mathcal{M}_m$ is smooth.

Lemma 3.1. *For $m = 3, 4$ there is a connected regular unramified holomorphic covering*

$$q_m : Q_m \longrightarrow \mathcal{M}_m$$

with deck group $\mathbf{Z}$, whose restriction to a smooth torus fibre is the covering corresponding to the subgroup $\Lambda_0 = \ker \gamma$ of the period lattice. For every $a_* \in \Delta$ there is a biholomorphism

$$Q_m \simeq Q_{m,a_*} \times \Delta \tag{3.2}$$

over $\mathbf{C}_b \times \Delta$, where $Q_{m,a_*} = Q_m|_{a_*}$.

Proof. On an ordinary or normalized finite-root chart, replacing Λ by Λ_0 gives the required local cyclic cover. Its periods are independent of a by (2.14). On a cusp chart the same cover is

$$(Y_\varepsilon / \langle \Psi_{(0,1)} \rangle) \times \Delta \longrightarrow (Y_\varepsilon \times \Delta)/\mathbf{Z}^2, \tag{3.3}$$

because $E_{21}(0, n)^t = 0$. All transition maps preserve Λ_0 and the positive generator of Λ/Λ_0. Their lifting obstruction is therefore a constant $\mathbf{Z}$-valued Čech 2-cocycle on $\mathbf{C}_b$, which vanishes since $H^2(\mathbf{C}, \mathbf{Z}) = 0$. The local covers consequently glue to the connected regular covering q_m.

It remains to remove the parameter dependence of the lifted transitions. The ordinary periods restricted to Λ_0 are independent of a. In the finite-root charts, the logarithmic translations in Lemma 2.7 vary linearly with a along e_2; on the cusp charts the retained subgroup satisfies $E_{21}(0, n)^t = 0$. The lifted transitions therefore commute with the action Θ in (2.14), which extends across the fillings. After choosing logarithms on the overlaps, the transition at a differs from the transition at a_* by

$$\Theta_{\exp(2\pi \mathrm{i}(a-a_*)h_{\alpha\beta}(b))}, \qquad h_{\alpha\beta} \in \mathcal{O}(V_\alpha \cap V_\beta),$$

where $h_{\alpha\beta}$ is independent of a. The transition identities and the effectiveness of Θ give

$$\exp\bigl(2\pi \mathrm{i}(a-a_*)(h_{\alpha\beta} + h_{\beta\gamma} - h_{\alpha\gamma})\bigr) = 1 \qquad \text{for every } a \in \Delta.$$

It follows that $h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}$. Cartan's Theorem B now gives $h_{\alpha\beta} = b_\alpha - b_\beta$. Rescaling the local trivializations by $\Theta_{\exp(-2\pi \mathrm{i}(a-a_*)b_\alpha)}$ makes every transition independent of a and proves (3.2). □

For a fixed $a_* \in \mathcal{A}$, the cyclic cover Q_{m,a_*} contains a conic band whose base is an elliptic surface. In this band construction, write z for the root coordinate b in (3.1), in agreement with the elliptic-surface equations below.

Lemma 3.2. *For $m = 3, 4$, the cyclic covering Q_{m,a_*} of Lemma 3.1 has an analytic-Zariski open subset Y_m. There is a smooth analytic surface $\mathcal{E}_m^{\mathrm{an}}$ proper over $\mathbf{C}_z$, with precisely the m nodal fibres $z^m = 1$, and a holomorphic line bundle $L_m \to \mathcal{E}_m^{\mathrm{an}}$ for which Y_m is a proper small resolution of*

$$H_m = \{A \otimes B = \Delta_m(z)\} \subset L_m \oplus L_m^{-1}, \qquad \Delta_3 = z^3 - 1, \quad \Delta_4 = 1 - z^4.$$

Its exceptional locus consists of one complete $\mathbf{P}^1$ over the node of each nodal fibre.

Proof. For $(m, j) = (3, 2), (4, 1)$, write $U_j = X_{a_*} \setminus S_{j,a_*}$. The normalized root maps $\rho_3(z) = z^3/(z^3 - 1)$ and $\rho_4(z) = 1/(1 - z^4)$ give the finite unramified covers $N_m = M_{m,a_*} \to U_j$ of degree m from Equation (3.1). The further cover $Q_{m,a_*} \to N_m$ retains precisely

$$\Lambda_0 = \ker \gamma = \mathbf{Z}\langle \widehat{u}, \widehat{w}, \widehat{\delta} \rangle.$$

Over an ordinary base chart its fibre is

$$\mathbf{C}^2/\mathbf{Z}\langle (\tau, \mu), (1, 0), (0, 1) \rangle. \tag{3.4}$$

At a cusp $z = \zeta$, $\zeta^m = 1$, the local model is $Y_\varepsilon/\langle \Psi_{(0,1)} \rangle$, as in Equation (3.3). Write its torus coordinates as (x_1, x_2, q). Since $B_0(0, 1)^t = (1, 0)^t$, the generator acts by

$$(x_1, x_2, q) \longmapsto \big(q\alpha(q)x_1, \beta(q)x_2, q\big), \tag{3.5}$$

where α and β are holomorphic units on the full cusp disc. The other deck generator has $B_0(1, 0)^t = (0, -1)^t$, and hence shifts the second, or row, index by one.

Step 1: the elliptic quotient. The fan of Proposition 2.8 has vertices $(i, j, 1)$. Projection

$$(i, j, r) \longmapsto (i, r) \tag{3.6}$$

takes its cones into those of the one-dimensional Tate fan. On the chart between columns i and $i + 1$, write

$$u_i = x_1 q^{-i}, \qquad v_i = x_1^{-1} q^{i+1}, \qquad u_i v_i = q. \tag{3.7}$$

The projected column transformation T is induced on the dense torus by

$$T(x_1, q) = (\vartheta(q)x_1, q), \qquad \vartheta(q) = q\alpha(q). \tag{3.8}$$

It sends the i-th node chart to the $(i + 1)$-st by

$$(u_i, v_i) \longmapsto (u_{i+1}, v_{i+1}) = (\alpha(q)u_i, \alpha(q)^{-1}v_i). \tag{3.9}$$

After shrinking the cusp disc, $t = \vartheta(q) = q\alpha(q)$ is a holomorphic coordinate with $|t| < 1$, since $\vartheta'(0) = \alpha(0) \neq 0$. The coordinate changes

$$\widetilde{u}_i = \alpha(q)^{-i}u_i, \qquad \widetilde{v}_i = \alpha(q)^{i+1}v_i, \qquad \widetilde{u}_i\widetilde{v}_i = t$$

are restrictions of $(x_1, q) \mapsto (x_1, t)$ on the dense torus, so they glue on the node charts. In these coordinates, T shifts the i-th chart to the $(i + 1)$-st while preserving $\widetilde{u}_i, \widetilde{v}_i$. Thus this is the standard analytic Tate construction with $b = 1$ [4, Subsection 2.1 and Remark 2.2]. Its column action is free and properly discontinuous, and the quotient is a Hausdorff complex surface with one irreducible nodal rational central fibre.

The quotient is proper over a smaller closed cusp disc. Indeed, over a compact subset of that disc, the closure in the toric model of the annular region

$$|\vartheta(q)| \leq |x_1| \leq 1$$

is contained in the union of two adjacent node charts and has compact closure: indeed, $|x_1| \leq 1$ bounds $u_0 = x_1$, while $|x_1| \geq |q\alpha(q)|$ bounds $v_0 = q/x_1$ because α is a unit. This closure meets every column orbit over $q \neq 0$ and every limiting orbit over $q = 0$. Its image in the column quotient is exactly the inverse image of the chosen compact base set and is compact. This proves properness. The local equation $u_i v_i = q$ shows that the total space is smooth.

On the ordinary charts the free x_2-scaling quotient of (3.4) is the elliptic curve $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$. The transitions used to construct Q_{m,a_*} commute with this scaling and their first coordinate is independent of x_2; the cusp transitions induce (3.7). They therefore glue the ordinary quotients and the nodal cusp quotients to a smooth surface $p_m : \mathcal{E}_m^{\mathrm{an}} \to \mathbf{C}_z$. Properness is local on the target and follows on the ordinary and cusp charts as just checked. Its fibres are connected; the only singular fibres occur at $z^m = 1$, each has one node and multiplicity one. In particular the fibration is relatively minimal.

At each cusp retain the central toric divisors in rows $j = 0, 1$ and delete those in every other row. Retain all noncentral points, and call the resulting subset of Q_{m,a_*} Y_m. The deleted divisors form a locally finite analytic

family before the column quotient. The quotient action only translates columns, so they remain locally finite after quotient; there are only finitely many cusps. Their union is consequently closed analytic. Thus Y_m is analytic-Zariski open in Q_{m,a_*}.

Step 2: descent of the line coordinate. On the two retained rows the characters

$$A = x_2, \qquad B^{\mathrm{tor}} = q/x_2 \tag{3.10}$$

have nonnegative pairing with every retained ray and hence extend holomorphically. Over $\Delta_m(z) \neq 0$, the free x_2-scaling gives a principal $\mathbf{C}^*$-bundle. Its associated line bundle extends over the cusp charts as follows. Use $A = x_2$ as the local line coordinate over the ordinary quotient charts and the node charts (3.7). The scaling-equivariant transitions of Q_{m,a_*} read

$$A_\alpha = g_{\alpha\beta} A_\beta,$$

with $g_{\alpha\beta}$ a holomorphic unit on the corresponding overlap in the elliptic base. Ordinary-to-cusp overlaps lie in $q \neq 0$. Between adjacent column charts, the transition is the unit $\beta(q)$ from (3.5), which is holomorphic and nonzero on the full cusp disc. The transition identities for Q_{m,a_*} give $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ on every triple overlap. These units are consequently the transition functions of a holomorphic line bundle L_m on all of $\mathcal{E}_m^{\mathrm{an}}$.

The global discriminant determines the dual coordinate. On $A \neq 0$, this coordinate is

$$B = \frac{\Delta_m(z)}{A}.$$

Near a node, the quotient

$$\delta(z) = \frac{\Delta_m(z)}{q} \tag{3.11}$$

is a holomorphic unit, and hence $B = \delta(z) B^{\mathrm{tor}}$ extends across the retained toric charts. Since Δ_m is global and $A_\alpha = g_{\alpha\beta} A_\beta$, one has $B_\alpha = g_{\alpha\beta}^{-1} B_\beta$. Thus B is the fibre coordinate of L_m^{-1}, and the characters give a global holomorphic map to

$$H_m = \{A \otimes B = \Delta_m(z)\} \subset L_m \oplus L_m^{-1}.$$

Step 3: the global contraction. On a node chart, (3.7) and (3.11) give

$$AB = \delta u_i v_i = \Delta_m(z).$$

The two triangles in each unit square select the same ruling as the incidence resolution

$$\{v_i s = At,\ Bs = \delta u_i t\} \subset \{AB = \delta u_i v_i\} \times \mathbf{P}^1_{[s:t]}.$$

Its two affine charts are

$$v_i = Ap, \quad B = \delta u_i p, \qquad u_i = \delta^{-1} Br, \quad A = v_i r, \qquad r = 1/p. \tag{3.12}$$

On an overlap of node charts write

$$u_\alpha = \kappa_{\alpha\beta} u_\beta, \qquad v_\alpha = \kappa_{\alpha\beta}^{-1} v_\beta, \qquad A_\alpha = g_{\alpha\beta} A_\beta, \qquad B_\alpha = g_{\alpha\beta}^{-1} B_\beta,$$

where both $\kappa_{\alpha\beta}$ and $g_{\alpha\beta}$ are holomorphic units. For adjacent column charts, $\kappa_{\alpha\beta} = \alpha(q)$ and $g_{\alpha\beta} = \beta(q)$ by (3.9) and (3.5). The projective transition is

$$[s_\alpha : t_\alpha] = [s_\beta : (\kappa_{\alpha\beta} g_{\alpha\beta})^{-1} t_\beta]. \tag{3.13}$$

Indeed, the two incidence equations transform as

$$v_\alpha s_\alpha = \kappa_{\alpha\beta}^{-1} v_\beta s_\beta = A_\alpha t_\alpha, \qquad B_\alpha s_\alpha = g_{\alpha\beta}^{-1} B_\beta s_\beta = \delta u_\alpha t_\alpha.$$

Thus the node-coordinate and line-coordinate changes are both included, and the local incidence resolutions glue. Their affine charts are exactly the two retained toric rows in (3.12); away from the nodes the contraction is an isomorphism. Hence they give a global map $Y_m \to H_m$.

Near each of the finitely many nodes the incidence space is a closed analytic subspace of $H_m \times \mathbf{P}^1$, so its projection is proper; away from those neighbourhoods the contraction is an isomorphism. Properness is local on the target, and therefore $Y_m \to H_m$ is proper. Its fibre over each node is the complete $\mathbf{P}^1$ obtained from the two affine exceptional lines in (3.12). This proves the assertion. □

The proper elliptic fibration $p_m : \mathcal{E}_m^{\mathrm{an}} \to \mathbf{C}_z$ is relatively minimal and has no multiple fibres by Lemma 3.2. The section theorem of Buzzard and Lu [5, Lemma 3.8] gives a holomorphic section O.

The period data identify the elliptic surface underlying the conic band with an explicit algebraic model. The relevant affine hypersurfaces are Danielewski surfaces:

$$W_3 = \{uv = w(w + 2\mathrm{i})\}, \qquad z = (4u^2 - v)/3, \tag{3.14}$$

$$W_4 = \{uv = w(4w^2 - 3)\}, \qquad z = (u - v)/(2\mathrm{i}). \tag{3.15}$$

Lemma 3.3. *There are fibre-preserving biholomorphisms $\mathcal{E}_m^{\mathrm{an}} \simeq E_m^{\mathrm{an}}$, where E_m is the Weierstrass surface over $\mathbf{A}_z^1$*

$$E_3 : \ y^2 = 4x^3 - 3zx - 1, \qquad E_4 : \ y^2 = 4x^3 - 3x - z^2, \tag{3.16}$$

with its section at infinity included. The complement of this section in E_m is the affine Danielewski surface W_m displayed in (3.14) or (3.15), respectively.

Proof. With the section O, Nakayama's Weierstrass-model theorem [21, Theorem 2.1] gives a proper bimeromorphic morphism from $\mathcal{E}_m^{\mathrm{an}}$ to its Weierstrass model over $\mathbf{C}$. Every fibre is irreducible and meets O, so no fibre component is contracted and this morphism is an isomorphism. The fundamental line bundle is trivial over $\mathbf{C}$, so the model has a global equation

$$y^2 = 4x^3 - g_2(z)x - g_3(z), \qquad g_2, g_3 \in \mathcal{O}(\mathbf{C}). \tag{3.17}$$

Alpöge's normalization $j(\tau) = 1728t$, together with the root maps $t = \rho_3(z) = z^3/(z^3 - 1)$ and $t = \rho_4(z) = 1/(1 - z^4)$ from Equation (3.1), gives

$$j_3(z) = 1728\,\frac{z^3}{z^3 - 1}, \qquad j_4(z) = \frac{1728}{1 - z^4}. \tag{3.18}$$

For $m = 3$, g_3 cannot vanish at a smooth fibre, since j_3 there never equals 1728; at a nodal fibre both g_2 and g_3 are nonzero. It is therefore a holomorphic unit. Its sixth root exists on simply connected $\mathbf{C}$, and scaling (x, y) makes $g_3 = 1$. The formula

$$j = 1728\,\frac{g_2^3}{g_2^3 - 27g_3^2} \tag{3.19}$$

then gives $g_2^3 = 27z^3$; a remaining constant root-of-unity scaling makes $g_2 = 3z$. For $m = 4$, j_4 never vanishes at a smooth fibre, so g_2 is a unit. A fourth-root scaling makes $g_2 = 3$, after which (3.19) gives $g_3^2 = z^4$. The remaining constant choice makes $g_3 = z^2$. This proves (3.16). Its discriminants are $27(z^3 - 1)$ and $27(1 - z^4)$, as required.

The complement of $O = [0 : 1 : 0]$ in E_3 has coordinates

$$u = x, \quad w = y - \mathrm{i}, \quad v = 4x^2 - 3z, \qquad uv = w(w + 2\mathrm{i}), \quad z = (4u^2 - v)/3.$$

For $E_4 \setminus O$, the coordinates

$$u = y + \mathrm{i}z, \quad v = y - \mathrm{i}z, \quad w = x, \qquad uv = w(4w^2 - 3), \quad z = (u - v)/(2\mathrm{i})$$

give the asserted models. □

Write $\varrho_m : Y_m \to H_m$ for the small resolution in Lemma 3.2. Since the conifold points lie in $\{A = B = 0\}$, the principal open

$$Y_{m,A} := \varrho_m^{-1}(H_m \cap \{A \neq 0\}) \simeq L_m^{\times} \tag{3.20}$$

is a holomorphic principal $\mathbf{C}^*$-bundle over $\mathcal{E}_m^{\mathrm{an}}$.

In the Danielewski coordinates, the algebraic open $Q^\circ = \{u \neq 0\} \subset W_m$ is isomorphic to $\mathbf{C}_u^* \times \mathbf{C}_w$. Since Q° is Stein and has the homotopy type of a circle, the Oka–Grauert principle [10, Theorem 5.3.1(ii)] makes $L_m|_{Q^\circ}$ holomorphically trivial. Choose such a trivialization, so that A is a scalar fibre coordinate. For a fixed $w_0 \in \mathbf{C}$, write $\eta = w - w_0$. Then

$$T = Q^\circ \setminus \{w = w_0\} \simeq (\mathbf{C}^*)^2_{u,\eta}, \qquad V = Y_{m,A}|_T \simeq (\mathbf{C}^*)^3_{u,\eta,A}. \tag{3.21}$$

The complement of T in E_m is algebraic, so V is analytic-Zariski open in $Y_{m,A}$.

Proposition 3.4. *For every $a_* \in \mathcal{A}$, the principal opens $Y_{m,A}$, the bands Y_m, the cyclic covers Q_{m,a_*}, the two complements $X_{a_*} \setminus S_{1,a_*}$ and $X_{a_*} \setminus S_{2,a_*}$, and X_{a_*} are Oka.*

Proof. The product $V \simeq (\mathbf{C}^*)^3$ in (3.21) is Oka. Analytic extension [2, Theorem 1.9] first makes $Y_{m,A}$ Oka and then Y_m Oka, since each contains the preceding space as an analytic-Zariski open subset. One band Y_m is itself analytic-Zariski open in Q_{m,a_*}, so the same theorem makes Q_{m,a_*} Oka. For $(m, j) = (4, 1)$ and $(3, 2)$, the composite

$$Q_{m,a_*} \longrightarrow M_{m,a_*} \longrightarrow X_{a_*} \setminus S_{j,a_*}$$

is an unramified holomorphic covering. Invariance of the Oka property under holomorphic coverings [9, Proposition 2.8] therefore makes both complements Oka. Finally, either complement is a nonempty analytic-Zariski open subset of X_{a_*}, so analytic extension proves that X_{a_*} is Oka. □

Proof of Theorem 1.1. The absolute assertions follow from Proposition 3.4. Fix $(m, j) = (4, 1)$ or $(3, 2)$. By Lemma 3.1, the composite covering over $\mathcal{U}_j$ has the form

$$P_m : Q_{m,a_*} \times \Delta \longrightarrow \mathcal{U}_j, \qquad \pi \circ P_m = \mathrm{pr}_\Delta . \tag{3.22}$$

Here pr_Δ is the projection onto Δ. Since Q_{m,a_*} is Oka, the fibre-bundle Oka principle [18, Corollary 5.7] gives this projection the parametric Oka property with approximation and interpolation (POPAI). The holomorphic covering P_m is a fibre bundle with discrete Oka fibres, so the same principle gives it POPAI. Since a covering is a Serre fibration, P_m is a surjective Oka map. Therefore [18, Corollary 5.6] gives POPAI for $\pi|_{\mathcal{U}_j}$.

The sets $\mathcal{U}_1$ and $\mathcal{U}_2$ form an analytic-Zariski open cover of $\mathfrak{X}_\Delta$, so [18, Corollary 5.13] gives POPAI for π. The relative compactification theorem makes π a proper holomorphic submersion, and Ehresmann's theorem makes it a Serre fibration. Hence π is an Oka map. □

4. Point modifications and band complements

For the point-deletion argument, fix $a \in \mathcal{A}$ and abbreviate $X = X_a$. Write $M_m = M_{m,a}$ and $Q_m = Q_{m,a}$ for the finite and cyclic covers from Equation (3.1) and Lemma 3.1, and let $\Xi_m \simeq \mathbf{Z}$ be the deck group of $Q_m \to M_m$. All noncentral points belong to every translated two-row band; this common principal smooth locus is denoted by $\mathcal{K}_m$.

If $R \subset M_m$ is finite and $\widehat{R} \subset Q_m$ is its complete inverse image, then for every two-row band $Y \simeq Y_m$ there is a decomposition

$$\widehat{R} \cap Y = D_Y \sqcup F_Y. \tag{4.1}$$

Here D_Y is a finite union of full Ξ_m-orbits through points of $\mathcal{K}_m$, whereas F_Y is finite. Indeed, the lift of each point is one deck orbit. A core orbit lies in every band, while a toroidal orbit meets a two-row band in at most two points because the deck generator shifts the row index by one.

Every orbit occurring in D_Y lies over one of finitely many curves

$$C_\nu = \{z = z_\nu\} \subset E_m, \qquad \Delta_m|_{C_\nu} = \Delta_m(z_\nu) \in \mathbf{C}^*. \tag{4.2}$$

The following elementary choice will separate their logarithmic lifts.

Lemma 4.1. *For finitely many reduced algebraic curves in $(\mathbf{C}^*)^2$, there is a monomial $M(u, v) = u^r v^s$ whose restriction to the normalization of every irreducible component is proper and nonconstant, and hence finite.*

Proof. Complete each normalized affine curve to a smooth projective curve. At every boundary point, the two orders of u and v do not both vanish; otherwise the branch would extend to a point of the original closed curve inside $(\mathbf{C}^*)^2$. Choose $(r, s) \in \mathbf{Z}^2$ outside the finitely many rational lines on which $r \operatorname{ord} u + s \operatorname{ord} v = 0$. The character then tends to 0 or ∞ at every boundary branch, so it is proper and nonconstant. It is therefore finite. □

Proposition 4.2. *Suppose $D \subset \mathcal{K}_m$ is a finite union of full Ξ_m-orbits and $F \subset Y_m$ is finite. Then*

$$Y_m \setminus (D \cup F) \quad \text{and} \quad Y_{m,A} \setminus (D \cup F)$$

are Oka.

Proof. Use the product open V from (3.21) and write $E = (D \cup F) \cap V$. A full deck orbit is closed and discrete in Q_m, so E is closed and discrete in V. Its complete inverse image $\widetilde{E} \subset \mathbf{C}^3_{x,y,\sigma}$ under

$$(x, y, \sigma) \longmapsto (u, \eta, A) = (\mathrm{e}^x, \mathrm{e}^y, \mathrm{e}^\sigma)$$

is also closed and discrete. Apply Lemma 4.1 to the components of $C_\nu \cap T$ and choose the resulting nonconstant character $M = u^r \eta^s$, with $(r, s) \neq (0, 0)$. If there are no such components, take $(r, s) = (1, 0)$. The projection

$$(x, y, \sigma) \longmapsto (rx + sy, \sigma)$$

maps $\widetilde{E}$ to a closed discrete subset of $\mathbf{C}^2$. Indeed, bounded projected logarithms keep M and the scalar coordinate A in compact annuli. Properness of M on the carrier curves and the finiteness of F confine the base points of E to a compact subset of T. The corresponding points of E therefore lie in a compact subset of $V = T \times \mathbf{C}^*$, so only finitely many occur. The projected logarithms of each point lie in a product of two discrete lattices.

The preceding projection discreteness and the Rosay–Rudin projection criterion [22, Theorem 3.9] make $\widetilde{E}$ tame, with the empty case understood trivially.

The tame-complement theorem [9, Proposition 2.24] makes $\mathbf{C}^3 \setminus \widetilde{E}$ Oka. Since $\widetilde{E}$ is the complete inverse image of E, the exponential map restricts to an unramified holomorphic covering $\mathbf{C}^3 \setminus \widetilde{E} \to V \setminus E$. Thus $V \setminus E$ is Oka by [9, Proposition 2.8]. This common analytic-Zariski open subset lies in both manifolds displayed in the statement. Applying analytic extension to each one [2, Theorem 1.9] completes the proof. □

Theorem 4.3. *For $m = 3, 4$ and every finite subset $R \subset M_m$, the manifold $M_m \setminus R$ is Oka.*

Proof. Let $\widehat{R}$ be the complete inverse image in Q_m and choose one two-row band $Y \simeq Y_m$. The decomposition (4.1) and Proposition 4.2 show that $Y \setminus \widehat{R}$ is Oka. It is analytic-Zariski open in $Q_m \setminus \widehat{R}$, so analytic extension makes the latter Oka. The restriction $Q_m \setminus \widehat{R} \to M_m \setminus R$ is an unramified holomorphic covering, so [9, Proposition 2.8] gives the conclusion. □

Proof of Theorem 1.2. Let $P \subset X$ be finite and take $U_1 = X \setminus S_{1,a}$. The degree four cover $M_4 \to U_1$ pulls $P \cap U_1$ back to a finite set R. By Theorem 4.3, $M_4 \setminus R$ is Oka. The restricted degree-four covering $M_4 \setminus R \to U_1 \setminus P$ therefore makes $U_1 \setminus P$ Oka by [9, Proposition 2.8]. This is analytic-Zariski open in $X \setminus P$, so [2, Theorem 1.9] proves that $X \setminus P$ is Oka.

Let E be the exceptional divisor of $\mathrm{Bl}_P X \to X$. Its complement is biholomorphic to $X \setminus P$ and is analytic-Zariski open in the blow-up. A final application of [2, Theorem 1.9] proves that $\mathrm{Bl}_P X$ is Oka. □

The same principal open also gives complements of curves in the band. Let $C_1, \dots, C_m \subset Y_m$ be the exceptional curves of the small resolution. The closed analytic curve $Z_m^0 = \{A = B = 0\} \subset H_m$ has complete inverse image

$$\mathfrak{A}_m = \varrho_m^{-1}(Z_m^0),$$

which contains the exceptional curves and the proper transforms of the remaining zero-axis components.

Theorem 4.4. *Suppose D, F satisfy the hypotheses of Proposition 4.2 and $I \subset \{1, \dots, m\}$. Then*

$$Y_m \setminus \left(D \cup F \cup \bigcup_{i \in I} C_i \right)$$

is Oka.

Proof. The Oka manifold $Y_{m,A} \setminus (D \cup F)$ from Proposition 4.2 is disjoint from all exceptional curves and is analytic-Zariski open in the displayed complement. Apply [2, Theorem 1.9]. □

In particular, deleting any subcollection of the exceptional curves from Y_m preserves the Oka property.

Theorem 4.5. *The manifold $Y_m \setminus \mathfrak{A}_m$ is Oka.*

Proof. The principal conic open $Y_{m,A}$ in (3.20) is Oka and analytic-Zariski open in $Y_m \setminus \mathfrak{A}_m$. Apply [2, Theorem 1.9]. □

Corollary 4.6. *For every $a \in \mathcal{A}$ and $p \in X_a$, the blow-up $\mathrm{Bl}_p X_a$ is an Oka complex manifold diffeomorphic to $\mathbf{P}^3$, but it is not biholomorphic to the standard projective threefold.*

Proof. The Oka assertion follows from Theorem 1.2. Since X_a is diffeomorphic to S^6, its smooth point blow-up is diffeomorphic to $\mathbf{P}^3$ [15]. If $E \simeq \mathbf{P}^2$ is the exceptional divisor, the canonical-bundle formula gives

$$K_{\mathrm{Bl}_p X_a} \simeq \pi^* K_{X_a} \otimes \mathcal{O}_{\mathrm{Bl}_p X_a}(2E).$$

Alpöge's smooth-type theorem gives $H^2(X_a, \mathbf{Z}) = 0$ [1, Theorem 8.1], while $[E]$ generates $H^2(\mathrm{Bl}_p X_a, \mathbf{Z}) \simeq \mathbf{Z}$. Thus $c_1(T_{\mathrm{Bl}_p X_a}) = -2[E]$ has divisibility 2, whereas $c_1(T_{\mathbf{P}^3}) = 4h$ has divisibility 4. Hence the two complex structures are not biholomorphic. □

Corollary 4.7. *If $C \subset X_a$ is a smooth compact complex curve, possibly disconnected, then $\mathrm{Bl}_C X_a$ is Oka–1. The same holds after any finite sequence of blow-ups along smooth centres.*

Proof. This follows from Proposition 3.4 and the blow-up invariance of the Oka–1 property [11, Theorem 3.1]. □

Proposition 4.8. *For $R = \mathrm{Tot}(\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1))$ with zero section $C \simeq \mathbf{P}^1$, one has*

$$\mathrm{Bl}_C R \simeq \mathrm{Tot}(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-1,-1)),$$

and this smooth algebraic threefold is algebraically elliptic and Oka.

Proof. The blow-up of a vector bundle along its zero section is the tautological line bundle on its projectivization. Here $\mathbb{P}(\mathcal{O}(-1) \otimes \mathbf{C}^2) = \mathbf{P}^1 \times \mathbf{P}^1$, giving the displayed isomorphism. The line bundle $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-1,-1)$ is trivial over each of the four standard affine charts $\mathbf{A}^2$ of $\mathbf{P}^1 \times \mathbf{P}^1$. Its total space is therefore covered by four Zariski opens isomorphic to $\mathbf{A}^3$, each algebraically elliptic. Gromov's algebraic localization lemma [16, Corollary 2.2] makes this total space algebraically elliptic. Gromov's theorem gives the Oka property [14]. □

Appendix A. Gluing and choice-independence

This appendix supplies the two global compatibility arguments deferred from Section 2.

Proof of Lemma 2.22. Write

$$C = \mathcal{J}_\Delta, \qquad N_j = \mathcal{N}_{j,\Delta}.$$

Let

$$O_j^C = C|_{D_j^* \times \Delta}, \qquad O_j^N = N_j|_{D_j^* \times \Delta}.$$

For uniform notation, let $\phi_j : O_j^C \to O_j^N$ be G_0 when $j = 0$ and G_j^{-1} when $j = 1, 2$. The gluing relation is generated by

$$x \sim \phi_j(x) \qquad (x \in O_j^C).$$

Because the discs D_0, D_1, D_2 are pairwise disjoint, the three open sets O_j^C are pairwise disjoint. No point of one filling piece is identified with a point of another filling piece. It follows directly that every equivalence class is either a singleton or a two-point set

$$\{x, \phi_j(x)\}$$

for a unique index j and a unique $x \in O_j^C$. In particular, an equivalence class contains at most one point from each individual piece. Hence the canonical map from every piece to the glued quotient is injective.

The image of each piece is open. For example, the saturation of the core is

$$C \sqcup O_0^N \sqcup O_1^N \sqcup O_2^N,$$

which is open in the disjoint union, and the saturation of N_j is $N_j \sqcup O_j^C$. The canonical injection of a piece is therefore a homeomorphism onto an open subset. Since the glued space is covered by four open subsets homeomorphic to second-countable spaces, the union of four countable bases is a countable basis for the glued space. The complex charts on the pieces have holomorphic transition functions because the only cross-piece transitions are the biholomorphisms ϕ_j and ϕ_j^{-1}. Thus they determine a unique complex-manifold structure once Hausdorffness has been proved.

Let $x \neq y$ be points of the glued space. The local maps to $B \times \Delta$ agree on the overlaps, so they determine a continuous map $\mathfrak{f}$. If $\mathfrak{f}(x) \neq \mathfrak{f}(y)$, choose disjoint neighbourhoods of their images in the Hausdorff base and take inverse images.

Suppose $\mathfrak{f}(x) = \mathfrak{f}(y) = (b, a)$. If $b \in D_j$, represent both points in the filling piece N_j; when $b \in D_j^*$ this is possible using ϕ_j. If b lies outside the three filling discs, represent both points in the core C. Thus there is one Hausdorff piece M containing distinct representatives $\widetilde{x}, \widetilde{y}$ of x, y.

Choose disjoint open neighbourhoods $U, V \subset M$ of $\widetilde{x}, \widetilde{y}$. Their saturations in the disjoint union are open. Explicitly, if $M = C$, then

$$\mathrm{Sat}(U) = U \sqcup \bigsqcup_j \phi_j(U \cap O_j^C),$$

and if $M = N_j$, then

$$\mathrm{Sat}(U) = U \sqcup \phi_j^{-1}(U \cap O_j^N);$$

the same formulas hold for V. These two saturations are disjoint. Indeed, an intersection point would determine one equivalence class meeting both U and V. That class would then contain two representatives in the same piece M, contrary to the one-representative property proved above and the disjointness of U, V. The quotient images of the two open saturations are therefore disjoint open neighbourhoods of x and y. Hence the glued space is Hausdorff. □

The remaining compatibility concerns the cusp cutoff and the parameter disc used in the relative construction.

Proof of Proposition 2.24. We first fix Δ and compare two radii $0 < \varepsilon' < \varepsilon$. For every valid cusp cutoff ρ for Δ, write

$$\mathcal{N}_{0,\Delta}^{(\rho)} = (Y_\rho \times \Delta)/\mathbf{Z}^2, \qquad \mathfrak{X}_\Delta^{(\rho)}$$

for the cusp quotient and the resulting global compactification, and write

$$G_0^{(\rho)} : \mathcal{J}_\Delta|_{D_\rho^* \times \Delta} \xrightarrow{\cong} \mathcal{N}_{0,\Delta}^{(\rho)}|_{D_\rho^* \times \Delta}$$

for the punctured cusp biholomorphism constructed in Proposition 2.21. All these objects are obtained by restricting the same action formula and the same exponential map. The core $\mathcal{J}_\Delta$ is dense in each compactification: the omitted fibres are locally given by $s_j^{m_j} = 0$ or $z_0 z_1 z_2 = 0$. Since the compactifications are Hausdorff by Lemma 2.22, two continuous maps to the same compactification that agree on the core agree everywhere. In particular, a comparison map restricting to the identity on the core is unique.

The subset $Y_{\varepsilon'} \times \Delta$ is open and invariant in $Y_\varepsilon \times \Delta$. The quotient map is open, and Lee's local quotient charts [19, Theorem 1.16] identify its quotient with an open subset of the larger quotient. Hence there is an open biholomorphic embedding

$$j_{\varepsilon',\varepsilon} : \mathcal{N}_{0,\Delta}^{(\varepsilon')} \hookrightarrow \mathcal{N}_{0,\Delta}^{(\varepsilon)}. \tag{A.1}$$

Its image is the inverse image of $D_{\varepsilon'} \times \Delta$.

Let $W_\Delta^{(\varepsilon)}$ be the central divisor in the larger cusp quotient. The two open subsets

$$j_{\varepsilon',\varepsilon}(\mathcal{N}_{0,\Delta}^{(\varepsilon')}) \quad \text{and} \quad \mathcal{N}_{0,\Delta}^{(\varepsilon)} \setminus W_\Delta^{(\varepsilon)}$$

cover $\mathcal{N}_{0,\Delta}^{(\varepsilon)}$. On the first one, the map to $\mathfrak{X}_\Delta^{(\varepsilon')}$ is the inverse of (A.1), followed by the inclusion of the smaller cusp filling. On the second one, the map is given by

$$\mathcal{N}_{0,\Delta}^{(\varepsilon)} \setminus W_\Delta^{(\varepsilon)} \xrightarrow{(G_0^{(\varepsilon)})^{-1}} \mathcal{J}_\Delta|_{D_\varepsilon^* \times \Delta} \longrightarrow \mathfrak{X}_\Delta^{(\varepsilon')},$$

where the last arrow is the inclusion of the common core.

These two definitions agree on their overlap. Indeed, both punctured cusp maps are induced by the same exponential formula (2.51); restricting that formula from D_ε^* to $D_{\varepsilon'}^*$ gives

$$G_0^{(\varepsilon)}|_{D_{\varepsilon'}^* \times \Delta} = j_{\varepsilon',\varepsilon} \circ G_0^{(\varepsilon')}. \tag{A.2}$$

In the smaller glued space, a core point and its image under $G_0^{(\varepsilon')}$ represent the same point. The two local maps therefore glue to a holomorphic map from the larger cusp filling to the smaller global compactification.

On the common core and the two finite fillings take the identity. On every cusp overlap this identity agrees with the map just constructed, again by (A.2). The gluing property therefore gives a holomorphic map

$$\Theta_{\varepsilon,\varepsilon'} : \mathfrak{X}_\Delta^{(\varepsilon)} \longrightarrow \mathfrak{X}_\Delta^{(\varepsilon')}$$

over $B \times \Delta$. Reversing the roles of the two compactifications gives an inverse: on the smaller cusp filling it is the open embedding (A.1), and on the core and finite pieces it is the identity. The two composites are the identity on every member of the open gluing cover. Hence $\Theta_{\varepsilon,\varepsilon'}$ is a biholomorphism. Uniqueness gives, for $\varepsilon'' < \varepsilon' < \varepsilon$,

$$\Theta_{\varepsilon',\varepsilon''} \circ \Theta_{\varepsilon,\varepsilon'} = \Theta_{\varepsilon,\varepsilon''}.$$

This proves part (i).

Now consider two parameter discs Δ_1, Δ_2 and a point $a_0 \in \Delta_1 \cap \Delta_2$. Choose a disc $V \Subset \Delta_1 \cap \Delta_2$ containing a_0. Let the two original valid cusp cutoffs be η_1 and η_2. Restricting from Δ_i to V can only decrease the supremum in condition (V2), so each η_i remains valid for V with its original auxiliary cutoff. Choose

$$0 < \rho < \min\{\eta_1, \eta_2\}.$$

By the monotonicity assertion of Lemma 2.10, ρ is a valid cusp cutoff for V.

Every lattice action, deck action, finite cyclic action, cusp action, and gluing map preserves the parameter. Consequently, restricting the construction over Δ_i to V gives exactly the construction obtained by applying the same formulas over V with the original cutoff η_i. In particular, the two restrictions have the same core, the same finite fillings, and the same period functions. By part (i), each is canonically biholomorphic to the construction over V made with cutoff ρ. In the notation of part (i), the comparison from the first restriction to the second is

$$C_{12}^{V,\rho} = (\Theta_{\eta_2,\rho})^{-1} \circ \Theta_{\eta_1,\rho}.$$

Different choices of ρ give comparison maps with the same restriction to the core, so uniqueness makes them equal. Comparisons constructed over different parameter neighbourhoods likewise agree on their overlaps and therefore glue to a canonical biholomorphism over $\Delta_1 \cap \Delta_2$. On a triple overlap, the composite comparison and the direct comparison both restrict to the identity on the core; uniqueness gives $C_{23} \circ C_{12} = C_{13}$. This proves part (ii).

Taking the fibre over a point a in part (ii) proves the final choice-independence assertion. □

Appendix B. The source complex structure on the six-sphere

The complex structure used throughout this paper originates in Alpöge's construction [1, Construction 6.1 and Theorem 8.1]. Its underlying idea is to complete a family of complex two-tori over a three-punctured projective line in three different ways. The finite monodromies at two punctures lead to smooth multiple fibres; the unipotent monodromy at the third leads to a reduced, nonnormal toric fibre. The choices of the three completions also determine the integral topology of the total space. We describe how these parts fit together, using the notation of Section 2.

Take $B = \mathbf{P}^1_t$ with $p_1 = 0$, $p_2 = 1$, $p_0 = \infty$, and $B^\circ = B \setminus \{p_0, p_1, p_2\}$. Alpöge [1, Proposition 2.11] uses the hyperbolic orbifold of signature $(0; 3, 4, \infty)$ on B. Its upper-half-plane uniformization U has group

$$\Gamma = \langle g_1, g_2 \mid g_1^3 = g_2^4 = 1 \rangle, \qquad g_0 = (g_1 g_2)^{-1},$$

where g_j fixes a point over p_j and g_0 is the primitive parabolic element at p_0. Removing the two finite-stabilizer orbits gives a covering $U^\circ \to B^\circ$. Thus the base simultaneously supplies an order-three point, an order-four point, and a cusp; each has a distinct local completion of the torus family.

The integral lattice $\Lambda = \mathbf{Z}\langle \widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta} \rangle$ will be the first homology of a smooth fibre. The representation $g_j \mapsto A_j$ in (2.2) has $A_1^3 = A_2^4 = I$, whereas $A_0 = (A_1 A_2)^{-1}$ is unipotent. The first-coordinate character $\gamma : \Lambda \to \mathbf{Z}$ is fixed by both finite monodromies; their coinvariants leave precisely the circle measured by γ. At the cusp, the primitive rank-two lattice

$$\Lambda_{\mathrm{tor}} = \ker(A_0 - I) = \mathrm{im}(A_0 - I) = \mathbf{Z}\langle \widehat{w}, \widehat{\delta} \rangle$$

is the lattice that will be exponentiated. On $\overline{\Lambda} = \Lambda/\Lambda_{\mathrm{tor}}$, the induced map to Λ_{tor} has matrix

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the bases $(\overline{\widehat{\gamma}}, \overline{\widehat{u}})$ and $(\widehat{w}, \widehat{\delta})$. Its unimodularity is what makes the cusp's periodic toric construction yield a single irreducible central fibre. These integral features are the link between the complex-geometric fillings and Alpöge's later topology calculation.

Alpöge [1, Theorem 3.4] realizes this monodromy by holomorphic period functions τ, μ, β on U. The first is a modular parameter satisfying $j(\tau) = 1728t \circ \pi$; its transformations under g_1, g_2 are $\tau \mapsto (\tau - 1)/\tau$ and $\tau \mapsto -1/\tau$. The branching of j at 0 and 1728 matches the order-three and order-four stabilizers: at the latter point the projective modular action has order two, while its integral lift has order four. The remaining two functions solve the affine descent equations forced by the same integral monodromy. In the present notation the resulting matrix is

$$\Omega_a(z) = \begin{pmatrix} 6\mu(z) & \tau(z) & 1 & 0 \\ \beta_0(z) + a & \mu(z) & 0 & 1 \end{pmatrix}, \qquad a \in \mathcal{A},$$

where the additive freedom in β is recorded by a. The inequality $\mathscr{D}_a < 0$ from (2.8) gives $\det_{\mathbf{R}} \Omega_a = \Im\tau\, \mathscr{D}_a \neq 0$. Hence its columns form a full lattice in $\mathbf{C}^2$. The equivariance $\Omega_a(gz) = R_g(z)\Omega_a(z)A(g)^{-1}$ makes $\mathbf{C}^2/\Omega_a(z)\Lambda$ descend from U° to a proper holomorphic torus family $J_a \to B^\circ$. In particular, the period functions provide both the complex structure of each fibre and the coordinate changes needed to carry it around the base.

At p_j, a local uniformizing coordinate s_j satisfies $s_j(g_j z) = \mathrm{e}^{-2\pi\mathrm{i}/m_j} s_j(z)$, where $(m_1, m_2) = (3, 4)$; the coarse coordinate is $t_j = s_j^{m_j}$. The pulled-back family extends over $s_j = 0$ with a smooth torus fibre. Alpöge's logarithmic transform [1, Theorem 5.4] divides this extension by the affine cyclic action

$$(z, \zeta) \longmapsto \left(g_j z, R_{g_j}(z)\zeta + \Omega_a(g_j z)\frac{v_j}{m_j} \right),$$

using the invariant vectors

$$v_1 = \widehat{\gamma} + 2\widehat{u} - 4\widehat{w}, \qquad v_2 = -\widehat{\gamma} - 3\widehat{u} + 3\widehat{w}.$$

The relations $A_j v_j = v_j$, $\gamma(v_1) = 1$, and $\gamma(v_2) = -1$ make the cyclic action free, as verified in Lemma 2.5. The quotient is therefore smooth; its reduced central surface S_j is a free cyclic quotient of a complex two-torus and is bielliptic. Because the map to the coarse base is $t_j = s_j^{m_j}$, its fibre is $m_j S_j$, namely $3S_1$ or $4S_2$. Over $s_j \neq 0$, translation by

$$\frac{\log s_j}{2\pi\mathrm{i}}\, \Omega_a(z) v_j$$

conjugates this affine action to the original monodromy action; the branch independence of the resulting identification with J_a is verified after (2.27).

The cusp requires a different operation. On a simply connected horodisc above p_0, choose a logarithmic coordinate s with $q = \mathrm{e}^{2\pi\mathrm{i}s} = 1/t$ and $s(g_0 z) = s(z) - 1$. The period block Z_a formed by the first two columns of Ω_a has the form

$$Z_a = sB_0 + C_a(q), \qquad C_a \in O(D_0, M_2(\mathbf{C})),$$

because $\tau - s$, μ, and $\beta_0 + a + \tau$ extend across $q = 0$. Quotienting first by the last two periods, those of Λ_{tor}, turns the fibre coordinates into $(\mathbf{C}^*)^2$. For $\bar{\lambda} \in \overline{\Lambda}$, the remaining period acts there by multiplication by

$$q^{B_0\bar{\lambda}} \exp\bigl(2\pi\mathrm{i} C_a(q)\bar{\lambda}\bigr).$$

The singular factor is thus an integral monomial in q, while the second factor is a holomorphic unit. This is the passage from the analytic period family to Alpöge's toric filling [1, Lemma 4.2 and Theorem 4.5].

More explicitly, triangulate $\mathbf{R}^2$ periodically by the two unimodular triangles in each unit square and cone the triangles from the origin in $\mathbf{R}^2 \oplus \mathbf{R}$ at height one. The resulting countable fan is smooth. Its toric threefold has a holomorphic height character q, and on each maximal affine chart this map reads $q = z_0 z_1 z_2$. Hence the central divisor upstairs is reduced with simple normal crossings. Translation of the fan by $B_0\bar{\lambda}$, together with the holomorphic unit in the preceding display, extends the residual $\overline{\Lambda}$-action over $q = 0$. Alpöge proves that, on a sufficiently small cusp disc, this action is free and properly discontinuous and its quotient is proper over the disc. Since $B_0\overline{\Lambda} = \mathbf{Z}^2$, all vertices of the periodic fan lie in one orbit. The quotient central fibre W is therefore irreducible and nonnormal, although it has normal-crossing singularities. Its normalization is the degree-six del Pezzo surface: the six boundary curves form a hexagon, and opposite sides are identified in W, producing three double curves and two triple points [1, Proposition 4.6]. Exponentiating the fibre coordinates identifies the toric quotient over $q \neq 0$ with the original family J_a over the punctured cusp disc [1, Proposition 4.7].

The three punctured identifications attach the two logarithmic fillings and the toric cusp filling to the same smooth torus family. Alpöge [1, Construction 6.1 and Theorem 6.2] shows that the resulting space is a compact Hausdorff complex threefold X^{A} with a proper map to $\mathbf{P}^1$ and exactly the fibres described above. The affine translations also enter its topology. In Alpöge's notation the cusp gluing is untranslated, and the character values are $(\ell_0, \ell_1, \ell_2) = (0, 1, -1)$; consequently

$$12\ell_0 - 4\ell_1 - 3\ell_2 = -1.$$

Alpöge [1, Theorem 7.17, Corollary 7.29, and Theorem 8.1] proves that these choices make X^{A} simply connected with the integral homology of S^6, and then identifies its underlying smooth manifold with S^6. Transporting the complex atlas along that diffeomorphism gives the complex structure on the standard smooth six-sphere.

Our family uses precisely the monodromy matrices, finite twists, period normalization, and untranslated cusp gluing just described. The constant a varies the invariant period $\beta_0 + a$ through the admissible half-plane; the member $a = a_{\mathrm{A}}$ is Alpöge's compact threefold by Lemma 2.27. Over each relatively compact parameter disc, the jointly holomorphic fillings give a proper submersion. Together with the choice-independence in

Proposition 2.24, Corollary 2.28 shows that all members are diffeomorphic to S^6. The normalization of W and its paired boundary curves are the source data used in the cusp filling of Subsection 2.2.

Acknowledgements

We thank F. Forstnerič for his correspondence with S.-Y. Xie concerning our Oka result and for bringing G. Etesi's preprint [8] to our attention.

S.-Y. Xie acknowledges partial support from the National Key R&D Program of China under Grants No. 2023YFA1010500 and No. 2021YFA1003100, and from the National Natural Science Foundation of China under Grants No. 12288201 and No. 12471081, as well as support from the Xiaomi Young Talents Program. Y. Kusakabe is supported by JSPS KAKENHI Grant Number JP24K16919.

AI Use Disclosure

The research questions, mathematical insights, and principal constructions are due to the authors. GPT-5.6-Sol was used as an auxiliary tool for specific technical problems, calculations, consistency checks, and the drafting and polishing of the exposition. The authors have examined all arguments and take full responsibility for the content of this paper.

References

[1] L. Alpöge, *The* $(3, 4, \infty)$ *modular family of* 2*-tori, completed at its three special points, is a complex structure on* S^6, manuscript, 2026, https://alpo.ge/s6.pdf. ↑ 1, 3, 4, 6, 10, 11, 17, 18, 19, 25, 27, 28

[2] S. An, Y.-H. Du, B. Guo, P.-C. Wang, and S.-Y. Xie, *A Local Approximation Characterization of Oka Manifolds*, arXiv:2610.10475v3, 2026, https://arxiv.org/abs/2610.10475v3. ↑ 2, 23, 24

[3] S. An, Y.-H. Du, B. Guo, P.-C. Wang, and S.-Y. Xie, *Oka manifolds are ubiquitous*, arXiv:2610.10493v3, 2026, https://arxiv.org/abs/2610.10493v3. ↑ 2

[4] H. Argüz, *Mirror symmetry for the Tate curve via tropical and log corals*, J. Lond. Math. Soc. (2) **105** (2022), no. 1, 343–411, https://doi.org/10.1112/jlms.12515. ↑ 20

[5] G. T. Buzzard and S. S.-Y. Lu, *Algebraic surfaces holomorphically dominable by* $\mathbf{C}^2$, Invent. Math. **139** (2000), no. 3, 617–659, https://doi.org/10.1007/s002220050021. ↑ 22

[6] Z. Chen, *Conditional Oka Complex Structures of the Six-Sphere*, arXiv:2609.26706v2, 2026, https://arxiv.org/abs/2609.26706v2. ↑ 2

[7] C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable*, Séminaire Bourbaki, vol. 1, exposé no. 24, pp. 153–168, Société mathématique de France, 1952, https://www.numdam.org/item/SB_1948-1951__1__153_0/. ↑ 18

[8] G. Etesi, *The complex structure on the six dimensional sphere*, arXiv:1509.02300v6, 2024, https://arxiv.org/abs/1509.02300. ↑ 29

[9] F. Forstnerič, *Oka manifolds: From Oka to Stein and back*, Annales de la Faculté des Sciences de Toulouse, Mathématiques **22** (2013), no. 4, 747–809, https://www.numdam.org/item/AFST_2013_6_22_4_747_0/. ↑ 1, 2, 23, 24

[10] F. Forstnerič, *Stein Manifolds and Holomorphic Mappings: The Homotopy Principle in Complex Analysis*, 2nd ed., Springer, Cham, 2017, https://doi.org/10.1007/978-3-319-61058-0. ↑ 1, 22

[11] F. Forstnerič and F. Lárusson, *Oka–1 manifolds: New examples and properties*, Math. Z. **309** (2025), article 26, https://doi.org/10.1007/s00209-024-03649-8. ↑ 2, 25

[12] W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, 1993. ↑ 9

[13] H. Grauert and R. Remmert, *Coherent Analytic Sheaves*, Grundlehren der mathematischen Wissenschaften, vol. 265, Springer, Berlin, 1984, https://doi.org/10.1007/978-3-642-69582-7. ↑ 14

[14] M. Gromov, *Oka's principle for holomorphic sections of elliptic bundles*, J. Amer. Math. Soc. **2** (1989), no. 4, 851–897, https://doi.org/10.1090/S0894-0347-1989-1001851-9. ↑ 1, 25

[15] A. T. Huckleberry, S. Kebekus, and T. Peternell, *Group actions on* S^6 *and complex structures on* $\mathbb{P}_3(\mathbb{C})$, Duke Math. J. **102** (2000), no. 1, 101–124, https://doi.org/10.1215/S0012-7094-00-10214-1. ↑ 24

[16] S. Kaliman and M. Zaidenberg, *Algebraic Gromov's ellipticity of cubic hypersurfaces*, Proc. Steklov Inst. Math. **329** (2025), 79–87, https://doi.org/10.1134/S008154382560067X. ↑ 25

[17] Y. Kusakabe, *Elliptic characterization and localization of Oka manifolds*, Indiana Univ. Math. J. **70** (2021), no. 3, 1039–1054, https://doi.org/10.1512/IUMJ.2021.70.8454. ↑ 1

[18] Y. Kusakabe, *Elliptic characterization and unification of Oka maps*, Math. Z. **298** (2021), 1735–1750, https://doi.org/10.1007/s00209-020-02661-y. ↑ 1, 2, 23

[19] J. M. Lee, *Introduction to Complex Manifolds*, Graduate Studies in Mathematics, vol. 244, American Mathematical Society, Providence, 2024, https://bookstore.ams.org/gsm-244. ↑ 5, 6, 7, 8, 15, 16, 26

[20] W. Liu and S. Rollenske, *Yet Another Family of Complex Structures on* S^6, arXiv:2609.33826v2, 2026, https://arxiv.org/abs/2609.33826. ↑ 2

[21] N. Nakayama, *On Weierstrass models*, in *Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata*, Vol. II, Academic Press, 1988, 405–431, https://doi.org/10.1016/B978-0-12-348032-3.50004-9. ↑ 22

[22] J.-P. Rosay and W. Rudin, *Holomorphic maps from* $\mathbf{C}^n$ *to* $\mathbf{C}^n$, Trans. Amer. Math. Soc. **310** (1988), no. 1, 47–86, https://doi.org/10.1090/S0002-9947-1988-0929658-4. ↑ 24
[23] J. Viaclovsky, *A two-parameter family of complex structures on* S^6, arXiv:2609.33785v1, 2026, https://arxiv.org/abs/2609.33785. ↑ 2

(Yun-Heng Du) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
Email address: duyunheng@amss.ac.cn

(Bin Guo) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
Email address: guobin@amss.ac.cn

(Yuta Kusakabe) Faculty of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
Email address: kusakabe@math.kyushu-u.ac.jp

(Song-Yan Xie) State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
Email address: xiesongyan@amss.ac.cn