A hyperbolic octic hypersurface in $\mathbb{P}^4$

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Abstract

We construct a Kobayashi hyperbolic hypersurface of degree eight in $\mathbb{P}^4(\mathbb{C})$. The proof combines persistence of intersections with the rigidity of Brody curves in the complement of three lines.

1 Introduction

Kobayashi's hyperbolicity conjecture for general projective hypersurfaces [9, Remark (3.3.16), Conjecture 1(i)] was proved in sufficiently large degree by Siu [13] and Brotbek [3]. For $n \geq 3$, a very general hypersurface of degree $d \geq 2n-1$ in $\mathbb{P}^n(\mathbb{C})$ is conjectured to be hyperbolic [16, p. 3].

Our construction and the explicit examples recalled below all build on Zaidenberg's deformation method [15, Theorem 3.1]. He used it to deform five lines in general position into a smooth plane quintic whose complement is hyperbolic. In $\mathbb{P}^3$, Shiffman and Zaidenberg [11, 12] used persistence of intersections to construct hyperbolic octics. Zaidenberg [17] obtained another by deforming a double cone, and Duval [5, 6] obtained a sextic by deforming six planes. Ciliberto and Zaidenberg [4] constructed sextics and septics from scrolls in $\mathbb{P}^3$ and recorded examples in $\mathbb{P}^4$ of every degree at least twelve by deforming unions of two cones.

Theorem 1. *There exists a Kobayashi hyperbolic hypersurface of degree eight in* $\mathbb{P}^4(\mathbb{C})$.

We deform a union of two quartics with equations q_1, q_2 by

$$X_t = (q_1 q_2 = to), \qquad 0 < |t| \ll 1. \tag{1}$$

where o is a homogeneous polynomial of degree eight. The auxiliary octic ($o = 0$) restricts where a Brody limit in one quartic can meet the other. We construct the quartics from two arrangements of four hyperplanes via a pencil of quadrics. A putative Brody curve in the relevant quadric complement projects to a curve omitting three lines and meeting the branch conic only nontransversely. The plane rigidity argument confines its lift to a ruling line punctured at three points.

2 Preliminaries

Throughout, $\mathbb{P}^m$ denotes complex projective space. For the auxiliary subsets, which need not be compact, hyperbolic means that every holomorphic map from $\mathbb{C}$ to the subset is constant.

2.1 Entire curves and persistence of intersections

An entire curve is a nonconstant holomorphic map from $\mathbb{C}$. We use the Fubini–Study metric on projective space. A Brody curve is an entire curve whose derivative is bounded in this metric. For compact analytic subsets of projective space, Brody's criterion [2] and its extension [15, Corollary 2.2] identify Kobayashi hyperbolicity with the absence of entire curves, equivalently of Brody curves.

If nonhyperbolic compact analytic subspaces $Y_\nu \subset \mathbb{P}^N$ tend to Y_0, Brody reparametrization gives entire curves $f_\nu : \mathbb{C} \to Y_\nu$ with

$$\|f_\nu'(0)\| = 1, \qquad \|f_\nu'(z)\| \le 1 \quad (z \in \mathbb{C}).$$

A subsequence converges locally uniformly to a Brody curve $f : \mathbb{C} \to Y_0$. The normalization keeps the limit nonconstant. For the noncompact complements appearing below, reparametrization is performed in the ambient projective space. The limit may lie in a deleted boundary stratum, which must then be considered separately.

Lemma 2. *Let f_ν converge locally uniformly to f, and let local equations g_ν of analytic hypersurfaces converge locally uniformly to g. If $g \circ f$ is not identically zero and $g(f(z_0)) = 0$, then there are, for all sufficiently large ν, points $z_\nu \to z_0$ with $g_\nu(f_\nu(z_\nu)) = 0$. A simple zero of $g \circ f$ is approximated by simple zeros.*

Proof. Choose a small disk about z_0 whose closure contains no other zero of $g \circ f$. Uniform convergence and Rouché's theorem preserve the number of zeros in the disk, counted with multiplicity. Shrinking the disk gives the first assertion. If the zero has multiplicity one, the preserved zero also has multiplicity one. □

A limiting curve either lies in the limiting divisor or retains its intersection restrictions. Local analytic factors can be treated separately when the approximating curves avoid each factor. Boundary containment is always handled on the lower-dimensional stratum itself.

The final step uses the following filtering principle, a form of the Shiffman–Zaidenberg deformation method; see also [5].

Lemma 3. *Let $A = (a = 0)$, $B = (b = 0)$, and $R = (r = 0)$ be hypersurfaces in $\mathbb{P}^N$ of respective degrees $d_A, d_B, d_A + d_B$. Suppose that*

$$A \cap B, \qquad A \setminus (B \setminus R), \qquad B \setminus (A \setminus R)$$

contain no entire curve. Then $(ab = tr)$ is Kobayashi hyperbolic for all sufficiently small nonzero t.

Proof. Otherwise, a sequence $t_\nu \to 0$ gives a nonconstant Brody limit in $A \cup B$, hence in A or B, say A. It cannot lie in $A \cap B$. On $(ab = t_\nu r)$, every intersection with B lies in R. Persistence of intersections therefore forces the limit into $A \setminus (B \setminus R)$, contradicting the hypothesis. □

For the quartics $Q_i = (q_i = 0)$, $i = 1, 2$, and the octic $O = (o = 0)$ in (1), the three sets to be controlled are consequently

$$Q_1 \setminus (Q_2 \setminus O), \qquad Q_2 \setminus (Q_1 \setminus O), \qquad Q_1 \cap Q_2. \tag{2}$$

It suffices to construct quartics Q_1 and $Q' = (q' = 0)$ for which

$$Q_1 \setminus (Q' \setminus O) \quad \text{and} \quad Q_1 \cap Q' \tag{3}$$

contain no entire curve. The passage from (3) to (2) uses a nearby quartic $Q_2 = (q_1 = sq')$ and is carried out in the final section. The parameter s is chosen after Q_1, Q', O, and t after s.

2.2 Hyperplane complements and algebraic curves

Hyperplanes in $\mathbb{P}^m$ are in general position if every $m+1$ have empty intersection. Fujimoto [7, Theorem 5.1] proved that the complement of $2m+1$ hyperplanes in general position in $\mathbb{P}^m$ is taut. In particular, it contains no entire curve; see also [10, Theorem 4.3.2]. We use $m=3$ and $m=2$.

Borel's identity theorem says that a relation among zero-free entire functions partitions into vanishing subsums whose terms have constant ratios; see [10, Corollary 4.2.18] and Green [8]. For four lines in general position in $\mathbb{P}^2$, apply it to the relation among their four linear forms, composed with a homogeneous representation of an entire curve omitting them. A single proportionality class would make the curve constant. The only remaining partition is into two pairs. The corresponding linear relation places the curve in the diagonal joining the intersections of those pairs. There are three such diagonals. An additional line, omitted except at one restored point, then excludes entire curves if it meets each diagonal away from its two vertices and from the restored point. This imposes finitely many additional incidence conditions.

The exponential normal form is due to Berteloot–Duval [1]; see also Tsukamoto [14]. In the complement of the three coordinate lines in $\mathbb{P}^2$, a Brody curve has the form

$$z \longmapsto [1 : e^{a_1 z+b_1} : e^{a_2 z+b_2}], \qquad a_1, a_2, b_1, b_2 \in \mathbb{C}.$$

At least one of a_1, a_2 is nonzero. After exchanging the two affine coordinates and making an affine change in z, the form is

$$z \longmapsto (e^z, ae^{\lambda z}), \qquad a \in \mathbb{C}^*. \tag{4}$$

The slope $\lambda \in \mathbb{C}$ may be zero, which includes the constant-coordinate cases. The bounded derivative condition is essential in this statement. The geometric consequences for additional punctured lines and for a branch conic are proved in the next section.

The one-dimensional boundary strata are checked by normalization and uniformization. If Γ is an irreducible projective curve, $\widetilde{\Gamma}$ its normalization of genus g, and $E \subset \Gamma$ a finite set, let $r = |\nu^{-1}(E)|$, where $\nu : \widetilde{\Gamma} \to \Gamma$ is the normalization map. The condition

$$2g - 2 + r > 0$$

implies that $\Gamma \setminus E$ admits no entire curve: a nonconstant map lifts to $\widetilde{\Gamma} \setminus \nu^{-1}(E)$, whose universal cover is the disk. In particular, three punctures suffice on a line or a smooth conic, and one puncture suffices when $g = 1$. An irreducible plane quartic with precisely two ordinary nodes and no other singularities has normalization of genus $3 - 2 = 1$; this is the curve criterion needed for the quartic boundary sections.

2.3 The two hyperplane arrangements

Let

$$H = \bigcup_{i=1}^{4} H_i, \qquad H_i = (h_i = 0), \qquad H' = \bigcup_{j=1}^{4} H'_j, \qquad H'_j = (h'_j = 0),$$

be two arrangements in $\mathbb{P}^4$. Their eight hyperplanes are in general position, with the further incidence conditions specified where they are used. For nonempty $I \subset \{1,2,3,4\}$, set

$$H_I = \bigcap_{i \in I} H_i, \qquad H^I = \bigcup_{i \notin I} H_i.$$

In particular, $H_i \simeq \mathbb{P}^3$, $H_{ij} \simeq \mathbb{P}^2$, $H_{ijk} \simeq \mathbb{P}^1$, and $v = H_{1234}$ is the vertex of H. The unions of double planes and triple lines are

$$\Pi = \bigcup_{i<j} H_{ij}, \qquad \Lambda = \bigcup_{i<j<k} H_{ijk}.$$

The primed notation has the same meaning for H'. Unprimed indices i, j, k, l appearing together are distinct. Primed indices in a specified intersection are distinct. In mixed expressions, the two index sets range independently.

The letters O, Q, C denote hypersurfaces of degrees eight, four, and two, respectively; the corresponding lowercase letters denote their homogeneous equations. The deformation parameters are nonzero and chosen successively. Each smallness condition may depend on all earlier choices. Within an induction, the symbols $\alpha, \beta, \delta, \eta$ refer to the current step, and their values are chosen separately at successive steps. After each induction step, the new octic or quartic is again denoted by $O = (o = 0)$ or $Q = (q = 0)$, respectively. The two quadrics in the pencil use the same parameter, as in their displayed equations.

3 Brody curves and plane configurations

The starting complements for the two arrangements are

$$H_i \setminus (H^i \cup H'), \tag{5}$$

$$H_{ij} \setminus ((H_l \setminus H_k) \cup H'), \tag{6}$$

$$(H_i \cap H'_j) \setminus (H^i \cup (H'^j \setminus H'_k)), \qquad j \neq k. \tag{7}$$

The other seven hyperplanes restrict to seven hyperplanes in general position in $H_i \simeq \mathbb{P}^3$, so Fujimoto's theorem gives hyperbolicity of (5). For (6), Borel's theorem places an entire curve in a diagonal of the four primed lines. Choose the unprimed arrangement so that H_l removes a third point on each diagonal, distinct from $v = H_{1234}$, the point restored on H_l; this imposes finitely many proper algebraic exclusions. In a mixed complement (7), the three unprimed hyperplanes give three omitted lines. The two primed components other than H'_j and H'_k give two further lines, each with its intersection with H'_k restored. Lemma 5(i) will treat these complements; part (ii) will be used for the quadric projection.

For a rational slope $\lambda = p/q$ in (4), where p, q are coprime integers with $0 < p < q$, the image satisfies $Y^q = a^q X^p Z^{q-p}$ in homogeneous coordinates $[Z : X : Y]$. Its intersections with a conic in the torus form a fiber of the quotient in the following lemma. A single choice of conic must work for all such slopes. Here "very general" means outside a countable union of proper algebraic subsets of the space of smooth conics.

Lemma 4. *For a very general smooth conic D transverse to the coordinate triangle and avoiding its vertices, every permutation of $[Z : X : Y]$, and every coprime pair $0 < p < q$, the function*

$$G = \frac{Y^q}{X^p Z^{q-p}}\bigg|_D$$

has simple critical points over $\mathbb{C}^*$ with distinct critical values.

Proof. For a fixed pair (p, q), these are algebraic open conditions. To prove nonemptiness, parametrize a conic by the binary quadratics whose affine expressions are

$$X = (t-1)(t-a), \qquad Y = t, \qquad Z = (t-b)(t-c), \qquad 1 < a < b < c.$$

With homogeneous parameter $[t : s]$, the middle quadratic is $Y = ts$. The coefficient determinant is $bc - a \neq 0$, so the conic is smooth and its six coordinate intersections are distinct. The logarithmic derivative of G is

$$R(t) = \frac{q}{t} - \frac{p}{t-1} - \frac{p}{t-a} - \frac{q-p}{t-b} - \frac{q-p}{t-c}.$$

Its numerator has degree four, with leading coefficient $-q$. The signs at the two ends of each interval

$$(-\infty, 0), \quad (1, a), \quad (a, b), \quad (b, c)$$

are opposite. Each interval contains a root, so the four roots are simple and account for all critical points over $\mathbb{C}^*$.

If two critical values coincide, follow the corresponding distinct critical points t_i, t_j as a varies. The derivative of the difference of local logarithms of the critical values is

$$p\left(\frac{1}{t_i - a} - \frac{1}{t_j - a}\right) \neq 0;$$

the terms arising from the motion of the critical points vanish since $G_t = 0$ there. A small change of a separates all coincident pairs while preserving simplicity. For each (p, q), the desired conditions therefore hold on a nonempty Zariski open set. The countable intersection over all pairs and the finitely many coordinate permutations is dense by Baire's theorem. □

Lemma 5. (i) *For a general configuration of three lines L_1, L_2, L_3 and two further lines M_1, M_2, with specified general points $p_r \in M_r$, there is no Brody curve in*

$$\mathbb{P}^2 \setminus (L_1 \cup L_2 \cup L_3 \cup (M_1 \setminus \{p_1\}) \cup (M_2 \setminus \{p_2\})).$$

(ii) *For a very general conic D relative to the three lines, every Brody curve in $\mathbb{P}^2 \setminus (L_1 \cup L_2 \cup L_3)$ that is nowhere transverse to D is contained in a line through a vertex of the three-line arrangement and tangent to D.*

Proof. Use the representation (4). First exclude nonreal slopes. If $\operatorname{Im} \lambda \neq 0$, translate the parameter by it_n, choosing the sign so that $\operatorname{Re}(i\lambda t_n) \to -\infty$. After taking a subsequence with $e^{it_n} \to b \in \mathbb{C}^*$, the curves tend to

$$z \longmapsto [1 : be^z : 0].$$

This curve fills a coordinate line minus its two vertices. It meets a general M_r away from p_r, and meets a conic transverse to the coordinate lines transversely. Simple intersections persist under locally uniform convergence, contradicting either hypothesis.

For a real slope, permute the homogeneous coordinates and rescale the parameter so that $0 \leq \lambda \leq 1$. In the affine coordinates $(x, y) = (X/Z, Y/Z)$, an irrational λ gives an image dense in

$$|y| = |a||x|^\lambda.$$

The other leaves in this real hypersurface are translation limits of the curve. On a general additional line, or on a smooth conic transverse to the coordinate lines, the function

$$\log|y| - \lambda \log|x|$$

is harmonic away from the coordinate lines. It tends to $-\infty$ at the points with $y = 0$, and to $+\infty$ at the points with $x = 0$ or at infinity. Each level therefore contains a regular

point; the critical points of a nonconstant harmonic function are isolated. At a regular point the line or conic is transverse to the corresponding leaf. In part (i), the regular point can also be chosen away from the restored point. Persistence gives the same contradiction. Thus the slope is rational in both assertions.

Write $\lambda = p/q$, where $0 \le p \le q$ and $\gcd(p,q) = 1$. With $t = e^{z/q}$, the image lies in the rational curve parametrized by

$$R: \quad t \longmapsto [1 : t^q : at^p], \qquad 0 \le p \le q, \quad \gcd(p,q) = 1. \tag{8}$$

Its degree is q. The cases $p = 0$ and $p = q$ give lines. If $0 < p < q$, the Wronskian is

$$W(1, t^q, at^p) = apq(p-q)t^{p+q-3},$$

which does not vanish for $t \in \mathbb{C}^*$. Thus R has no inflection point away from the coordinate lines.

(i) Suppose first that $q > 1$. Then $0 < p < q$, and R meets the coordinate triangle only at two vertices. Each M_r avoids these vertices, so its intersection with R must be concentrated at p_r. The absence of an inflection point bounds the multiplicity by two; hence $q = 2$ and $p = 1$. Up to a coordinate permutation these conics have equation $Y^2 = cXZ$. The condition that p_1 lie on such a conic determines c. Requiring M_1 to be tangent there is a proper algebraic condition on (M_1, p_1), excluded for a general configuration.

If $q = 1$, the image is a line through a coordinate vertex P, with P and one point Q on the opposite side omitted. Each M_r must meet this line at p_r or at Q. The two intersections cannot both be Q, since the two extra lines and a coordinate side are not concurrent. Once one restored point is used, the line is determined by that point and P. It would then have to pass through either the other restored point or the intersection of the other extra line with the opposite side. These finitely many collinearities are proper algebraic conditions, again excluded by the general choice.

(ii) In homogeneous coordinates $[Z : X : Y]$, consider

$$G = \frac{Y^q}{X^p Z^{q-p}}\bigg|_D.$$

If $q > 1$, all intersections of R with D lie in the torus. The degree of G is $2q$, and its critical points over $\mathbb{C}^*$ are simple with distinct critical values by Lemma 4. Since R is the level $G = a^q$, absence of transverse intersections would make every point of this fiber critical. There can be at most one such point, with multiplicity two, contradicting $2q > 2$.

The remaining curve R is a line through a coordinate vertex P, omitting P and a point Q on the opposite side. The conic avoids P. If the line were not tangent to D, its two distinct intersections with D would include one different from Q. That point is on the Brody curve and is a transverse intersection, a contradiction. Thus the line is tangent to D. □

The first assertion excludes all entire curves in the stated complement, not only Brody curves. Indeed, Brody reparametrization gives a nonconstant limit in $\mathbb{P}^2$. If the limit lies on a coordinate line, the other two coordinate lines and either additional line remove at least three distinct points. The restored points are outside the coordinate triangle. If it lies on an additional line, its three intersections with the coordinate lines are omitted. Picard's theorem excludes these boundary limits. Every other limit is a Brody curve in the stated complement and is excluded by the lemma.

4 The auxiliary octic

To deform the second arrangement H' to a pair of quadrics, we need an octic O for which the three families

$$H_i \setminus \big(H^i \cup (H' \setminus O)\big), \tag{9}$$
$$H_{ij} \setminus \big((H_l \setminus H_k) \cup (H' \setminus O)\big), \tag{10}$$
$$(H_i \cap H'_j) \setminus \big(H^i \cup (H'^j \setminus (H'_k \cup O))\big), \qquad j \neq k, \tag{11}$$

are hyperbolic. In the first family, the unprimed boundary H^i remains omitted, while intersections with H' are allowed only on O. We introduce these allowed intersections first on the six double planes of H' and then on its four hyperplanes. The initial octic avoids all quadruple points of $H \cup H'$. At each later stage, excluding finitely many parameter values preserves this avoidance condition.

Let P be the union of a collection of double planes of H'. The induction preserves hyperbolicity of

$$H_i \setminus \big(H^i \cup (H' \setminus (O \cap P))\big), \tag{12}$$
$$(H_i \cap H'_j) \setminus \big(H^i \cup (H'^j \setminus (H'_k \cup (O \cap P)))\big). \tag{13}$$

For $P = \varnothing$, these are (5) and (7). To add H'_{12}, take $P_+ = P \cup H'_{12}$ and replace O by

$$O_\alpha = \big((h'_3)^4 (h'_4)^4 = \alpha o\big). \tag{14}$$

For α sufficiently small, (12)–(13) remain hyperbolic after (O, P) is replaced by (O_α, P_+). Indeed, on every previously selected double plane the monomial in (14) vanishes, so the old intersections with O are unchanged. The new intersections on H'_{12} tend to

$$H'_{123} \cup H'_{124}.$$

Consider a nonconstant Brody limit in H_i which is not contained in any component of H'. If an intersection of the limit with H'_1 or H'_2 is approached by newly allowed points, apply persistence to H'_3 or H'_4. The restriction there forces the limiting point to be an old allowed intersection, so the limit lies in (12).

If the limit lies in exactly one H'_j, the same argument applies to the other primed hyperplanes, with one component left free: take $k = 2$ when $j = 1$, $k = 1$ when $j = 2$, and $k = 1$ when $j = 3$ or 4. The limit then lies in (13). A limit contained in two primed hyperplanes lies in a line $H_i \cap H'_{jk}$, from which H^i removes three distinct points. More primed hyperplanes leave only a point. For curves initially in $H_i \cap H'_j$, the same restriction argument gives the preceding mixed complement; an additional primed hyperplane again leaves a line with three unprimed points removed.

It remains to consider limits in an unprimed boundary hyperplane. In H_{ij}, the old allowed intersections on P are absent because O avoids all quadruple points. By general position, the new limiting intersections on $H'_{123} \cup H'_{124}$ do not lie in H_{ij}. Such a limit omits the four primed lines and the remaining unprimed lines, and is excluded by (6). In a mixed boundary line $H_i \cap H_j \cap H'_a$, the two remaining unprimed hyperplanes and at least two of the other primed hyperplanes remove distinct points. Deeper strata are punctured lines or points. Picard's theorem excludes the line cases. The other double planes are added by relabeling. After six steps, $P = \Pi'$; thus (11) has been obtained and the first family is

$$H_i \setminus \big(H^i \cup (H' \setminus (O \cap \Pi'))\big). \tag{15}$$

In (15), the allowed intersections with H' are still restricted to Π'. We now extend them to the individual hyperplanes. Let S be the union of a collection of components of H'. The next induction preserves the hyperbolicity of

$$H_i \setminus \big(H^i \cup (H' \setminus (O \cap (S \cup \Pi')))\big), \tag{16}$$

$$H_{ij} \setminus \big((H_l \setminus H_k) \cup (H' \setminus (O \cap S))\big). \tag{17}$$

For $S = \varnothing$, these are (15) and (6). For $S_+ = S \cup H'_1$, the deformation is

$$O_\beta = \big((h'_2)^2 (h'_3)^2 (h'_4)^4 = \beta o\big). \tag{18}$$

For sufficiently small β, replacing (O, S) by (O_β, S_+) preserves hyperbolicity in (16)–(17). The old restrictions on $S \cup \Pi'$ are unchanged. The new allowed intersections on H'_1 tend to its intersections with $H'_2 \cup H'_3 \cup H'_4$. A limit not contained in a primed hyperplane lies in the preceding complement, by persistence of intersections with the other primed hyperplane through each new limiting intersection. For a limit in a single primed hyperplane, use the mixed complements (11). If the containing hyperplane differs from H'_1, leave H'_1 free; if it is H'_1, any other index can be left free because the remaining restrictions are unchanged. Two primed hyperplanes leave a line with three unprimed points removed.

For a limit in H_{ij}, the second induction invariant (17) gives the same conclusion unless the limit lies in a primed line. On $H_{ij} \cap H'_1$, the other three primed hyperplanes remove three points. On $H_{ij} \cap H'_a$ with $a \neq 1$, at most the point on H'_1 is newly allowed; the other two primed hyperplanes and H_l still remove three distinct points. The point removed by H_l is not restored by H_k, by general position. Finally, a limit contained in the unprimed line H_{ijl} omits all four primed hyperplane points: the old octic avoids these quadruple points, and none is a new limiting intersection. Picard's theorem excludes the remaining cases. At $S = H'$, the two families give (9) and (10). Moreover,

$$O_\beta \cap \Pi' = O \cap \Pi' \qquad (\beta \neq 0), \tag{19}$$

because the monomial in (18) vanishes identically on every double plane of H'. Each mixed complement (11) depends only on this restriction: on $H_i \cap H'_j$, the octic is used only where another component of H' meets H'_j. Thus the mixed complements are unchanged during the second induction.

5 The quadric pencil

The two quadrics lie in the pencil generated by $h'_1 h'_2$ and $h'_3 h'_4$. For a small nonzero parameter u, write

$$C = (h'_1 h'_2 = u h'_3 h'_4), \qquad C' = (h'_3 h'_4 = u h'_1 h'_2), \qquad u^2 \neq 1. \tag{20}$$

Deforming the two pairs of hyperplanes by (20) and using (9)–(11) gives hyperbolicity of

$$H_i \setminus \big(H^i \cup ((C \cup C') \setminus O)\big), \tag{21}$$

$$H_{ij} \setminus \big((H_l \setminus H_k) \cup ((C \cup C') \setminus O)\big). \tag{22}$$

The mixed complements control curves whose limits lie in a component of the second arrangement during these two quadric deformations. More precisely, a limit not contained in H' lies in the corresponding octic complement. If it lies in H'_1 or H'_2, persistence against C' forces it to avoid $H'_3 \cup H'_4$ outside O; the other member of the pair is the

component left free in (11). For a limit in H_3' or H_4', interchange the two pairs. Two primed hyperplanes leave a line in H_i, punctured by the three unprimed hyperplanes. On $H_{ij} \cap H_a'$, the opposite primed pair removes two points and H_l removes a third. The octic does not restore these points, since they are quadruple points of the arrangement. Limits in H_{ijl} omit the four primed hyperplane points. The same puncture counts exclude limits in unprimed boundary strata.

The parameters in these finite constructions can be chosen uniformly for H in a neighborhood of its initial value H_0. Failure of a common neighborhood and a common small parameter bound at any stage would give a sequence $H_\nu \to H_0$ together with stage parameters $\tau_\nu \to 0$ for which the required complement contains an entire curve. Brody reparametrization and persistence for the convergent linear equations give exactly the limit cases just excluded at H_0. The finite avoidance and transversality conditions are open. Choose the current parameter within the resulting bound and shrink the neighborhood before the next stage. Since there are finitely many stages, the two complement families (21)–(22), with C, C', O fixed, hold throughout a common nonempty open neighborhood of H.

We also need the following complements on quadric sections to be hyperbolic, for $D = C$ or C' and $a = 1$ or 2:

$$(H_i \cap D) \setminus \big(H^i \cup (H_a' \setminus (\Lambda' \cup O))\big). \tag{23}$$

Choose H in this neighborhood so that all the conic conditions of Lemma 4 hold simultaneously. For each i and each $D = C, C'$, fix a smooth section $H_i \cap D$ and a projection center in H_i outside the section. The other three hyperplanes through the center project to an arbitrary triangle. Since projective transformations act transitively on smooth conics, each fixed-slope good condition pulls back to a dense algebraic open condition on the arrangements. By Baire's theorem, the countably many slope conditions for the finitely many choices of i and D hold simultaneously for some arrangement in the same open neighborhood.

The finitely many additional conditions on ruling lines can be imposed in the same neighborhood. As a triple line moves through the projection center, its intersection points with the quadric vary over the quadric, so both families of ruling lines vary freely. Their intersections with H_a' sweep the two lines of $H_i \cap C \cap H_a'$. Neither line is contained in O, since the octic avoids the mixed quadruple points. Thus the conditions that these intersections lie outside $\Lambda' \cup O$ and be distinct from the two unprimed boundary points are proper algebraic exclusions. The same argument applies to C' and $a = 2$. It is enough to give the projection proof for $D = C$ and $a = 1$.

Take $v = H_{1234}$ outside C. For a plane $P \subset H_i$ not containing v, linear projection from v gives

$$\pi : H_i \setminus \{v\} \longrightarrow P.$$

Its restriction to the smooth quadric surface $H_i \cap C$ is a finite double cover branched along a conic $D_\pi \subset P$. Finiteness and the bound on its derivative over the compact surface ensure that the restriction sends Brody curves to Brody curves.

If (23) contained an entire curve, Brody reparametrization would produce a nonconstant limit in the quadric surface. A limit in an unprimed boundary hyperplane lies on a conic punctured in four points by the other two unprimed hyperplanes. A limit in $H_a' \cap C$ lies on one of its two ruling lines, each punctured in three distinct points by the unprimed hyperplanes. Picard's theorem excludes both boundary cases. We therefore obtain a Brody curve B in (23). The projected curve $\pi \circ B$ omits the three lines $P \cap H^i$ and is immersed by the exponential normal form. Since it lifts through the double cover, every

isolated intersection with D_π has even multiplicity, so the curve is nowhere transverse to D_π. By Lemma 5(ii), its image lies in a line through a vertex of the three-line arrangement and tangent to D_π.

The plane spanned by v and this line contains one of the triple lines H_{ijk} and is tangent to $H_i \cap C$. Its intersection with the quadric is a pair of ruling lines, and the image of B lies in one of them. The ruling line passes through one of the two points of $H_{ijk} \cap C$ and meets H^i in two points. By the incidence conditions imposed above, its intersection with H'_1 is a third distinct point outside $\Lambda' \cup O$. All three points are omitted, contrary to Picard's theorem. The same argument applies to the other quadric and to H'_2.

With c, c' the homogeneous equations of C, C' in (20), deform their union by

$$Q' = (cc' = \gamma(h'_1)^4), \qquad 0 < |\gamma| \ll 1. \tag{24}$$

For distinct members of the pencil,

$$C \cap C' = (H'_1 \cup H'_2) \cap (H'_3 \cup H'_4). \tag{25}$$

We use the base locus and the transverse double locus of Q' in $H'_1 \setminus \Lambda'$ to prove hyperbolicity of

$$H_i \setminus (H^i \cup (Q' \setminus O)), \tag{26}$$
$$H_{ij} \setminus ((H_l \setminus H_k) \cup (Q' \setminus O)), \tag{27}$$
$$(H_i \cap Q') \setminus H^i \tag{28}$$

by applying (21), (22), and (23) to the Brody limits as $\gamma \to 0$.

In (28), the approximating curves lie in Q', so their limit lies in $C \cup C'$. Suppose that it lies in C but not in $C \cap C'$. On Q', the equation $c' = 0$ forces $h'_1 = 0$. Persistence of intersections therefore excludes intersections of the limit with C' away from H'_1. By (25), the part of $C \cap C'$ outside H'_1 is the part lying on H'_2, and

$$C \cap H'_1 \cap H'_2 \subset \Lambda'.$$

Consequently the limit avoids $H'_2 \setminus \Lambda'$. If the limit also avoids H^i, it is excluded by (23) with $a = 2$.

In (26) and (27), the approximating curves avoid $Q' \setminus O$. Suppose again that their limit lies in C but not in $C \cap C'$. Near a point of

$$C \cap C' \cap H'_1 \setminus (\Lambda' \cup O),$$

the two components of $C \cup C'$ meet transversely. The equation $cc' - \gamma(h'_1)^4 = 0$ has two local analytic factors tending, up to nonvanishing factors, to c and c'. These factors describe the two branches of Q' along this transverse double locus. On the approximating curves both factors are nonzero in a neighborhood disjoint from O. Hurwitz's theorem applied to the factor tending to c' excludes an intersection of the limit with C' there. Since $C \cap H'_1 \subset C \cap C'$, the limit avoids $H'_1 \setminus (\Lambda' \cup O)$. Here the relevant complement is (23) with $a = 1$.

A limit contained in $C \cap C'$ lies in one of the planes in (25). Its intersection with H_i is a line; the other three unprimed hyperplanes remove three distinct points for a general configuration. Picard's theorem excludes this case. The arguments with C and C' interchanged are identical.

For a limit in H_{ij} contained neither in $C \cup C'$ nor in a triple line, apply (22). Limits from $H_i \setminus H^i$ omit both remaining unprimed hyperplanes; for curves initially in H_{ij}, persistence

preserves omission of $H_l \setminus H_k$, which is sufficient here. A limit in $H_{ij} \cap C$ or $H_{ij} \cap C'$ lies on a smooth conic. For curves initially avoiding H^i, the two remaining unprimed hyperplanes remove four distinct points. For curves initially in H_{ij}, H_l and the applicable H'_a each remove two points. The latter points lie on the pencil base and are outside O and Λ' by the finite general-position conditions. A limit in H_{ijl} omits the four points of $(C \cup C') \cap H_{ijl}$. The conic and triple-line cases therefore have at least three punctures on a rational curve.

The quartic deformations below use the one-dimensional sections of Q' to exclude limits in double planes. Writing $x = h'_1$, $y = h'_2$, $z = h'_3$, and $w = h'_4$, the equation of Q' becomes

$$-ux^2y^2 + (1+u^2)xyzw - uz^2w^2 - \gamma x^4 = 0.$$

For the nonzero small parameters under consideration, the singular locus consists of the two planes H'_{13} and H'_{14}. The transverse singularities away from Λ' are ordinary nodes. The polynomial is irreducible: its discriminant as a quadratic in y is

$$x^2\big((1-u^2)^2z^2w^2 - 4u\gamma x^4\big),$$

which is not a square in $\mathbb{C}(x,z,w)$. A general plane H_{ij} therefore cuts out an irreducible quartic with exactly two ordinary nodes. Its normalization has genus $3-2=1$. The complement of a nonempty finite set in this curve contains no nonconstant entire curve. The arrangement is chosen so that these sections are general, each triple line meets Q' in four distinct points outside O, and $v \notin Q'$.

6 Quartic deformations and the final construction

6.1 The auxiliary quartic

With Q' and O fixed, we seek a quartic $Q = (q = 0)$ for the deformation of H to Q_1. Hyperbolicity of the following three families will give (3):

$$H_i \setminus \big((H^i \setminus Q) \cup (Q' \setminus O)\big), \tag{29}$$
$$H_{ij} \setminus \big((H^{ij} \setminus Q) \cup (Q' \setminus O)\big), \tag{30}$$
$$(H_i \cap Q') \setminus (H^i \setminus Q). \tag{31}$$

The quartic is chosen to avoid the finite set $\Lambda \cap Q'$ and the vertex v. These conditions are retained at every deformation by excluding finitely many parameter values. As in the construction of O, we introduce the allowed boundary intersections on Q first on the four triple lines of H, then on its six double planes.

Let L be the union of a collection of triple lines. The induction preserves hyperbolicity of

$$H_i \setminus \big((H^i \setminus (Q \cap L)) \cup (Q' \setminus O)\big), \tag{32}$$
$$H_{ij} \setminus \big((H_l \setminus (H_k \cup (Q \cap L))) \cup (Q' \setminus O)\big). \tag{33}$$

For $L = \varnothing$, these are (26)–(27). To add H_{123}, take $L_+ = L \cup H_{123}$ and replace Q by

$$Q_\delta = (h_4^4 = \delta q). \tag{34}$$

For sufficiently small δ, replacing (Q, L) by (Q_δ, L_+) preserves hyperbolicity in (32)–(33). Every previously selected triple line lies in H_4, so $Q_\delta \cap L = Q \cap L$. The new points on H_{123}

tend only to v, and v itself is not on Q_δ. For $i \neq 4$, a limit not contained in an unprimed boundary hyperplane cannot acquire a new intersection at v: persistence of intersections with H_4 would force $q(v) = 0$. It therefore lies in the preceding complement (32). For $i = 4$, the new line meets H_4 only at $v \notin Q_\delta$, so no new point is restored there. A limit in a double plane is controlled by (33); the only new limiting point is v, already on its restored component H_k. For curves initially in that double plane, the same argument applies unless the limit lies in H_l. On the resulting triple line the limit omits the four points of $Q' \setminus O$.

Limits contained in Q' require a separate check. The old allowed points on L miss Q', and the new ones tend to $v \notin Q'$. A limit in $H_i \cap Q'$ that is not contained in a double plane thus avoids H^i and is excluded by (28). A limit in $H_{ij} \cap Q'$ omits the intersections with a remaining unprimed hyperplane, since these lie on $\Lambda \cap Q'$ and Q avoids that finite set. Its genus-one normalization is punctured, so it too admits no nonconstant entire curve. A deeper intersection with Q' is finite. These alternatives exhaust the limiting curves. After the four triple lines have been added, the first family is

$$H_i \setminus \big((H^i \setminus (Q \cap \Lambda)) \cup (Q' \setminus O)\big), \tag{35}$$

and the second is

$$H_{ij} \setminus \big((H_l \setminus (H_k \cup Q)) \cup (Q' \setminus O)\big). \tag{36}$$

Here the intersections with Q on $H_l \cap H_{ij}$ lie on a triple line, so the restriction to Λ is redundant in (36).

We next allow further intersections with the boundary H^i at points of Q on the double planes, retaining those already allowed on the triple lines. Let P be the union of a collection of double planes of H. The required families are

$$H_i \setminus \big((H^i \setminus (Q \cap (P \cup \Lambda))) \cup (Q' \setminus O)\big), \tag{37}$$

$$(H_i \cap Q') \setminus (H^i \setminus (Q \cap P)). \tag{38}$$

For $P = \varnothing$, these are (35) and (28). For $P_+ = P \cup H_{12}$, the deformation is

$$Q_\eta = (h_3^2 h_4^2 = \eta q). \tag{39}$$

For sufficiently small η, replacing (Q, P) by (Q_η, P_+) preserves hyperbolicity in (37)–(38). The old restrictions on $P \cup \Lambda$ are unchanged, and the new allowed points on H_{12} tend to $H_{123} \cup H_{124}$. For curves initially in H_3 or H_4, these are already allowed triple-line intersections, so the complements are unchanged. For curves initially in H_1 or H_2 whose limit is not contained in H_3 or H_4, persistence against those hyperplanes forces every new limiting intersection to be an old allowed one. If the limit lies in a double plane contained in H_3 or H_4, use (36), leaving the component with possible new intersections free. In H_{12} both remaining hyperplanes have unchanged restrictions. In H_{34} a new limiting point could only be v, which is already on the component left free in the boundary-plane complement. Triple lines omit the four points of $Q' \setminus O$.

For limits contained in Q', the same persistence argument gives the preceding complement (38) unless the limit lies in a double plane. In that case it is contained in the plane quartic $H_{ij} \cap Q'$. The new points tend to triple lines, while $Q \cap \Lambda \cap Q' = \varnothing$; persistence against a remaining unprimed hyperplane leaves a nonempty puncture set on its genus-one normalization. If both H_3 and H_4 contain the limit, the new limiting point is $v \notin Q'$, so the old punctures remain. Deeper intersections with Q' are finite. Thus both (37) and (38) remain hyperbolic. After six steps, $P = \Pi$, and these families become (29) and (31). The boundary-plane complements (36) remain unchanged, since

$$Q_\eta \cap \Lambda = Q \cap \Lambda \qquad (\eta \neq 0). \tag{40}$$

Indeed, every triple line lies in H_3 or H_4. In (36), the quartic is used only on the triple line H_{ijl}, so (40) preserves the complement at each step. The complement (30) is contained in (36) and is therefore hyperbolic as well.

6.2 The final hypersurface

With this choice of Q, deform the first arrangement by

$$Q_1 = (h_1h_2h_3h_4 = rq), \qquad 0 < |r| \ll 1. \tag{41}$$

Persistence of intersections reduces (3) to (29)–(31). Indeed, at an intersection of Q_1 with any H_j, equation (41) forces $q = 0$. A Brody limit in H_i that is not contained in a double plane therefore lies in (29), unless it is contained in Q', in which case (31) applies. A limit in H_{ij} and not in Q' lies in (30). A limit in $H_{ij} \cap Q'$ is excluded by the punctured genus-one normalization just considered: the intersections of the quartic section with the remaining hyperplanes cannot be restored because Q avoids $\Lambda \cap Q'$. A triple line omits the four points of $Q' \setminus O$, and the quadruple intersection is a point. For curves initially in $Q_1 \cap Q'$, the same classification starts in $H_i \cap Q'$ and uses (31). Thus no nonconstant limit curve can occur in either family of (3).

After fixing Q_1, Q', O, take

$$Q_2 = (q_1 = sq'), \qquad 0 < |s| \ll 1. \tag{42}$$

On Q_1, the equations of Q_2 and Q' agree up to a nonzero factor. Hence

$$Q_1 \setminus (Q_2 \setminus O) = Q_1 \setminus (Q' \setminus O), \qquad Q_1 \cap Q_2 = Q_1 \cap Q'.$$

On Q_2, the equation $q_1 = sq'$ likewise gives

$$Q_2 \setminus (Q_1 \setminus O) = Q_2 \setminus (Q' \setminus O).$$

For sufficiently small s, the last complement is hyperbolic by deformation from Q_1 with Q' fixed. If a Brody limit is not contained in Q', Hurwitz's theorem forces it to avoid $Q' \setminus O$. A limit contained in Q' lies in the hyperbolic intersection $Q_1 \cap Q'$.

Proof of Theorem 1. The three sets in (2) are hyperbolic. Lemma 3, applied to $A = Q_1$, $B = Q_2$, and $R = O$, therefore gives Kobayashi hyperbolicity of $X_t = (q_1q_2 = to)$ for all sufficiently small nonzero t. Its degree is eight. □

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