ANALYTIC CONSTRUCTION OF RATIONAL CURVES ON FANO MANIFOLDS

YUN-HENG DU, BIN GUO, AND SONG-YAN XIE

Abstract. Inspired by constructions of entire curves in Oka geometry, we construct rational curves on complex Fano manifolds by holomorphic disk surgeries. Positive Ricci curvature guides an area-decreasing deformation, and an affine-lift perturbation enlarges the source. Explicit estimates control the area and weighted derivative changes of both operations. A finite hierarchy of principal-map recoveries then gives a sphere meeting a prescribed compact fiber without being contained in it, with a sharp metric area bound. The construction uses analytic compactness to pass from the disks to a sphere.

Contents

1. Introduction and main results

1.1. The analytic construction problem

We construct rational curves on complex Fano manifolds by explicit analytic operations, with control of the intermediate disks. A Fano manifold X is connected, smooth, and compact, with ample anticanonical line bundle K_X^{-1}.

Rational curves are central to Mori's solution [47] of Hartshorne's conjecture on ample tangent bundles. In his monograph *Ample subvarieties of algebraic varieties*, Hartshorne [33] asked whether a smooth complex projective variety with ample tangent bundle must be projective space. Mori's construction combines deformation theory and bend-and-break, which also apply in characteristic zero, with Frobenius in positive characteristic and specialization back to $\mathbb{C}$.

On the analytic side, Siu and Yau [55] used energy-minimizing harmonic spheres in their 1980 proof of the Frankel conjecture. Their argument uses positive holomorphic bisectional curvature, a stronger condition than the Ricci positivity that Yau's prescribed-Ricci theorem provides on every Fano manifold [61]. In 1982, Yau asked whether any two points of a compact Kähler manifold with positive Ricci curvature could be joined by a chain of rational curves [62, p. 44]. Rational connectedness of Fano manifolds was subsequently established by Campana and by Kollár, Miyaoka,

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and Mori [7, 43]. An analytic construction remained a separate challenge, which Siu later proposed to approach through complex Monge–Ampère equations, without reduction to positive characteristic [54].

Our approach to this problem draws on methods for constructing entire curves in Oka geometry, in particular those of Xie [58] (see also [10, 35, 57]). These methods build entire curves by successively modifying holomorphic disks and enlarging their domains. Guo and Xie [31, Section 5] suggested pursuing such constructions on Fano manifolds with a slow-growth condition, with the aim of producing both entire and rational curves analytically. The construction developed here gives a concrete realization of this vision.

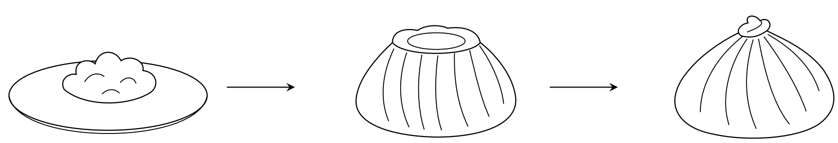
A nonconstant entire curve of finite area gives a rational curve by removal of the singularity at infinity (Theorem 2.4). We therefore try to construct such an entire curve as a limit of holomorphic disks with radii tending to infinity. Brody's lemma [6] suggests the basic strategy: normalize the derivative at the origin and obtain convergence from derivative bounds on compact subsets. Here we must also keep the total area uniformly bounded. Positive Ricci curvature provides the area-decreasing deformations used for this purpose. Derivative concentration may prevent direct convergence, so we also analyze the spheres arising through bubbling and retain their required incidence properties.

To carry out these deformations, we need room to vary holomorphic disks in the target. In the Fano setting, the following folklore conjecture in Oka geometry gives reason to expect such flexibility; see [25, p. 660].

Conjecture A. *Every Fano manifold is Oka.*

Our recent theorem that rationally connected smooth projective manifolds are Oka–1 [17, Corollary 1.8] gives a first step toward the conjecture, since it includes every Fano manifold. This result encourages us to seek considerable freedom in deforming holomorphic disks. Its rational-connectedness hypothesis means that it provides motivation here, rather than an input to the rational-curve construction or its applications. Known cases of the Oka conjecture include smooth quadric [28], cubic [39], and quartic [18] Fano hypersurfaces; see also [59]. For background on Oka and Oka–1 theory, we refer to [24, 1].

This suggests an analogy with making a baozi, a Chinese steamed bun. The dough is repeatedly stretched and gathered around its filling, just as we deform holomorphic maps on successively larger disks. Sealing the bun corresponds to compactifying the resulting finite-area entire curve at infinity.



(a) Wrapper and filling (b) Gather the pleats (c) Seal the baozi

FIGURE 1. Shaping a baozi by gathering a dough wrapper around a filling and sealing the pleats. This traditional Chinese craft suggests an analogy for closure. In the proof, a finite-area entire principal map extends across infinity; bubbling is treated separately by marked compactness.

To carry out this construction, two difficulties must be resolved. The first is area control: holomorphic approximation alone supplies no uniform bound as the source radii tend to infinity.

The second difficulty is to retain location in the limit. Classical Brody reparametrization moves the source centers [6], so even disks through a fixed point need not yield an entire curve through that point. Duval addressed this loss of location by replacing derivative selection with control of where area accumulates [19]; see also [20]. For a compact set charged by an Ahlfors current, his theorem produces an entire curve whose intersection with that set has positive area. Duval's localization viewpoint also guides our construction. In our finite-area setting, we retain position through a base constraint and two marked values, rather than invoke the Ahlfors-current theorem itself.

Even when the source center is fixed, bubbling can separate its prescribed local data from the nonconstant sphere components. The analysis of bubbling goes back to Sacks and Uhlenbeck's work on harmonic maps [52] and Gromov's theory of pseudoholomorphic curves [29]. Area may also concentrate near the source boundary. We must therefore obtain a nonconstant sphere satisfying both the required incidence conditions and the area bound.

We use Ricci positivity to deform disks so that their area decreases above an explicit threshold. The key is a Ricci-trace estimate, realized through approximation and interpolation by holomorphic families that preserve a scalar derivative condition and satisfy uniform estimates. To control where the resulting curve lies, we work relative to a compact fiber F of a local holomorphic projection q. We allow the center of a disk f to move within F, while fixing one nonzero scalar component of the derivative of $q \circ f$. This keeps the initial tangent vector outside the tangent space of F without fixing the entire vector. Together with marked compactness, this normalization produces a sphere that meets F but is not contained in it, with a sharp area bound. Taking F to be a single point gives a rational curve through that point. The two surgeries control the area and weighted derivative of the intermediate disks, while the recovery levels account for bubbling. The curves are constructed analytically over $\mathbb{C}$.

1.2. **Main results**

For rational connectedness, we need a rational curve that meets a fiber of the rational quotient without remaining inside it. The main theorem therefore treats a compact fiber of a holomorphic map q defined on an open neighborhood N. Here $F = q^{-1}(y_0)$ denotes the entire fiber over y_0 in N, including all its components. We use the Ricci and area conventions in (1).

Theorem 1.1. *Let (X, ω) be a compact Kähler n-fold satisfying*

$$\mathrm{Ric}(\omega) \geq \kappa\omega, \qquad \kappa > 0.$$

For an open subset $N \subset X$ and a complex manifold Y of positive dimension, consider a holomorphic map $q : N \to Y$ and a point $y_0 \in Y$. Suppose that the full fiber $F := q^{-1}(y_0)$ is nonempty and compact and that q is a submersion on a neighborhood of F in N. Then there is a nonconstant holomorphic map $u : \mathbb{P}^1 \to X$ such that

$$u(\mathbb{P}^1) \cap F \neq \varnothing, \qquad u(\mathbb{P}^1) \not\subset F, \qquad \int_{\mathbb{P}^1} u^*\omega \leq \frac{2\pi(n+1)}{\kappa}.$$

Every Fano manifold admits a metric with this Ricci lower bound, by Yau's prescribed-Ricci theorem [61]; a Kähler–Einstein metric is not required. See Proposition 3.1. Conversely, the lower bound gives positive curvature on K_X^{-1}, hence ampleness and projectivity; see Remark 3.2. Thus the reduced image of the sphere in the theorem is a rational curve.

The projective-space example in Remark 2.11 shows that the coefficient $2\pi(n + 1)$ cannot be improved in a universal metric area bound.

The relative viewpoint has a classical antecedent in Campana's generic-fiber escape criterion and his Fano theorem [7, Proposition 2.7 and Theorem 3.1]. Here we work near a specified compact full fiber of a holomorphic submersion and obtain a metric area bound.

During the construction, the center is allowed to move within F. If compactness has not already produced a sphere meeting and leaving F, this freedom allows the normalized base derivative to pass to the principal map. The final limit used to obtain the sharp area bound retains the two incidence properties, but need not retain the derivative normalization. Thus the theorem prescribes neither a tangent direction nor a contact order.

Taking q to be a holomorphic coordinate map near x gives the fiber $F = \{x\}$, so the relative theorem immediately yields the following pointwise form.

Theorem 1.2. *Let (X, ω) be a positive-dimensional compact Kähler manifold of complex dimension n, and suppose $\mathrm{Ric}(\omega) \geq \kappa\omega$ for some $\kappa > 0$. Then, for every $x \in X$, there is a nonconstant holomorphic map $u_x : \mathbb{P}^1 \to X$ such that*

$$x \in u_x(\mathbb{P}^1), \qquad \int_{\mathbb{P}^1} u_x^*\omega \leq \frac{2\pi(n+1)}{\kappa}.$$

Classical bend-and-break produces a rational curve C through a prescribed point of a Fano n-fold with

$$0 < -K_X \cdot C \leq n + 1.$$

Mori's account [48, p. 749] attributes this form of the result to Kollár, by a technique similar to that of [47]. Indeed, integration over C gives

$$\kappa \int_C \omega \leq \int_C \mathrm{Ric}(\omega) = 2\pi(-K_X \cdot C).$$

The classical result controls the anticanonical degree, giving more information than the area bound for a fixed metric. Our estimate has the same order of growth in n. Our contribution is the analytic construction: the disk deformations give control of area and weighted derivatives during the process. For the classical argument, see [13, Chapter 3].

1.3. **The surgeries and their quantitative control**

For a holomorphic map $f : \mathbb{D}_R \to X$, write

$$\mathrm{Area}_R(f) = \int_{\mathbb{D}_R} f^*\omega, \qquad B_R(f) = \sup_{z \in \mathbb{D}_R} (R - |z|)|f_z(z)|_\omega.$$

A bound $B_R(f) \leq B_0$ gives $|f_z(z)|_\omega \leq B_0/(R - |z|)$, and hence a derivative bound on every smaller disk. This is the boundary-distance weighting used in Brody's reparametrization argument [6]. We use bounded values of B to choose uniform surgery parameters, and sequences with B tending to infinity to extract bubbles by point selection.

The first surgery lowers area while fixing the center and one scalar component of the derivative. We choose holomorphic variation fields satisfying these constraints and sum their second area variations along the real and imaginary parameter directions. The trace calculation gives

$$4\pi(n+1) - 2\int_{\mathbb{D}_R} f^* \mathrm{Ric}(\omega) \leq 4\pi(n+1) - 2\kappa \, \mathrm{Area}_R(f).$$

Thus, above $A_* = 2\pi(n+1)/\kappa$, at least one real direction has negative second variation. Choosing the sign of the parameter also makes its first-order contribution nonpositive. In a holomorphic frame with unitary boundary values, the constraints are imposed by one quadratic and $n-1$ linear multiplier components, with the least possible Dirichlet cost $\pi(n+1)$ under the prescribed interpolation and boundary conditions; see the calculation after (34). A projective lift, a corona theorem, and a holomorphic retraction realize these fields by actual families; uniform parameter bounds and a cubic remainder then give a definite area decrease.

For the second surgery, choose a holomorphic lift H of f into an affine space and a holomorphic map Π back to X, defined near the lifted image, with $\Pi \circ H = f$. For sufficiently small $t > 0$, set

$$h_t(z) = \Pi(H(z) + tzH'(0)), \qquad G_t(z) = h_t(z/(1+t)).$$

The affine perturbation fixes the center and multiplies $f_z(0)$ by $1 + t$. Rescaling the source then restores the original derivative and increases the source radius from R to $(1+t)R$. Proposition 4.10 controls the changes in area and B^2 on the respective source disks:

$$|\,\mathrm{Area}_{(1+t)R}(G_t) - \mathrm{Area}_R(f)| \leq C_{II}t, \qquad |B_{(1+t)R}(G_t)^2 - B_R(f)^2| \leq C_{II}t/\pi,$$

with constants uniform when the area and weighted derivative are bounded. An alternative scalar repair gives a linear cost in the absolute radius increment.

Under a fixed derivative bound, the two surgeries have uniform positive steps. Alternating enlargement and area reduction reaches each prescribed derivative threshold with the explicit count of Proposition 2.9. The radii may nevertheless approach a finite limit. If this happens, compactness either produces the required sphere or leaves a normalized map on the original source disk, called the principal map. Point selection on the same sequence accounts for at least a fixed positive amount e_X of area carried by a sphere, even when concentration occurs near the source boundary. We can enlarge the principal map and continue.

Since enlargement can add area, we record each principal map before this repair. Further limits of the recorded maps lower their common area cap by another e_X. The repair estimates also force radius progress when the returned derivatives stay bounded. This bounds the depth of nested recoveries and yields either the required sphere or any prescribed finite source radius; the construction still allows subsequential limits. Section 2.3 gives the construction and its depth bound.

Recovery also admits the finite-catalogue description of Section 6. A finite cover selects a collection of replacement disks. Once this catalogue is chosen, explicit rescaling restores the derivative cap and lowers area. Together with the two surgeries, it gives a bounded iteration for each prescribed scalar-derivative target.

If the source radii become arbitrarily large, compactness gives either the required sphere or a normalized entire map of finite area, which extends across infinity. To obtain the sharp bound, we then let the area cap decrease to A_*. On each resulting sphere, mark one point mapping into F and another mapping to the boundary of a fixed neighborhood of F. The limiting curve may have several components, but marked compactness retains a chain connecting these two target sets, with total area at most A_*. Its first component not contained in F meets F and has the required area bound. The compactness statements used here are derived from Ivashkovich and Shevchishin [37] in Section 5.

Figure 2 depicts the passage from the expanded disks to the sphere and the final marked limit, for a point fiber.

The controlled holomorphic deformations developed here also suggest an approach to Conjecture A. We hope that further developments will lead from deformations of disks to dominating holomorphic sprays on Fano manifolds.

Plan of the paper.

Section 2 proves the relative sphere theorem and records the metric sharpness example. The Ricci-trace calculation, uniform holomorphic realization, and marked compactness are established in Sections 3–5. Section 6 gives the finite-catalogue alternative. The final section briefly recalls how the construction combines with classical algebraic arguments to recover rational connectedness and Mori's ample-tangent characterization of projective space.

2. Explicit construction of a relative sphere

We construct the disks themselves and retain their area and weighted derivative costs. Finite stages reach any chosen derivative threshold. At a finite limiting radius we recover the principal map, record an area loss, and repair it before continuing. A finite bound on the depth of such recoveries gives the relative sphere.

2.1. **Normalized disks and the two operations**

We fix the metric normalization and recall the area and weighted derivative quantities from the Introduction.

Unless otherwise specified, (X, ω) is a compact Kähler manifold of complex dimension $n \geq 1$. In holomorphic coordinates $w^1, \ldots, w^n$, let $h = (h_{\alpha\bar{\beta}})$ be the Gram matrix of the coordinate frame.

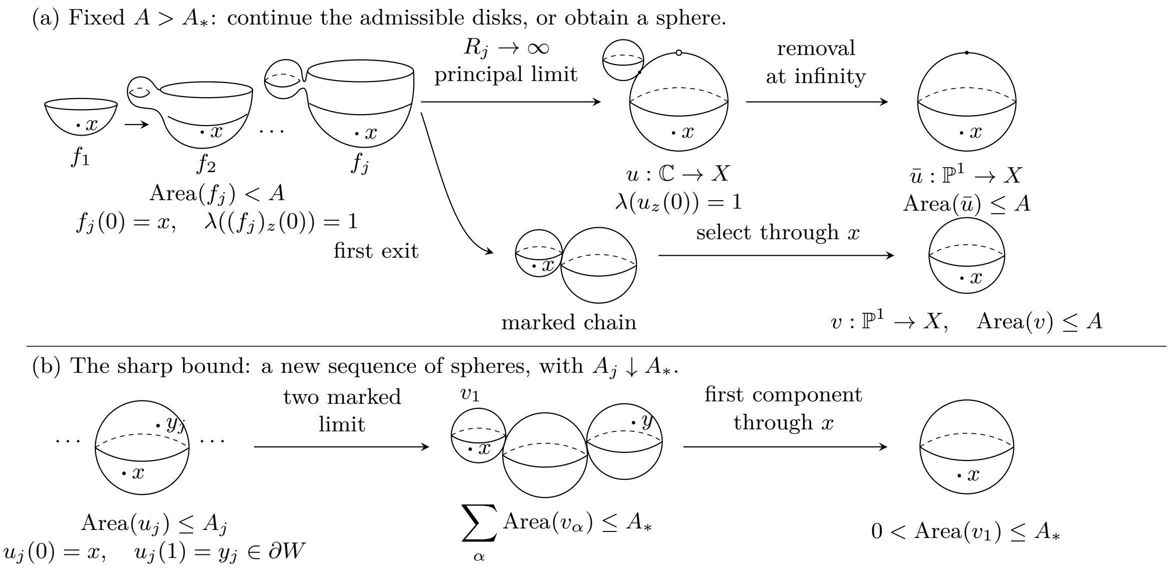


FIGURE 2. The two limiting stages, pictured for $F = \{x\}$. (a) Under a fixed cap, the compactness alternative supplies a marked chain and a sphere through x, or a normalized principal map. A normalized principal map is enlarged, restricted, and used to continue the construction; on expanding sources, finite-area removal closes the entire principal map at infinity. The bulge and neck illustrate possible concentration; a sphere at the node represents a bubble obtained after reparametrization. (b) As the area caps A_j decrease to A_*, two marked values retain a chain from x to $y \in \partial W$, where W is a relatively compact neighborhood of x with smooth boundary inside a coordinate chart. Its first component has area at most A_*; it need not contain y or retain the derivative normalization. All contours are schematic.

Our Hermitian inner products are complex-linear in the first variable and conjugate-linear in the second [14, Ch. V, §12]. Thus $h_{\alpha\bar{\beta}} = \langle \partial/\partial w^\alpha, \partial/\partial w^\beta \rangle_\omega$, and coefficient columns a, b pair as $a^\mathsf{T} h \bar{b}$, where T denotes ordinary transpose. For a holomorphic source map f, write $f_z = df(\partial/\partial z)$. With repeated coordinate indices summed and dA denoting Euclidean area measure, our normalizations are

$$\omega = \frac{i}{2} h_{\alpha\bar{\beta}}\, dw^\alpha \wedge d\bar{w}^\beta, \qquad f^*\omega = |f_z|_h^2 \, dA, \qquad \mathrm{Ric}(\omega) = -i\partial\bar{\partial} \log \det h. \tag{1}$$

For a holomorphic map $f : S \to X$ from a Riemann surface, set $\mathrm{Area}(f) := \int_S f^*\omega$; this counts geometric area with the multiplicity of the map. A holomorphic sphere is a map $u : \mathbb{P}^1 \to X$. When u is nonconstant, we call its reduced image a rational curve, while $\mathrm{Area}(u)$ continues to include the mapping multiplicity. For $a \in \mathbb{C}$ and $r > 0$, write $\mathbb{D}_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$, and abbreviate $\mathbb{D}_R := \mathbb{D}_R(0)$ and $\mathbb{D} := \mathbb{D}_1(0)$; bars denote closure, and $\partial\mathbb{D}_R$ is the boundary circle. We use the area on $\mathbb{D}_R$ and the boundary-distance-weighted derivative bound

$$\mathrm{Area}_R(f) := \int_{\mathbb{D}_R} f^*\omega, \qquad B_R(f) := \sup_{z \in \mathbb{D}_R} (R - |z|)\, |f_z(z)|_\omega \,. \tag{2}$$

Thus $B_R(f)$ weights the derivative by its distance to the source boundary. Write $\mathcal{O}(V, X)$ for the holomorphic maps from an open set $V \subset \mathbb{C}$ to X, and $K \Subset U$ when the closure of K is compact and contained in U. Smooth convergence means uniform convergence of all derivatives on compact source subsets.

Under the Ricci bound $\mathrm{Ric}(\omega) \geq \kappa\omega$, $\kappa > 0$, set

$$A_* = \frac{2\pi(n+1)}{\kappa}. \tag{3}$$

Positive Ricci curvature makes the anticanonical bundle positive, so Kodaira's theorem gives a projective embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$; see Remark 3.2. We fix one such embedding to carry out the two disk operations. It remains fixed as the map and source radius vary, and the constants may depend on it. All areas and target derivative norms are still measured using ω.

For the relative constraint, start with $q : N \to Y$, y_0, and the full compact fiber $F = q^{-1}(y_0)$ in Theorem 1.1. We shrink the source and target neighborhoods so that fibers over a small ball cannot reach the source boundary. This makes the restricted submersion proper. Choose $F \Subset U \Subset N$ with smooth boundary, keeping $\overline{U}$ inside the submersion neighborhood. The full-fiber hypothesis gives $y_0 \notin q(\partial U)$. We can therefore choose a coordinate ball Y' about y_0 such that $\overline{Y'} \cap q(\partial U) = \varnothing$ and $Y' \subset q(U)$. After replacing N by $U \cap q^{-1}(Y')$ and Y by Y', the map q is proper: the inverse image of a compact subset of Y' is closed in $\overline{U}$ and avoids ∂U. In the restricted coordinate ball write $d_Y = \dim Y$ and let $\psi : N \to \mathbb{C}^{d_Y}$ be the coordinate expression of q, translated so that $\psi^{-1}(0) = F$. Fix $0 \neq \eta \in (\mathbb{C}^{d_Y})^*$ and set

$$\lambda^F = (\lambda^F_x)_{x\in F}, \qquad \lambda^F_x := \eta \circ d\psi_x \in T^*_x X.$$

The covectors λ^F_x are the differentials of the same scalar function $\eta \circ \psi$ at points $x \in F$. Thus, even when $f(0)$ moves in F, normalization always means $(\eta \circ \psi \circ f)_z(0) = 1$, with the composition defined near zero. In particular, $d\psi_{f(0)}(f_z(0)) \neq 0$, so the initial tangent vector is not tangent to F. The submersion property makes λ^F_x nonzero for every $x \in F$. The classes below consist of maps on $\mathbb{D}_R$ that extend holomorphically to some open neighborhood of $\overline{\mathbb{D}}_R$. The neighborhood may depend on the map. Set

$$\mathcal{A}^{F,\lambda^F}_R = \left\{ \begin{array}{l} f \in \mathcal{O}(\mathbb{D}_R, X) : f \text{ extends holomorphically near } \overline{\mathbb{D}}_R, \\ f(0) \in F, \quad \lambda^F_{f(0)}(f_z(0)) = 1 \end{array} \right\}. \tag{4}$$

We call a holomorphic disk *normalized* if its center lies in F and its base scalar derivative equals 1. Membership in $\mathcal{A}^{F,\lambda^F}_R$ requires, in addition, holomorphic extension across the source boundary. In the iteration, A and B always refer to the current map and its current source radius.

Choose $x_0 \in F$ and a holomorphic coordinate chart χ with $\chi(x_0) = 0$. Since $\lambda^F_{x_0}$ is nonzero, choose $v \in \mathbb{C}^n$ such that $\lambda^F_{x_0}(d(\chi^{-1})_0(v)) = 1$. For sufficiently small R, the map $f_{\mathrm{seed}}(z) = \chi^{-1}(zv)$ is holomorphic near $\overline{\mathbb{D}}_R$ and belongs to $\mathcal{A}^{F,\lambda^F}_R$. Its area tends to zero as R tends to zero, so we choose R with $\mathrm{Area}_R(f_{\mathrm{seed}}) < A_*$.

To relate the source radius to the weighted derivative, let

$$\ell_F := \max_{x\in F} \|\lambda^F_x\|_{\omega,*},$$

the maximal dual norm of the covector field λ^F along F. Here $\|\cdot\|_{\omega,*}$ is the dual norm induced by ω. Since F is compact and $\lambda^F_x \neq 0$ on F, we have $0 < \ell_F < \infty$. For every normalized disk, the scalar condition gives $1 = |\lambda^F_{f(0)}(f_z(0))| \leq \ell_F |f_z(0)|_\omega$. Evaluating the supremum defining $B_R(f)$ at $z = 0$, we obtain

$$B_R(f) \geq R|f_z(0)|_\omega \geq R/\ell_F. \tag{5}$$

Each surgery replaces the current disk by a new holomorphic map with the same center and normalized scalar derivative. Principal recovery and catalogue replacement may move the center within F, while retaining the scalar normalization. The new maps need not agree with their predecessors on the original source disks: enlargement is achieved by deformation, not by analytic continuation of a single map. Fix also $F \Subset V \Subset N$ with smooth boundary.

Fix $c > 0$ with $\omega \geq c\iota^*\omega_{\mathrm{FS}}$, normalizing a projective line to have Fubini–Study area π, and set

$$e_X = c\pi > 0. \tag{6}$$

If $u : \mathbb{P}^1 \to X$ is nonconstant, the projective map $\iota \circ u$ has an integer degree $d \geq 1$. Hence

$$\mathrm{Area}(u) \geq c \int_{\mathbb{P}^1} u^*\iota^*\omega_{\mathrm{FS}} = c\pi d \geq e_X.$$

The two operations have complementary roles. The first lowers area on a fixed source disk, using the Kähler variation identity of Lemma 3.3 above the Ricci threshold. The second enlarges the source with a linear area cost. Under fixed area and weighted-derivative caps, the first operation has a uniform step and a uniform weighted-derivative cost.

Proposition 2.1. *Assume that ω is Kähler and $\mathrm{Ric}(\omega) \geq \kappa\omega$ with $\kappa > 0$, and keep the fixed embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$. For every $A_0, B_0, \delta > 0$ there are $\sigma > 0$ and $\beta_I < \infty$ such that each $f \in \mathcal{A}_R^{F,\lambda^F}$ with*

$$A_* + \delta \leq \mathrm{Area}_R(f) \leq A_0, \qquad B_R(f) \leq B_0$$

admits a map $g \in \mathcal{A}_R^{F,\lambda^F}$ satisfying

$$g(0) = f(0), \qquad \mathrm{Area}_R(g) \leq \mathrm{Area}_R(f) - \sigma, \qquad |B_R(g)^2 - B_R(f)^2| \leq \beta_I. \tag{7}$$

The map is a member of the explicit affine-lift family in Theorem 4.8, selected by a real area Hessian and the sign of its first derivative. Formula (64) *specifies its step and the guaranteed area decrease. The same constants work when the scalar derivative* 1 *is replaced by any prescribed value k, with that value preserved by the operation.*

The second surgery perturbs a lifted disk in affine space, projects back to X, and dilates the source.

Lemma 2.2. *Fix an area bound A_0 and a weighted-derivative bound B_0, and keep the embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$ fixed. There are constants $0 < \bar{t} \leq 1$ and $C_{II} < \infty$, independent of the radius and the individual map, with the following property. Let $h : \mathbb{D}_R \to X$ be holomorphic on a finite open disk, with $\mathrm{Area}_R(h) \leq A_0$ and $B_R(h) \leq B_0$. There exist a holomorphic lift $H : \mathbb{D}_R \to \mathbb{C}^{2m}$ and a holomorphic map Π from a neighborhood of $H(\mathbb{D}_R)$ to X, with $\Pi \circ H = h$, such that the formulas*

$$h_t(z) = \Pi(H(z) + tzH'(0)), \qquad G_t(z) = h_t(z/(1+t)), \qquad 0 < t \leq \bar{t}. \tag{8}$$

define holomorphic maps $h_t : \mathbb{D}_R \to X$ and $G_t : \mathbb{D}_{(1+t)R} \to X$. The map G_t satisfies

$$|\,\mathrm{Area}_{(1+t)R}(G_t) - \mathrm{Area}_R(h)| \leq C_{II}t, \tag{9}$$

$$|B_{(1+t)R}(G_t)^2 - B_R(h)^2| \leq C_{II}t/\pi. \tag{10}$$

These same estimates compare h_t and h on $\mathbb{D}_R$, and $(h_t)_z(0) = (1+t)h_z(0)$. If h extends past its closed disk, the maps h_t, G_t extend past their corresponding closed disks. For a map given only on the open disk, set

$$g = G_t|_{\mathbb{D}_{R'}}, \qquad R' = (1 + t/2)R. \tag{11}$$

Since $R < R' < (1+t)R$, the resulting map g has a larger source than h and is holomorphic on a neighborhood of its closed disk. It satisfies

$$\mathrm{Area}_{R'}(g) \leq \mathrm{Area}_R(h) + C_{II}t, \qquad B_{R'}(g)^2 \leq B_R(h)^2 + C_{II}t/\pi. \tag{12}$$

The maps G_t and g preserve the center and the derivative at 0*:*

$$G_t(0) = g(0) = h(0), \qquad (G_t)_z(0) = g_z(0) = h_z(0).$$

Proposition 2.1 and Lemma 2.2 are proved in Section 4.4. The proofs give the parameter choices, the common B^2 costs, the arbitrary-scalar descent, and the open-disk lift.

For each recovered principal map h on a finite disk, first take $B_0 = B_R(h)$, which is finite by Lemma 2.7. Apply Lemma 2.2 with this derivative bound and the current area bound. Thus the enlargement parameters are chosen after h is obtained and may vary between recoveries. The two-sided B^2 estimate (10) applies on the full enlarged disk; after restriction to a smaller source, we use the upper bound in (12).

Figure 3 separates the affine perturbation from the source dilation.

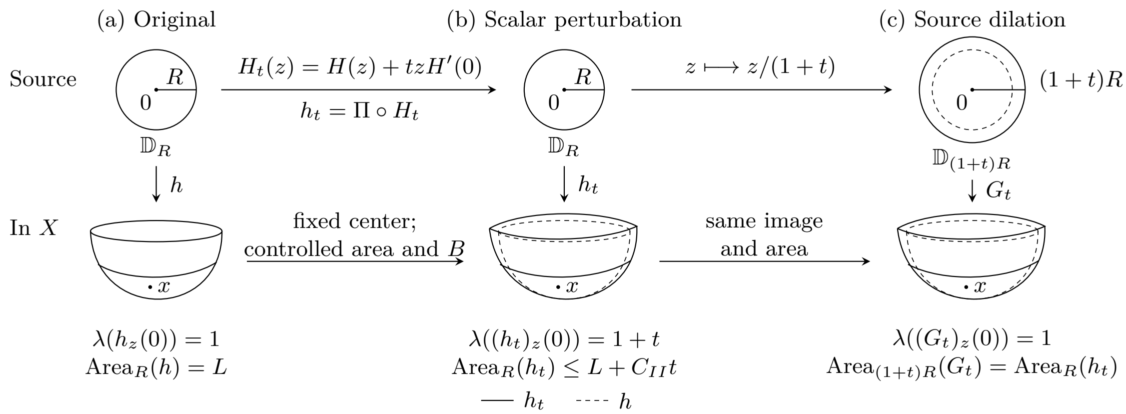


FIGURE 3. Enlargement of one finite-area disk at a fixed center $x \in F$, with $\lambda = \lambda_x^F$. The affine-lift perturbation $h_t = \Pi(H + tzH'(0))$ acts on the same disk and increases the scalar derivative from 1 to $1+t$. Dilation gives $G_t(z) = h_t(z/(1+t))$ on $\mathbb{D}_{(1+t)R}$, with scalar derivative 1 and unchanged total area. The area and B^2 changes obey the common linear bounds in Lemma 2.2. The outlines are schematic; dashed target outlines show h.

2.2. **Principal limits and marked values**

We need two outputs from compactness: a principal map in the original source coordinate, and a chain retaining two marked target values when bubbling occurs. If the principal map is entire and has finite area, removal at infinity extends it to a sphere. We derive these statements from the compactness and removal theorems of Ivashkovich and Shevchishin [36, 37]. The proof locations are specified below.

Proposition 2.3. *Let (X,ω) be a compact Hermitian manifold. For each j, let $R_j > 0$ and let $f_j : U_j \to X$ be holomorphic, where $U_j \subset \mathbb{C}$ is open and $\overline{\mathbb{D}}_{R_j} \Subset U_j$. Assume that $\sup_j \mathrm{Area}_{R_j}(f_j) < \infty$, and set $A_\infty = \liminf_j \mathrm{Area}_{R_j}(f_j)$. Choose $R_\infty \in (0,\infty]$ with $R_\infty \le \liminf_j R_j$, interpreting $\mathbb{D}_\infty$ as $\mathbb{C}$. After passing to a subsequence realizing A_∞ and then to a further subsequence, there are a finite set $\Sigma_\infty \subset \mathbb{D}_{R_\infty}$ and a holomorphic map $f_\infty : \mathbb{D}_{R_\infty} \to X$ such that, in the original source coordinate,*

$$f_j \longrightarrow f_\infty \quad \text{smoothly on compact subsets of } \mathbb{D}_{R_\infty} \setminus \Sigma_\infty, \qquad \mathrm{Area}_{R_\infty}(f_\infty) \le A_\infty.$$

The possibly constant map f_∞ is holomorphic across Σ_∞. This extension does not assert convergence of f_j at the concentration points.

The proof of Proposition 2.3 is in Section 5.2.

Theorem 2.4. *Let (X,ω) be a compact Hermitian manifold. A holomorphic map from a punctured disk to X with finite ω-area extends holomorphically over the puncture. Consequently, a finite-area holomorphic map $u : \mathbb{C} \to X$ extends to a holomorphic map $\mathbb{P}^1 \to X$.*

The proof of Theorem 2.4 is in Section 5.1.

Lemma 2.5. *Let (X,ω) be a compact Hermitian manifold, and let $K_0, K_1 \subset X$ be disjoint compact subsets. Suppose $R_j > 1$, $R_j \to \infty$, and the holomorphic maps $f_j : U_j \to X$, where $U_j \subset \mathbb{C}$ is open and $\overline{\mathbb{D}}_{R_j} \Subset U_j$, satisfy*

$$f_j(0) \in K_0, \qquad f_j(1) \in K_1, \qquad \sup_j \operatorname{Area}_{R_j}(f_j) < \infty.$$

Then there are nonconstant holomorphic maps $u_1, \dots, u_\ell : \mathbb{P}^1 \to X$ whose images form a chain: the first meets K_0, the last meets K_1, and consecutive images intersect. Their total area, counted with the multiplicities of these maps, satisfies

$$\sum_{i=1}^{\ell} \operatorname{Area}(u_i) \leq \liminf_j \operatorname{Area}_{R_j}(f_j).$$

The proof of Lemma 2.5 is in Section 5.3.

The fixed source marks and the disjoint target sets play different roles. Since the marks remain at 0 and 1, stabilization cannot produce an attached tree made entirely of constant components: such a stable tree would need at least two marks converging to the same source point. Since K_0 and K_1 are disjoint, the nodal path between the marked values must contain a nonconstant component.

The scalar derivative is a derivative of the bounded local base map. If first-exit points from a fixed neighborhood of F approach zero, rescaling and marked compactness give a sphere meeting and leaving F. Otherwise, a subsequence stays in that neighborhood on a fixed source disk, allowing the scalar normalization to pass to the principal map.

Lemma 2.6. *Let (X,ω) be a compact Hermitian manifold, and retain F, N, ψ, and λ^F from (4). Suppose $R_j \to R_\infty \in (0,\infty]$ and $f_j \in \mathcal{A}_{R_j}^{F,\lambda^F}$, with*

$$\operatorname{Area}_{R_j}(f_j) \longrightarrow E < \infty.$$

Then at least one of the following conclusions holds:

(i) *There is a nonconstant sphere $v : \mathbb{P}^1 \to X$ such that*

$$v(\mathbb{P}^1) \cap F \neq \varnothing, \qquad v(\mathbb{P}^1) \not\subset F, \qquad \operatorname{Area}(v) \leq E.$$

(ii) *A subsequence has a principal limit $h : \mathbb{D}_{R_\infty} \to X$ in the original source coordinate, as in Proposition 2.3, satisfying*

$$\operatorname{Area}_{R_\infty}(h) \leq E, \qquad h(0) \in F, \qquad \lambda^F_{h(0)}(h_z(0)) = 1.$$

On some fixed disk about zero, the base compositions $\psi \circ f_j$ converge to $\psi \circ h$ smoothly on compact subsets. Here $\mathbb{D}_\infty = \mathbb{C}$.

Proof. Choose $F \Subset V \Subset N$ with smooth boundary, and fix $r > 0$ such that $R_j > r$ for all sufficiently large j. There are two possibilities after passage to a subsequence.

First, suppose there are $z_j \to 0$ with $z_j \neq 0$ and $f_j(z_j) \in \partial V$. Such points can be chosen as first exits along segments from zero whenever no fixed small closed source disk has image in V along an infinite subsequence. The rescaled maps

$$v_j(\zeta) \coloneqq f_j(z_j \zeta), \qquad |\zeta| < R_j/|z_j|,$$

extend past their closed source disks and satisfy

$$v_j(0) \in F, \qquad v_j(1) \in \partial V, \qquad \operatorname{Area}_{R_j/|z_j|}(v_j) = \operatorname{Area}_{R_j}(f_j) \longrightarrow E.$$

Their source radii tend to infinity. Lemma 2.5 gives a chain of nonconstant spheres from F to ∂V, of total area at most E. Take its first component not contained in F. It meets F, either at the first marked value or at its intersection with the preceding component, and has area at most E. This proves (i).

Otherwise, for some $0 < \rho < r$, an infinite subsequence satisfies $f_j(\overline{\mathbb{D}}_\rho) \subset V$. Apply Proposition 2.3 on $\mathbb{D}_{R_\infty}$ to this subsequence. Its principal map h takes values in $\overline{V}$ on $\mathbb{D}_\rho$: this holds away from the finite concentration set by convergence, and at that set by continuity. In particular, $\psi \circ h$ is defined there. The bounded holomorphic maps $\psi \circ f_j$ have a further locally uniformly convergent subsequence. Principal convergence identifies its limit with $\psi \circ h$ away from the concentration set; the identity theorem identifies them throughout $\mathbb{D}_\rho$. Cauchy estimates give smooth convergence on compact subdisks. Passing the value and scalar derivative at zero to the limit yields

$$\psi(h(0)) = 0, \qquad (\eta \circ \psi \circ h)_z(0) = \lim_j (\eta \circ \psi \circ f_j)_z(0) = 1.$$

The area bound follows from principal compactness, proving (ii). $\square$

The principal map in conclusion (ii) retains a definite amount of area in the scalar base coordinate. On the fixed disk $\mathbb{D}_\rho$ from the proof, set $b_j = \eta \circ \psi \circ f_j$ and $b = \eta \circ \psi \circ h$. For every $0 < r < \rho$, the submean inequality and $b_j'(0) = 1$ give

$$\int_{\mathbb{D}_r} |b_j'|^2 \, dA \geq \pi r^2, \qquad \int_{\mathbb{D}_r} |b'|^2 \, dA = \lim_j \int_{\mathbb{D}_r} |b_j'|^2 \, dA \geq \pi r^2. \tag{13}$$

The equality follows from smooth convergence on the smaller disk. This is area in the fixed scalar base coordinate. It is carried by the retained principal map and does not locate a bubble selected elsewhere.

The following lemma supplies the weighted derivative bound for the principal map at a finite endpoint.

Lemma 2.7. *Let (X, ω) be a compact Hermitian manifold, let $0 < R < \infty$, and let $h : \mathbb{D}_R \to X$ be holomorphic with $\mathrm{Area}_R(h) < \infty$. Then $B_R(h) < \infty$.*

The proof in Section 5.4 uses point selection: an unbounded weighted derivative for this fixed map would place a positive amount of area in arbitrarily thin boundary collars.

Recovery with an area loss.

Lemma 2.8. *Let f_j be normalized maps holomorphic near $\overline{\mathbb{D}}_{R_j}$, where $R_j \to R \in (0, \infty)$, $\mathrm{Area}_{R_j}(f_j) \to E < \infty$, and $B_{R_j}(f_j) \to \infty$. One may select either a sphere meeting and escaping F of area at most E, or a normalized principal map $h : \mathbb{D}_R \to X$ satisfying*

$$\mathrm{Area}_R(h) \leq E - e_X, \qquad B_R(h) < \infty. \tag{14}$$

The assertion also holds for finite-area maps on the open disks, without extension across the boundary.

Proof. Apply the relative compactness alternative just stated. Retain the principal branch when that is the selected alternative, and its subsequence. Use the point selection of Lemma 5.2 on this same subsequence, obtaining a nonconstant sphere u. If w_j is its selected point and $\Lambda_j = |(f_j)_z(w_j)|_\omega$, then

$$\Lambda_j \geq B_{R_j}(f_j)/R_j \longrightarrow \infty.$$

Pass to a subsequence with $w_j \to w \in \overline{\mathbb{D}}_R$. The point w lies in the finite concentration set of the principal convergence or on $\partial \mathbb{D}_R$: local smooth convergence at any other interior point bounds the derivatives there.

Fix a compact region Q in the principal disk away from its concentration set, and fix $S < \infty$. The region $\mathbb{D}_{S/\Lambda_j}(w_j)$ representing $u_j|_{\mathbb{D}_S}$ shrinks to w. Since Q stays a positive distance from w, these regions are disjoint for large j; both lie in $\mathbb{D}_{R_j}$. Their areas therefore give, on taking the limit,

$$\int_Q h^* \omega + \int_{\mathbb{D}_S} u^* \omega \leq E.$$

First exhaust the principal disk outside its finite exceptional set and then let $S \to \infty$. The result is $\mathrm{Area}_R(h) + \mathrm{Area}(u) \leq E$, including when w is a source boundary point. Equation (6) proves the first inequality in (14). A finite-area holomorphic map on a finite disk has finite weighted derivative by Lemma 2.7.

For open-disk maps choose a point w_j with $(R_j - |w_j|)|(f_j)_z(w_j)| \geq B_{R_j}(f_j)/2$, and restrict to a radius $r_j < R_j$ so close to R_j that

$$R_j - r_j < 1/j, \quad \mathrm{Area}_{R_j}(f_j) - \mathrm{Area}_{r_j}(f_j) < 1/j, \quad r_j - |w_j| \geq (R_j - |w_j|)/2.$$

The restricted maps extend past their closed disks, have radii tending to R, areas tending to E, and weighted derivatives tending to infinity. The preceding proof applies to these maps. □

In the principal-map alternative, recovery retains the whole principal map, discards the rescaled sphere, and permits the center to move in F. It requires a sequence and a subsequential limit. It is not a cutting and gluing operation on one given disk, and it makes no claim that the recovered map is close to that disk near a bubble.

2.3. **Iteration and the relative sphere**

Fix $\delta > 0$ and set

$$L = A_* + \delta, \qquad \eta_0 = \delta/2, \qquad C = A_* + 2\delta. \tag{15}$$

For a stage threshold $M > 0$, choose all constants with the same area and derivative caps C, M. Use the descent step σ from (64) and set

$$\tau_M = \min\left\{\frac{\bar{t}}{2}, \frac{\eta_0}{\max\{1, C_{II}\}}\right\} > 0. \tag{16}$$

One enlargement multiplies the radius by $1 + \tau_M$ and adds at most η_0 to the area. Since $L + \eta_0 = A_* + 3\delta/2 < C$, enlarging a disk of area below L keeps it below the cap C. Apply the first applicable rule, restarting with the updated disk after each surgery:

(1) If $B_R(f) \geq M$, record the disk and end the stage.
(2) If $B_R(f) < M$ and $\mathrm{Area}_R(f) \geq L$, perform Surgery I.
(3) If $B_R(f) < M$ and $\mathrm{Area}_R(f) < L$, perform Surgery II with $t = \tau_M$.

Proposition 2.9. *Let the starting disk of a stage have radius R_s, area $A_s < C$, and satisfy* (4)*. The preceding rules produce a disk with $A < C$ and $B \geq M$ in finitely many surgeries. If $B_{R_s}(f_s) < M$, define*

$$N_M = \left\lceil \frac{\log(\ell_F M/R_s)}{\log(1 + \tau_M)} \right\rceil, \tag{17}$$

$$J_0 = \begin{cases} 0, & A_s < L, \\ \lfloor (A_s - L)/\sigma \rfloor + 1, & A_s \geq L, \end{cases} \tag{18}$$

$$J = \lfloor \eta_0/\sigma \rfloor + 1. \tag{19}$$

An explicit upper bound for the total number of surgeries is

$$T_M = J_0 + N_M(1 + J). \tag{20}$$

If the starting disk already has $B \geq M$, no surgery is required.

Proof. Both surgeries preserve normalization. Descent lowers area, and enlargement starts below L and ends below $L + \eta_0 < C$. Thus $A < C$ throughout. Each operation starts with $B < M$, so the chosen constants apply until the stage ends.

At most J_0 initial descents bring $A < L$ or end the stage. After each enlargement, at most J descents do the same; the extra 1 in the counts covers equality at L. Allow at most N_M such blocks. At the N_Mth enlargement, if needed,

$$R_s(1 + \tau_M)^{N_M} \geq \ell_F M,$$

so (5) gives $B \geq M$. This proves the bound T_M. Summing the area changes over any completed prefix, with N_I, N_{II} denoting its surgery counts, also gives

$$0 \leq A_{\text{current}} \leq A_s - N_I\sigma + N_{II}\eta_0, \qquad N_I\sigma \leq A_s + N_{II}\eta_0. \tag{21}$$

□

A finite hierarchy of principal recoveries.

The stages may reach arbitrarily large B while their radii remain bounded. Recovery then loses at least e_X of area, but later enlargements can replenish it. We therefore record the principal maps *before* enlarging them and count the depth of successive limits of these recorded maps. The procedures below allow countable repetitions and selections of convergent subsequences.

Proposition 2.10. *Fix a finite target radius $T > 0$. For every integer $m \geq 0$, there is a procedure $\mathcal{P}_m$ which starts from any normalized disk of area less than C, holomorphic near its closed source disk, and returns one of the following:*

(i) *a sphere meeting and escaping F, of area at most C;*
(ii) *a normalized disk of radius at least T and area at most C, possibly defined only on an open disk;*
(iii) *a normalized principal map $h : \mathbb{D}_R \to X$, with $R < T$, satisfying*

$$\mathrm{Area}_R(h) \leq C - (m+1)e_X. \tag{22}$$

The disk radii never decrease. The procedure uses the finite stages of Proposition 2.9, the repair (11)*, and at most $m+1$ nested levels of recovery by Lemma 2.8. In particular, with $N_C = \lfloor C/e_X \rfloor$, the procedure $\mathcal{P}_{N_C}$ returns* (i) *or* (ii)*.*

Proof.

Level zero.

Retain the initial disk if its radius is at least T. Otherwise run the finite stages with $M = 1, 2, 3, \ldots$, carrying each recorded disk to the next stage. After each surgery, return (ii) if the radius has reached T. If all stages are performed, the recorded disks have area less than C, satisfy $B_{R_j}(f_j) \geq j$, and have nondecreasing radii tending to $R \in (0, T]$. Select a subsequence whose areas converge. Lemma 2.8 gives either (i) or a normalized principal map of area at most $C - e_X$. Retain that map as (ii) if $R = T$, and as (iii) otherwise. This defines $\mathcal{P}_0$.

One additional level.

Suppose $\mathcal{P}_{m-1}$ is defined, with $m \geq 1$. Apply it to the current closed disk, retaining any result of type (i) or (ii). Record a result of type (iii) as (h_j, R_j); then

$$R_j < T, \qquad \mathrm{Area}_{R_j}(h_j) \leq C - me_X. \tag{23}$$

Its derivative bound $b_j = B_{R_j}(h_j)$ is finite by Lemma 2.7.

For each integer $q \geq 1$, preselect the enlargement constants $\bar{t}_q, C_{II,q}$ with caps C, q, and put

$$q_j = \max\{1, \lceil b_j \rceil\}, \qquad t(q) = \min\left\{\bar{t}_q/2, \frac{e_X}{2\max\{1, C_{II,q}\}}\right\} > 0. \tag{24}$$

The repair (11) with $t = t(q_j)$ gives a map holomorphic near its closed disk of radius

$$R'_j = (1 + t(q_j)/2)R_j.$$

It remains normalized, has area at most $C - me_X + e_X/2 < C$, and satisfies the B^2 estimate (12). Retain it as (ii) if $R'_j \geq T$; otherwise it is an admissible input for another call to $\mathcal{P}_{m-1}$.

Repeat this rule, retaining any result of type (i) or (ii). If the returns (h_j, R_j) continue indefinitely, then

$$R_{j+1} \geq (1 + t(q_j)/2)R_j, \tag{25}$$

and $R_j \uparrow R \leq T$. To select large-B returns explicitly, fix an integer $Q \geq 1$ and set

$$a_Q = \min_{1 \leq q \leq Q} t(q)/2 > 0.$$

Each return with $b_j \leq Q$ increases the next radius by a factor at least $1 + a_Q$. A scan starting at radius R_j therefore either terminates or finds $b > Q$ within

$$1 + \left\lceil \frac{\log(T/R_j)}{\log(1 + a_Q)} \right\rceil$$

further returns. Taking $Q = 1, 2, 3, \ldots$, each scan beginning strictly after the previously selected index, selects a subsequence with $b_j \to \infty$ and radii tending to R.

Select a further subsequence whose areas tend to E. These are the recorded maps *before repair*, so (23) gives $E \leq C - me_X$. The open-disk part of Lemma 2.8 yields either (i) or a normalized principal map satisfying

$$\mathrm{Area}_R(h) \leq E - e_X \leq C - (m+1)e_X.$$

Return it as (ii) if $R = T$, and as (iii) otherwise. This defines $\mathcal{P}_m$ with one additional recovery level. Finite depth allows only countably many individual surgeries and subsequence selections. Finally, for $m = N_C$, the bound in (iii) is $C - (N_C + 1)e_X < 0$, leaving precisely (i) and (ii). □

Proof of Theorem 1.1. Fix $\delta > 0$ and $C = A_* + 2\delta$. Apply Proposition 2.10 to the chosen seed with targets $T = 2j$, $j = 1, 2, 3, \ldots$, retaining any sphere returned. If all results are disks, restrict them to $\mathbb{D}_j$. These maps extend past their closed disks, remain normalized, and have area at most C. After selecting a subsequence whose areas converge, Lemma 2.6 gives either the desired sphere or a normalized entire principal map of area at most C. Theorem 2.4 extends the latter to a sphere. Its scalar base derivative is 1 at zero, so it meets F and is not contained in F.

For the sharp bound, repeat with $\delta_j = 1/(2j)$ to obtain spheres of area at most $A_* + 1/j$. Each meets both F and ∂V, where V is the fixed neighborhood chosen above: a sphere contained in V has constant ψ-image, and meeting $F = \psi^{-1}(0)$ would force it to lie in F. Pass to a subsequence with convergent areas, reparametrize the spheres so that 0 maps to F and 1 to ∂V, and restrict to disks of radii tending to infinity. Lemma 2.5 gives a chain between these sets of total area at most A_*. Its first component not contained in F meets F at the first marked value or at the preceding node, and is the required sphere. □

Proof of Theorem 1.2. Given $x \in X$, let N be a coordinate neighborhood of x and let $q : N \to Y \subset \mathbb{C}^n$ be its coordinate map. The full fiber over $q(x)$ is $F = \{x\}$, and q is a submersion. Theorem 1.1 therefore gives a nonconstant sphere through x with area at most A_*. □

Remark 2.11. On $\mathbb{P}^n$, use the Fubini–Study form

$$\omega_{\mathrm{FS}} = \frac{i}{2}\partial\bar{\partial}\log(1 + |w^1|^2 + \cdots + |w^n|^2).$$

In the normalization (1), $\mathrm{Ric}(\omega_{\mathrm{FS}}) = 2(n+1)\omega_{\mathrm{FS}}$ and $\int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = \pi$ on a projective line. Every nonconstant sphere in $\mathbb{P}^n$ has positive integer degree d and area $d\pi$. Thus for $\kappa = 2(n+1)$ the bound $2\pi(n+1)/\kappa = \pi$ is attained by a line and cannot be decreased.

3. Ricci traces and scalar-constrained variations

This section explains why positive Ricci curvature can force area decrease above A_*. The calculation has two ingredients: a boundary trace identity and a choice of holomorphic multipliers that fix the center and one scalar derivative. Their Dirichlet cost $\pi(n+1)$, combined with Proposition 3.4, gives the threshold $A_* = 2\pi(n+1)/\kappa$. Section 4 realizes the resulting variation fields by holomorphic families with uniform estimates. We retain the normalizations (1)–(2).

For $z = x + iy$, the complex derivative and the Euclidean Laplacian are

$$\partial_z = \tfrac{1}{2}(\partial_x - i\partial_y), \qquad \Delta = \partial_x^2 + \partial_y^2 = 4\partial_z\partial_{\bar z}.$$

On f^*T_X, ∇_z denotes the Chern covariant derivative induced by ω in the source direction. On $\partial\mathbb{D}_R$, ν is the outward Euclidean unit normal, ∂_ν the corresponding derivative, and ds Euclidean arclength. Let A^* denote the conjugate transpose of A. We use the Hilbert–Schmidt norm and the operator norm

$$\|A\|_{\mathrm{HS}}^2 := \operatorname{tr}(AA^*) = \sum_{j,k} |A_{jk}|^2, \qquad \|A\|_{\mathrm{op}} := \sup_{|v|=1} |Av|. \tag{26}$$

The symbol Id_n denotes the identity matrix. Distances and diameters use the specified metric.

For a Hermitian holomorphic line bundle (L, h), let $\Theta(L)$ denote the curvature of its Chern connection. In Demailly's convention, the real curvature form is $i\Theta(L) = -i\partial\bar\partial \log |s|_h^2$, where s is a nonvanishing local holomorphic frame, and its cohomology class is $2\pi c_1(L)$ [14, Ch. V, (12.8), (13.3)]. We write $c_1(X) = c_1(K_X^{-1})$.

Yau's theorem supplies the Ricci lower bound that allows the relative sphere theorem to be applied to Fano manifolds.

Proposition 3.1. *Every complex Fano manifold admits a Kähler metric ω and a constant $\kappa > 0$ such that*

$$\operatorname{Ric}(\omega) \geq \kappa\omega. \tag{27}$$

Proof. Ampleness of K_X^{-1} gives a positive closed $(1,1)$-form ρ representing $2\pi c_1(X)$. In a fixed Kähler class, Yau's prescribed-Ricci theorem [61] gives a metric ω with $\operatorname{Ric}(\omega) = \rho$. The least eigenvalue of ρ relative to ω is positive and continuous on the compact manifold. Its positive minimum therefore supplies κ with $\operatorname{Ric}(\omega) \geq \kappa\omega$. □

Remark 3.2. A connected compact Hermitian manifold with a metric satisfying (27) is Fano. The induced Hermitian metric on K_X^{-1} has positive Chern curvature $\operatorname{Ric}(\omega)$, so Kodaira's embedding theorem makes K_X^{-1} ample and X projective [41, pp. 314–316].

In Euclidean space, the paired second area variation is Green's formula for the squared norm of the variation field. The *Kähler condition* $d\omega = 0$ makes the same boundary identity exact for a curved target, by the classical coefficient symmetries of closed $(1,1)$-forms [14, Ch. VI, §4]; the geometry is retained in the metric norm.

Lemma 3.3. *Assume that ω is Kähler, and fix $R > 0$. Let $\Delta_t \subset \mathbb{C}$ be a parameter disk centered at zero, $U \subset \mathbb{C}$ an open neighborhood of $\overline{\mathbb{D}}_R$, and $\mathcal{F} : U \times \Delta_t \to X$ a holomorphic map. Write $f_t(z) := \mathcal{F}(z, t)$ and $\xi := \partial_t \mathcal{F}(\cdot, 0)$. Denote the second derivatives along the two real parameter axes by*

$$H_{\mathbb{R}} := \left.\frac{d^2}{ds^2}\right|_{s=0} \operatorname{Area}_R(f_s), \qquad H_{i\mathbb{R}} := \left.\frac{d^2}{ds^2}\right|_{s=0} \operatorname{Area}_R(f_{is}).$$

Then

$$H_{\mathbb{R}} + H_{i\mathbb{R}} = \int_{\partial\mathbb{D}_R} \partial_\nu \, |\xi|_\omega^2 \, ds. \tag{28}$$

In particular, two holomorphic families with the same central map f_0 and the same tangent field ξ have the same paired second variation $H_{\mathbb{R}} + H_{i\mathbb{R}}$.

Proof. Set $\widetilde{\omega} := \mathcal{F}^*\omega$, and write its Hermitian coefficient matrix in the source coordinates $\zeta^1 = z$, $\zeta^2 = t$ as

$$\widetilde{\omega} = \frac{i}{2} \sum_{\alpha,\beta=1}^{2} \widetilde{\omega}_{\alpha\bar\beta} \, d\zeta^\alpha \wedge d\overline{\zeta^\beta}.$$

The Kähler condition gives $d\widetilde{\omega} = \mathcal{F}^*(d\omega) = 0$. Closedness relates derivatives in the parameter direction to derivatives in the source direction. Comparing the $(2,1)$ and $(1,2)$ coefficients yields

$$\partial_t \widetilde{\omega}_{z\bar{z}} = \partial_z \widetilde{\omega}_{t\bar{z}}, \qquad \partial_{\bar{t}} \widetilde{\omega}_{t\bar{z}} = \partial_{\bar{z}} \widetilde{\omega}_{t\bar{t}}.$$

Differentiating these identities and commuting mixed derivatives gives

$$\partial_t \partial_{\bar{t}} \widetilde{\omega}_{z\bar{z}} = \partial_z \partial_{\bar{t}} \widetilde{\omega}_{t\bar{z}} = \partial_z \partial_{\bar{z}} \widetilde{\omega}_{t\bar{t}}.$$

At $t = 0$, the coefficient $\widetilde{\omega}_{t\bar{t}}$ is $|\xi|^2_\omega$. Consequently

$$\partial_t \partial_{\bar{t}} \operatorname{Area}_R(f_t)\big|_{t=0} = \int_{\mathbb{D}_R} \partial_z \partial_{\bar{z}} |\xi|^2_\omega \, dA = \frac{1}{4} \int_{\partial \mathbb{D}_R} \partial_\nu |\xi|^2_\omega \, ds.$$

The sum of the second derivatives along the real and imaginary parameter axes is the parameter Laplacian $4\partial_t \partial_{\bar{t}}$. Hence

$$H_{\mathbb{R}} + H_{i\mathbb{R}} = 4 \, \partial_t \partial_{\bar{t}} \operatorname{Area}_R(f_t)|_{t=0} \, ,$$

which yields (28). $\square$

The classical Riemannian second-variation formula contains interior curvature and second-fundamental-form terms; on Kähler submanifolds, holomorphic normal fields are Jacobi fields [53, §§3.2, 3.5]. Here holomorphicity and $d\omega = 0$ give the explicit paired boundary formula (28), allowing moving boundary values and branch points without a boundary stationarity assumption.

Fix $R > 0$ and a map f holomorphic on a neighborhood of $\overline{\mathbb{D}}_R$. To select variation fields before realizing them by holomorphic families, define the boundary quadratic form on holomorphic sections ξ of $f^* T_X$ that are smooth on the closed disk:

$$\mathcal{H}_{R,f}(\xi) := \int_{\partial \mathbb{D}_R} \partial_\nu |\xi|^2_\omega \, ds. \tag{29}$$

When ξ is the tangent field of a complex holomorphic family, Lemma 3.3 identifies $\mathcal{H}_{R,f}(\xi)$ with the sum of the second area derivatives along the real and imaginary parameter axes.

We use the Wiener–Masani boundary normalization in the form employed by Berndtsson and Rosay [3, Theorem 2.1 and p. 888]; Lemma 4.1 recalls the construction of the resulting boundary-unitary holomorphic frame. The following computation combines this normalization with Green's formula and the classical determinant-curvature identities [14, Ch. V, (13.3); Ch. VIII, (6.6)–(6.7)]. We record the matrix form that separates the Ricci integral from the multiplier's Dirichlet cost and will give the descent threshold below.

Proposition 3.4. *Let $e = (e_1, \ldots, e_n)$ be a holomorphic frame of $f^* T_X$, smooth on the closed disk and unitary on $\partial \mathbb{D}_R$. For a positive integer N, let Φ be a holomorphic $\operatorname{Mat}_{n \times N}(\mathbb{C})$-valued map on a neighborhood of $\overline{\mathbb{D}}_R$, satisfying the boundary coisometry condition*

$$\Phi(\zeta)\Phi(\zeta)^* = \operatorname{Id}_n \quad (|\zeta| = R).$$

The columns Φ_a give sections $(e\Phi)_a = \sum_{j=1}^n e_j \Phi_{ja}$ satisfying the trace identity

$$\sum_{a=1}^N \mathcal{H}_{R,f}((e\Phi)_a) = 4 \int_{\mathbb{D}_R} \|\Phi'\|^2_{\mathrm{HS}} \, dA - 2 \int_{\mathbb{D}_R} f^* \operatorname{Ric}(\omega). \tag{30}$$

Here Φ' is the entrywise derivative and $\|\cdot\|_{\mathrm{HS}}$ is the norm in (26).

Proof. The Gram matrix $G := (\langle e_j, e_k \rangle_\omega)_{j,k}$ represents squared norms by $|ec|^2_\omega = c^{\mathsf{T}} G \bar{c}$. Thus

$$\sum_{a=1}^N |(e\Phi)_a|^2_\omega = \sum_{a=1}^N \Phi_a^{\mathsf{T}} G \overline{\Phi_a} = \operatorname{tr}(\Phi^{\mathsf{T}} G \bar{\Phi}) = \operatorname{tr}(G \bar{\Phi} \Phi^{\mathsf{T}}).$$

The first trace sums the diagonal entries of $\Phi^{\mathsf{T}} G \bar{\Phi}$; the last equality uses cyclicity of the trace. Since finite summation commutes with the normal derivative and integration, (29) gives the boundary integral of $\partial_\nu \operatorname{tr}(G\bar{\Phi}\Phi^{\mathsf{T}})$. On the boundary, $G = \bar{\Phi}\Phi^{\mathsf{T}} = \mathrm{Id}_n$, but their normal derivatives need not vanish. The product rule and Jacobi's classical determinant formula (see, for instance, [4]) give, on $\partial\mathbb{D}_R$,

$$\partial_\nu \operatorname{tr}(G\bar{\Phi}\Phi^{\mathsf{T}}) = \operatorname{tr}(\partial_\nu G) + \operatorname{tr}(\partial_\nu(\bar{\Phi}\Phi^{\mathsf{T}})),$$
$$\partial_\nu \log\det G = \operatorname{tr}(G^{-1}\partial_\nu G) = \operatorname{tr}(\partial_\nu G).$$

The boundary normalization is essential in the second line: tr and log det have the same first differential at Id_n. The divergence theorem and $\Delta \operatorname{tr}(\bar{\Phi}\Phi^{\mathsf{T}}) = 4\|\Phi'\|^2_{\mathrm{HS}}$ give the multiplier term. Since $\det G$ is the metric coefficient on $f^*K_X^{-1}$, (1) gives

$$f^* \operatorname{Ric}(\omega) = -\frac{1}{2}\Delta \log\det G \, dA.$$

Green's formula therefore turns the determinant term $\partial_\nu \log\det G$ into $-2\int f^*\operatorname{Ric}(\omega)$. □

We first construct variations that fix the value of a disk at 0 and one nonzero scalar component of its derivative. Fix $x_0 \in X$ and a nonzero complex-linear covector

$$\lambda \in T^*_{x_0}X \setminus \{0\}. \tag{31}$$

The scalar condition $\lambda(f_z(0)) = 1$ fixes one component of the initial derivative and gives $|f_z(0)|_\omega \geq \|\lambda\|^{-1}_{\omega,*}$, where the norm on the covector is dual to ω at x_0. The multiplier below imposes the linearized value and scalar-derivative constraints. Proposition 2.1 realizes these constraints by holomorphic families that preserve $f(0) = x_0$ and $\lambda(f_z(0)) = 1$ exactly, while leaving the other derivative components free. In Section 2, the center varies in a compact fiber and the covector is pulled back from the base. In the principal branch of Lemma 2.6, this base normalization passes to the limit. For a disk satisfying $f(0) = x_0$ and $\lambda(f_z(0)) = 1$, the frame value $e(0) : \mathbb{C}^n \to T_{x_0}X$ identifies the scalar constraint with the nonzero covector $L_f := \lambda \circ e(0)$. Choose $U_f \in U(n)$ so that its last $n-1$ columns form an orthonormal basis of $\ker L_f$. All columns must vanish at zero to fix the center. In the last $n-1$ directions, which lie in $\ker L_f$, a simple zero also preserves the scalar derivative to first order. In the remaining direction we impose a double zero so that its derivative at zero vanishes. This leads to the multiplier

$$\Phi_f(z) := U_f \operatorname{diag}\left((z/R)^2, z/R, \ldots, z/R\right). \tag{32}$$

The columns of Φ_f vanish at zero, so $(e\Phi_f)'(0) = e(0)\Phi'_f(0)$. The first column of Φ_f has zero derivative at zero; the derivatives of its remaining columns belong to $\ker L_f \subset \mathbb{C}^n$. Thus $\Phi_f(0) = 0$ and $L_f\Phi'_f(0) = 0$ impose the infinitesimal constraints. On $|z| = R$, the diagonal entries have modulus one, so $\Phi_f\Phi_f^* = \mathrm{Id}_n$. For each integer $d \geq 1$, scaling $z = R\zeta$ and integrating in polar coordinates gives $\int_{\mathbb{D}_R} |((z/R)^d)'|^2 \, dA = \pi d$, independently of R. Since U_f is unitary, the quadratic column contributes 2π and each of the $n-1$ linear columns contributes π. Hence

$$\int_{\mathbb{D}_R} \left\|\Phi'_f\right\|^2_{\mathrm{HS}} dA = 2\pi + (n-1)\pi = \pi(n+1). \tag{33}$$

If $\operatorname{Ric}(\omega) \geq \kappa\omega$, then

$$\sum_{a=1}^{n} \mathcal{H}_{R,f}((e\Phi_f)_a) \leq 4\pi(n+1) - 2\kappa \operatorname{Area}_R(f). \tag{34}$$

The cost in (33) is optimal within the prescribed multiplier class. After a unitary change of rows, Parseval's identity gives unit total squared coefficient norm in each row, while the vanishing conditions give its lowest possible degree. Indeed, fix $R > 0$, $N \geq 1$, and $0 \neq L \in (\mathbb{C}^n)^*$, and let Φ be holomorphic near $\overline{\mathbb{D}}_R$ with values in $\mathrm{Mat}_{n\times N}(\mathbb{C})$, satisfying

$$\Phi\Phi^* = \mathrm{Id}_n \quad \text{on } \partial\mathbb{D}_R, \qquad \Phi(0) = 0, \qquad L\Phi'(0) = 0.$$

The boundary identity forces $N \geq n$. Choose $Q \in U(n)$ with first row $L/\|L\|$, where $\|L\|$ is the Euclidean dual norm. Multiplication by Q preserves that identity and the Hilbert–Schmidt Dirichlet integral. The rows r_i of $Q\Phi$ have norm one on $\partial\mathbb{D}_R$ and vanish to orders at least $m_1 = 2$ and $m_i = 1$ for $i \geq 2$. Writing $r_i(z) = \sum_{k\geq m_i} a_{ik}(z/R)^k$, with $a_{ik} \in \mathbb{C}^N$, Parseval's identity and radial integration give

$$\sum_{k\geq m_i} \|a_{ik}\|^2 = 1, \qquad \int_{\mathbb{D}_R} |r_i'|^2\, dA = \pi \sum_{k\geq m_i} k\|a_{ik}\|^2 \geq \pi m_i.$$

Summing proves the lower bound $\pi(n+1)$. It is attained by $U\operatorname{diag}((z/R)^2, z/R, \dots, z/R)$ when the last $n-1$ columns of $U \in U(n)$ span $\ker L$, appending $N-n$ zero columns if $N > n$. This multiplier optimality leaves the holomorphic realization and its remainder estimates to the next section.

4. Uniform holomorphic realization and disk operations

We realize the trace calculation by holomorphic families and then prove the two disk operations: uniform descent and scalar enlargement. Holomorphic mapping spaces and Stein neighborhoods of graphs provide a qualitative deformation theory [21]. Here the parameter radius and cubic area remainder must remain uniform as the disk and its source radius vary. We obtain this control by lifting to a fixed Stein submanifold of affine space, perturbing there, and projecting back to X.

Assume in this section that X is projective. Keep the projective embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$ fixed as in Section 2; it will be used to represent $\iota \circ f$ by homogeneous coordinates and to construct the lift space $\mathcal{Y}$. The embedding ι is kept fixed as f and R vary, so the constants below may depend on ι.

We use the standard L^p and C^k norms on source regions, with Euclidean area measure. Vector and operator norms are induced by the fixed Hermitian and Euclidean metrics; matrix-valued L^2 estimates use the Hilbert–Schmidt norm.

4.1. **Frame and lifting estimates**

Bounds on area and weighted derivative do not control derivatives on the source boundary. The following example explains why uniform realization requires direct estimates. For $0 < r < 1$ and $\varepsilon > 0$, the functions $h_r : \mathbb{D} \to \mathbb{C}$ given by

$$h_r(z) := \varepsilon \frac{1-r^2}{1-rz}$$

are holomorphic near $\overline{\mathbb{D}}$ and satisfy

$$\int_{\mathbb{D}} |h_r'|^2 dA = \pi\varepsilon^2 r^2, \qquad B_1(h_r) \leq \frac{\varepsilon}{2}, \qquad |h_r'(1)| = \varepsilon \frac{r(1+r)}{1-r} \longrightarrow \infty.$$

Thus bounded area and bounded B_1 do not imply relative compactness in $C^1(\overline{\mathbb{D}})$ (with the Euclidean metric on $\mathbb{C}$); the uniform estimates below must be proved directly.

Boundary spectral factorization and a Green potential bound control a normalized frame. Homogeneous coordinates and Nicolau's corona theorem then give a uniformly bounded affine lift with bounded Dirichlet integral.

Fix $A_0, B_0 \geq 0$. In the auxiliary results below, f is holomorphic near $\overline{\mathbb{D}}_R$ and satisfies $\operatorname{Area}_R(f) \leq A_0$ and $B_R(f) \leq B_0$. Norms on fixed finite-rank bundles are chosen once and for all.

The spectral factorization theorem is due to Wiener and Masani [56, Theorem 7.13]. We use the smooth boundary version stated by Berndtsson and Rosay [3, Theorem 2.1]. It makes the frame in Theorem 4.7 unitary on the boundary, as required by the Ricci-trace identity.

Lemma 4.1. *Let $H : \partial\mathbb{D} \to \operatorname{Mat}_{n\times n}(\mathbb{C})$ be a smooth positive-definite Hermitian matrix function. There is a map $A : \mathbb{D} \to \operatorname{GL}_n(\mathbb{C})$, holomorphic in $\mathbb{D}$ and smooth on $\overline{\mathbb{D}}$ with invertible boundary values, such that*

$$A(\zeta)^* H(\zeta) A(\zeta) = \operatorname{Id}_n \quad (|\zeta| = 1).$$

Proof. The Wiener–Masani theorem in the smooth boundary form of [3, Theorem 2.1] gives $H = h^*h$, where h and h^{-1} are holomorphic on $\mathbb{D}$ and smooth on $\overline{\mathbb{D}}$. Thus $A := h^{-1}$ has the required boundary normalization. $\square$

For $z \in \mathbb{D}$, let δ_z be the unit point mass at z. The Dirichlet Green function of the unit disk, with the sign and normalization used here, is [14, Ch. I, §4.B]

$$g(z,w) = \log\left|\frac{1-\bar{z}w}{z-w}\right|, \qquad -\Delta_w g = 2\pi\delta_z.$$

On $\mathbb{D}_R$ we use $g_R(z,w) := g(z/R, w/R)$.

The Green potential controls both the frame and the homogeneous lift. The following elementary estimate separates a neighborhood of the logarithmic singularity, controlled by B_0, from its complement, controlled by the total area.

Lemma 4.2. *Let $\varepsilon_f^\omega(w) := |f_z(w)|_\omega^2$ be the area density of f, so that $f^*\omega = \varepsilon_f^\omega\, dA$ and $\mathrm{Area}_R(f) = \int_{\mathbb{D}_R} \varepsilon_f^\omega\, dA$. If $\mathrm{Area}_R(f) \le A_0$ and $B_R(f) \le B_0$, then*

$$\sup_{z\in\mathbb{D}_R} \int_{\mathbb{D}_R} g_R(z,w)\varepsilon_f^\omega(w)dA(w) \le C_g(A_0,B_0) := C_G(B_0^2 + A_0), \tag{35}$$

where C_G is a universal constant, independent of R and f.

Proof. First reduce to the unit disk. Under the dilation $h(\zeta) := f(R\zeta)$, the identities

$$\mathrm{Area}_1(h) = \mathrm{Area}_R(f), \qquad B_1(h) = B_R(f), \qquad \varepsilon_h^\omega(\eta) = R^2\varepsilon_f^\omega(R\eta)$$

show that the hypotheses are unchanged. Since $g_R(R\zeta, R\eta) = g(\zeta,\eta)$, change of variables also preserves the Green-potential integral. We may therefore assume $R = 1$.

Fix $z \in \mathbb{D}$ and set $\delta := 1 - |z|$, its distance to the boundary. Split the integral into the disk $|w - z| < \delta/2$ and its complement in $\mathbb{D}$. On the first region,

$$1 - |w| \ge 1 - |z| - |w - z| > \frac{\delta}{2}, \qquad |f_z(w)|_\omega \le \frac{B_0}{1-|w|}.$$

Thus $\varepsilon_f^\omega(w) \le 4B_0^2/\delta^2$. To bound the Green kernel, write

$$1 - \bar{z}w = (1 - |z|^2) + \bar{z}(z - w).$$

Since $1 - |z|^2 = (1 + |z|)\delta \le 2\delta$, the triangle inequality gives

$$|1 - \bar{z}w| \le 2\delta + |z - w| \le \frac{5\delta}{2}, \qquad g(z,w) \le \log\frac{5\delta}{2|z-w|}.$$

Integration in polar coordinates bounds this contribution by

$$\frac{4B_0^2}{\delta^2}\, 2\pi \int_0^{\delta/2} r\log\frac{5\delta}{2r}\, dr = \pi B_0^2\left(\log 5 + \frac{1}{2}\right).$$

On the complementary region $|z - w| \ge \delta/2$, the same triangle inequality instead yields

$$\frac{|1-\bar{z}w|}{|z-w|} \le 1 + \frac{2\delta}{|z-w|} \le 5.$$

Thus $g(z,w) \le \log 5$ there, and the remaining contribution is at most

$$(\log 5)\int_{\mathbb{D}} \varepsilon_f^\omega\, dA \le A_0 \log 5.$$

Adding the two bounds proves (35); one may take the universal constant $C_G := \pi(\log 5 + 1/2)$, independently of z, R, and f. $\square$

We use the following Chern connection and curvature convention [14, Ch. V, (12.2)–(12.4)]. In the coordinates of (1), a section with coefficient column c has squared norm $c^{\mathsf{T}} h\bar{c}$. Writing $\Theta(T_X) = \nabla^2$, the connection and curvature matrices are

$$\nabla' c = \partial c + \bar{h}^{-1}(\partial\bar{h})c, \qquad \nabla'' c = \bar{\partial} c, \qquad \Theta(T_X) = \bar{\partial}(\bar{h}^{-1}\partial\bar{h}).$$

The coefficient of $dw^\alpha \wedge d\bar{w}^\beta$ in $\Theta(T_X)$ is $-\partial_{\bar{w}^\beta}(\bar{h}^{-1}\partial_{w^\alpha}\bar{h})$; the minus sign comes from reversing the order of the differentials. For $v = \sum_\alpha v^\alpha \partial_{w^\alpha}$ and a tangent vector u with coefficient column c, our curvature tensor is

$$R^\omega(v, \bar{v}, u, \bar{u}) := \sum_{\alpha,\beta} v^\alpha \overline{v^\beta}\, c^{\mathsf{T}} \big[-\partial_{w^\alpha}\partial_{\bar{w}^\beta} h + (\partial_{w^\alpha} h) h^{-1}(\partial_{\bar{w}^\beta} h)\big]\bar{c}. \tag{36}$$

Thus positive holomorphic bisectional curvature means that (36) is positive for every pair of nonzero tangent vectors v, u at the same point. For a holomorphic disk f, the curvature endomorphism $\mathcal{R}^\omega_f$ is characterized by

$$\langle \mathcal{R}^\omega_f v, v\rangle_\omega = R^\omega(f_z, \overline{f_z}, v, \bar{v}). \tag{37}$$

The Bochner formula and the curvature formula for the determinant bundle [14, Ch. VIII, proof of Lemma 7.2; Ch. V, (4.6)] give, for a holomorphic section s of f^*T_X,

$$\partial_z\partial_{\bar{z}}|s|^2_\omega = |\nabla_z s|^2_\omega - \langle \mathcal{R}^\omega_f s, s\rangle_\omega, \qquad f^*\operatorname{Ric}(\omega) = 2\operatorname{tr}(\mathcal{R}^\omega_f)\, dA. \tag{38}$$

Choose C_ω so that

$$\|\mathcal{R}^\omega_f(z)\|_{\mathrm{op}} \le C_\omega \varepsilon^\omega_f(z).$$

Compactness of X gives such a constant.

For a frame $e = (e_1, \dots, e_n)$, write $G = (\langle e_j, e_k\rangle_\omega)$. Matrix inequalities refer to Hermitian quadratic forms. The Green bound and Bochner identity control G throughout the disk and bound the frame's covariant Dirichlet integral. Both estimates enter the lifted-field bounds in Theorem 4.7. The two-sided metric control follows the boundary normalization and curvature-potential method of Berndtsson and Rosay [3, Lemma 2.2 and Proposition 2.3]. We give the calculation with the caps A_0, B_0 and record also the covariant Dirichlet bound.

Proposition 4.3. *There are constants $\Gamma \ge 1$ and $D_1 < \infty$, depending only on (X, ω, A_0, B_0), such that every holomorphic frame e of f^*T_X that is smooth on the closed disk and unitary on its boundary has Gram matrix G satisfying*

$$\Gamma^{-1}\operatorname{Id}_n \le G \le \Gamma \operatorname{Id}_n. \tag{39}$$

Moreover

$$\sum_{a=1}^n \int_{\mathbb{D}_R} |\nabla_z e_a|^2 dA \le D_1. \tag{40}$$

In particular, these constants are independent of R, f, and the boundary-unitary frame.

Proof. Set

$$u_G := \operatorname{tr} G = \sum_{a=1}^n |e_a|^2_\omega > 0, \qquad q := \sum_{a=1}^n |\nabla_z e_a|^2_\omega.$$

The curvature bound and (38) give

$$\begin{aligned}\partial_z\partial_{\bar{z}} u_G &= q - \sum_a \langle \mathcal{R}^\omega_f e_a, e_a\rangle_\omega \\ &\ge q - C_\omega \varepsilon^\omega_f \sum_a |e_a|^2_\omega = q - C_\omega \varepsilon^\omega_f u_G.\end{aligned}$$

Because e_a is holomorphic and the connection preserves the metric, $\partial_z u_G = \sum_a \langle \nabla_z e_a, e_a \rangle_\omega$. Cauchy–Schwarz, applied to the direct sum of the n fibers, gives

$$|\partial_z u_G|^2 \leq \left(\sum_a |e_a|_\omega^2 \right) \left(\sum_a |\nabla_z e_a|_\omega^2 \right) = u_G q.$$

Apply the classical logarithmic norm calculation to the section $(e_1, \ldots, e_n)$ of the direct-sum bundle [14, Ch. VIII, proof of Lemma 7.2]. Since $u_G > 0$, it gives

$$\begin{aligned} \partial_z \partial_{\bar{z}} \log u_G &= \frac{\partial_z \partial_{\bar{z}} u_G}{u_G} - \frac{|\partial_z u_G|^2}{u_G^2} \\ &\geq \frac{q}{u_G} - C_\omega \varepsilon_f^\omega - \frac{q}{u_G} = -C_\omega \varepsilon_f^\omega. \end{aligned}$$

Hence

$$\Delta \log u_G \geq -4 C_\omega \varepsilon_f^\omega, \qquad u_G|_{\partial \mathbb{D}_R} = n.$$

Green representation, with $-\Delta_w g_R(z,w) = 2\pi \delta_z$, reads

$$\log \frac{u_G(z)}{n} = -\frac{1}{2\pi} \int_{\mathbb{D}_R} g_R(z,w) \Delta_w \log u_G(w) \, dA(w).$$

Since $g_R \geq 0$, Lemma 4.2 yields

$$\log \frac{u_G(z)}{n} \leq \frac{2C_\omega}{\pi} \int_{\mathbb{D}_R} g_R(z,w) \varepsilon_f^\omega(w) \, dA(w) \leq \frac{2 C_\omega C_g(A_0, B_0)}{\pi}.$$

The algebraic dual holomorphic frame $e^a(e_b) = \delta_b^a$ has Gram matrix $G^{-\mathsf{T}} := (G^{-1})^{\mathsf{T}}$, where T denotes ordinary transpose. The curvature of the dual bundle is the negative transpose of the original curvature [14, Ch. V, (4.3′)], so its operator norm is bounded by $C_\omega \varepsilon_f^\omega$. Applying the logarithmic trace estimate to the dual frame gives

$$\operatorname{tr} G \leq \Gamma, \qquad \operatorname{tr}(G^{-1}) \leq \Gamma, \qquad \Gamma := n \exp\left(\frac{2 C_\omega C_g(A_0, B_0)}{\pi} \right).$$

For every eigenvalue $\mu > 0$ of G, $\mu \leq \operatorname{tr} G \leq \Gamma$ and $\mu^{-1} \leq \operatorname{tr}(G^{-1}) \leq \Gamma$. These two inequalities give (39).

It remains to bound the frame's covariant Dirichlet integral. We first sum the Bochner identity; after integration, the boundary determinant calculation from Proposition 3.4 will identify its left-hand side:

$$\partial_z \partial_{\bar{z}} \operatorname{tr} G = \sum_a |\nabla_z e_a|^2 - \operatorname{tr}(\mathcal{R}_f^\omega \mathsf{S}), \qquad \mathsf{S} := \sum_a e_a \otimes e_a^\flat.$$

Here $e_a^\flat(v) = \langle v, e_a \rangle_\omega$ denotes metric duality. Since $G = \mathrm{Id}_n$ on $\partial \mathbb{D}_R$, Jacobi's formula gives

$$\partial_\nu \log \det G = \operatorname{tr}(G^{-1} \partial_\nu G) = \partial_\nu \operatorname{tr} G \quad \text{on } \partial \mathbb{D}_R.$$

Consequently Green's formula and (1) give

$$\int_{\mathbb{D}_R} \partial_z \partial_{\bar{z}} \operatorname{tr} G \, dA = \frac{1}{4} \int_{\partial \mathbb{D}_R} \partial_\nu \log \det G \, ds = -\frac{1}{2} \int_{\mathbb{D}_R} f^* \operatorname{Ric}(\omega).$$

Integrating the Bochner identity and solving for the nonnegative derivative term therefore gives

$$\sum_a \int_{\mathbb{D}_R} |\nabla_z e_a|^2 dA = \int_{\mathbb{D}_R} \operatorname{tr}(\mathcal{R}_f^\omega \mathsf{S}) dA - \frac{1}{2} \int_{\mathbb{D}_R} f^* \operatorname{Ric}(\omega).$$

Now $\operatorname{tr} \mathsf{S} \leq n\Gamma$, while the absolute Ricci density is at most $2n C_\omega \varepsilon_f^\omega$, since $f^* \operatorname{Ric} = 2 \operatorname{tr}(\mathcal{R}_f^\omega) \, dA$. Taking absolute values and using $\operatorname{Area}_R(f) \leq A_0$ gives

$$\sum_a \int_{\mathbb{D}_R} |\nabla_z e_a|^2 dA \leq n C_\omega (\Gamma + 1) A_0.$$

Thus $D_1 \coloneqq nC_\omega(\Gamma+1)A_0$ is an admissible choice in (40). $\square$

Recall the projective embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$ chosen at the beginning of this section. Let ω_{FS} be the Fubini–Study form on its target, with normalization

$$\omega_{\mathrm{FS}} = \frac{i}{2}\partial\bar\partial \log \sum_{j=1}^{m} |Z_j|^2.$$

Choose $c_\iota > 0$ such that $\iota^*\omega_{\mathrm{FS}} \le c_\iota \omega$; such a constant exists by compactness of X. The displayed Fubini–Study formula is read in homogeneous coordinates $Z = (Z_1, \dots, Z_m) \neq 0$; it is unchanged as a form on projective space when the homogeneous representative is multiplied by a nonvanishing holomorphic function.

In rank one, boundary normalization subtracts a harmonic function from the logarithm of the metric, as in Berndtsson and Rosay [3, p. 886]. Together with the classical Fubini–Study potential [14, Ch. VI, (4.4)], this gives homogeneous coordinates with the bounds needed for the corona theorem. We record the area identity and the uniform lower bound.

Lemma 4.4. *Every disk satisfying* $\mathrm{Area}_R(f) \le A_0$ *and* $B_R(f) \le B_0$ *has a homogeneous lift* $\mathbf{s}$ *from an open neighborhood of* $\overline{\mathbb{D}}_R$ *to* $\mathbb{C}^m \setminus \{0\}$*, holomorphic on that neighborhood, with* $[\mathbf{s}] = \iota \circ f$ *and* $|\mathbf{s}| = 1$ *on the boundary. It satisfies*

$$|\mathbf{s}| \le 1, \qquad \int_{\mathbb{D}_R} |\mathbf{s}'|^2 dA = \int_{\mathbb{D}_R} f^*\iota^*\omega_{\mathrm{FS}} \le c_\iota A_0, \tag{41}$$

and

$$|\mathbf{s}(z)| \ge \delta_{\mathbf{s}} > 0, \tag{42}$$

where $\delta_{\mathbf{s}} = \delta_{\mathbf{s}}(X, \omega, \iota, A_0, B_0)$ *is independent of* R *and* f*.*

Proof. The tautological line bundle on $\mathbb{P}^{m-1}$ has fiber $\mathbb{C}Z$ over $[Z]$. Trivialize its pullback by $\iota \circ f$ on a slightly larger disk, using the Oka–Grauert principle [28, Satz 6], and choose a nonvanishing section, represented by a lift v in $\mathbb{C}^m \setminus \{0\}$. The positive real-analytic boundary function $|v|$ has a logarithm whose harmonic extension is the real part of a holomorphic function a on a slightly larger disk. Then $\mathbf{s} \coloneqq e^{-a}v$ has boundary norm one, and the maximum principle gives $|\mathbf{s}| \le 1$.

The area density of $\iota \circ f$ for the Fubini–Study metric is $\varepsilon_f^{\mathrm{FS}} \coloneqq |(\iota\circ f)'|^2_{\mathrm{FS}}$, so that $f^*\iota^*\omega_{\mathrm{FS}} = \varepsilon_f^{\mathrm{FS}}\, dA$. Set $u \coloneqq \log |\mathbf{s}|^2$. The boundary normalization $|\mathbf{s}| = 1$ gives $u = 0$ and $\partial_\nu u = \partial_\nu |\mathbf{s}|^2$ there. Since $\mathbf{s}$ has no common zero, the Fubini–Study formula gives $\Delta u = 4\varepsilon_f^{\mathrm{FS}}$, while holomorphicity gives $\Delta |\mathbf{s}|^2 = 4|\mathbf{s}'|^2$. Green's formula therefore yields

$$\int_{\mathbb{D}_R} |\mathbf{s}'|^2 dA = \frac{1}{4}\int_{\partial\mathbb{D}_R} \partial_\nu |\mathbf{s}|^2 ds = \frac{1}{4}\int_{\partial\mathbb{D}_R} \partial_\nu u \, ds = \int_{\mathbb{D}_R} \varepsilon_f^{\mathrm{FS}} dA,$$

which proves (41). Green representation for u with zero boundary values [14, Ch. I, (4.5)] gives

$$-u(z) = \frac{2}{\pi}\int g_R(z,w)\varepsilon_f^{\mathrm{FS}}(w)\,dA(w) \le \frac{2c_\iota C_g(A_0, B_0)}{\pi}.$$

Thus (42) holds with $\delta_{\mathbf{s}} = \exp(-c_\iota C_g(A_0,B_0)/\pi)$. $\square$

Write $H^\infty(\mathbb{D})$ for the bounded holomorphic functions with the supremum norm, and let

$$\mathcal{D}(\mathbb{D}) \coloneqq \left\{ h : \mathbb{D} \to \mathbb{C} \text{ holomorphic} : \int_{\mathbb{D}} |h'|^2 dA < \infty \right\}.$$

The Dirichlet algebra is

$$\mathfrak{A} \coloneqq H^\infty(\mathbb{D}) \cap \mathcal{D}(\mathbb{D}), \qquad \|h\|_{\mathfrak{A}} \coloneqq \|h\|_\infty + \left(\int_{\mathbb{D}} |h'|^2 dA\right)^{1/2}.$$

To construct the affine lift, we need a holomorphic dual vector χ with $\chi \cdot \mathbf{s} = 1$ and bounds on both its supremum norm and Dirichlet integral. Nicolau's theorem and the bound at the beginning of its proof [49, pp. 135–136] give the following statement.

Theorem 4.5. *For every $m \geq 1$ and $A, \delta > 0$ there is $C_N(m, A, \delta) < \infty$ such that, whenever $u_1, \ldots, u_m \in \mathfrak{A}$ satisfy*

$$\|u_j\|_{\mathfrak{A}} \leq A, \qquad \max_j |u_j(z)| \geq \delta \quad (z \in \mathbb{D}),$$

there are $v_1, \ldots, v_m \in \mathfrak{A}$ with

$$\sum_j u_j v_j = 1, \qquad \|v_j\|_{\mathfrak{A}} \leq C_N(m, A, \delta).$$

The same statement holds on every disk after dilation.

Proof. The case $\delta > A$ is vacuous; assume $\delta \leq A$. For $p = 2$, Nicolau uses the norm

$$\|h\|_{\mathrm{Nic}} := \|h\|_\infty + \left(\frac{1}{\pi}\int_{\mathbb{D}} |h'|^2 dA\right)^{1/2}, \qquad \|h\|_{\mathrm{Nic}} \leq \|h\|_{\mathfrak{A}} \leq \sqrt{\pi}\,\|h\|_{\mathrm{Nic}}.$$

His corona condition is the same maximum lower bound as above [49, p. 135, (1)]. The generators u_j/A therefore have Nicolau norm at most 1 and lower bound δ/A. The estimate at the beginning of his proof [49, p. 136] gives solutions b_j with $\sum_j (u_j/A) b_j = 1$ and $\|b_j\|_{\mathrm{Nic}} \leq K_m(\delta/A)$, where K_m depends only on the indicated data. Setting $v_j := b_j/A$ gives $\sum_j u_j v_j = 1$ and

$$\|v_j\|_{\mathfrak{A}} \leq \frac{\sqrt{\pi}}{A} K_m(\delta/A),$$

which is an admissible choice of $C_N(m, A, \delta)$. For a disk of radius R,

$$\|h(R\,\cdot)\|_{\infty,\mathbb{D}} = \|h\|_{\infty,\mathbb{D}_R}, \qquad \int_{\mathbb{D}} |(h(R\zeta))'|^2\, dA(\zeta) = \int_{\mathbb{D}_R} |h'(z)|^2\, dA(z),$$

so the same bound applies after dilation. ∎

For a normalized lift $\mathbf{s}$ from Lemma 4.4, its coordinates have algebra norm at most $C_{\mathbf{s}} := 1 + \sqrt{c_\iota A_0}$, and at least one has modulus at least $\delta_{\mathbf{s}}/\sqrt{m}$ at every point, because $|\mathbf{s}|^2 \leq m \max_j |\mathbf{s}_j|^2$. Theorem 4.5 gives a holomorphic map $\chi : \mathbb{D}_R \to (\mathbb{C}^m)^*$, whose components belong to the algebra on $\mathbb{D}_R$ obtained from $\mathfrak{A}$ by dilation, satisfying

$$\chi \cdot \mathbf{s} = 1, \qquad \|\chi\|_\infty + \|\chi'\|_2 \leq C_\chi \tag{43}$$

for a constant $C_\chi = C_\chi(X, \omega, \iota, A_0, B_0)$. The dot denotes the bilinear pairing $\chi \cdot \mathbf{s} = \sum_{j=1}^m \chi_j \mathbf{s}_j$, without complex conjugation. The constant absorbs the fixed dimension factors in passing from scalar to vector norms.

To extend the lift past $\overline{\mathbb{D}}_R$, dilate χ radially while keeping $\mathbf{s}$ fixed, then restore their pairing by division. For $0 < \rho < 1$ sufficiently close to 1, set

$$\chi_\rho(z) := \chi(\rho z), \qquad \gamma_\rho := \chi_\rho \cdot \mathbf{s}, \qquad \widehat{\chi} := \frac{\chi_\rho}{\gamma_\rho}.$$

Since $\chi(\rho z) \cdot \mathbf{s}(\rho z) = 1$, we have

$$\gamma_\rho(z) - 1 = \chi(\rho z) \cdot \big(\mathbf{s}(z) - \mathbf{s}(\rho z)\big).$$

The fixed lift $\mathbf{s}$ is uniformly continuous on $\overline{\mathbb{D}}_R$, and χ is bounded. Thus $\gamma_\rho \to 1$ uniformly there as $\rho \uparrow 1$, and we may require $|\gamma_\rho| \geq 1/2$. The quotient is therefore holomorphic near $\overline{\mathbb{D}}_R$ and satisfies $\widehat{\chi} \cdot \mathbf{s} = 1$ and $\|\widehat{\chi}\|_\infty \leq 2C_\chi$. Dilation gives $\|\chi_\rho'\|_2 \leq C_\chi$; the product and quotient rules give

$$\begin{aligned} &\|\gamma_\rho'\|_2 \leq C_\chi(1 + \sqrt{c_\iota A_0}), \\ &\|\widehat{\chi}'\|_2 \leq 2C_\chi + 4C_\chi^2(1 + \sqrt{c_\iota A_0}) =: D_\chi. \end{aligned} \tag{44}$$

All norms here are on $\mathbb{D}_R$. The bound depends only on the fixed target, embedding, and A_0, B_0; the base lift $\mathbf{s}$ is unchanged.

4.2. **Uniform lifts and holomorphic variations**

The affine model below is a homogeneous version of the classical incidence-complement construction known as the Jouanolou trick; see [23, §2.6, p. 771]. The pairs $(\mathbf{s}, \chi)$ lie in this fixed model: the equation $\chi \cdot \mathbf{s} = 1$ excludes the cone vertex. A holomorphic right inverse lifts tangent fields to this model. The model depends only on the embedding; the area and weighted-derivative caps determine a compact subset and a retraction near it.

Lemma 4.6. *For the projective embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$ chosen above, there are a closed Stein submanifold $\mathcal{Y} \subset \mathbb{C}^{2m}$, a holomorphic submersion $\pi_X : \mathcal{Y} \to X$, and a holomorphic bundle map $\mathcal{L} : \pi_X^* T_X \to T_\mathcal{Y}$ such that $d\pi_X \circ \mathcal{L} = \mathrm{Id}$. These objects depend only on the fixed embedding.*

Proof. Consider the affine cone of the embedding,

$$\widehat{X} := \{0\} \cup \{\mathbf{s} \in \mathbb{C}^m \setminus \{0\} : [\mathbf{s}] \in \iota(X)\},$$

and set

$$\mathcal{Y} := \{(\mathbf{s}, \chi) \in \widehat{X} \times (\mathbb{C}^m)^* : \chi \cdot \mathbf{s} = 1\}, \qquad \pi_X(\mathbf{s}, \chi) := \iota^{-1}[\mathbf{s}]. \tag{45}$$

The defining equation excludes the cone vertex. On a chart $\mathbf{s}_j \neq 0$, local coordinates consist of coordinates on X, a nonzero scale factor, and $m-1$ unconstrained dual coordinates. Hence $\mathcal{Y}$ is a smooth closed affine algebraic variety and therefore Stein, while π_X is a holomorphic submersion.

For the holomorphic splitting, set $V := \mathbb{C}^m$ and let $\ell := \iota^* \mathcal{O}_{\mathbb{P}^{m-1}}(-1)$ be the tautological line bundle on X; its fiber $\ell_x \subset V$ is the line representing $\iota(x)$. Define

$$\mathcal{U} := \{(x, \chi) \in X \times V^* : \chi|_{\ell_x} \neq 0\}.$$

The map

$$\Theta : \mathcal{Y} \longrightarrow \mathcal{U}, \qquad (\mathbf{s}, \chi) \longmapsto (\pi_X(\mathbf{s}, \chi), \chi),$$

is biholomorphic. Indeed, if s_0 is a local nonvanishing holomorphic section of ℓ, then

$$\sigma(x, \chi) := \frac{s_0(x)}{\chi(s_0(x))}$$

is independent of the choice of s_0, is holomorphic, lies in ℓ_x, and satisfies $\chi(\sigma(x, \chi)) = 1$. Thus

$$\Theta^{-1}(x, \chi) = (\sigma(x, \chi), \chi).$$

Since $\mathcal{U}$ is open in $X \times V^*$, its tangent space at (x, χ) is $T_x X \oplus V^*$. Transporting the vectors $(v, 0)$ by $d\Theta^{-1}$ gives the required holomorphic right inverse

$$\mathcal{L} : \pi_X^* T_X \to T_\mathcal{Y}, \qquad \mathcal{L}_y(v) := d(\Theta^{-1})_{\Theta(y)}(v, 0), \qquad d\pi_X \circ \mathcal{L} = \mathrm{Id}. \tag{46}$$

$\square$

We assemble three ingredients for the holomorphic variations: a controlled boundary-unitary frame, a lift of the disk, and lifts of the frame vectors. Under fixed area and weighted-derivative caps, the lifted disks lie in a common compact subset of the fixed affine model, and the estimates for all three ingredients are uniform.

Theorem 4.7. *Fix the affine model $\mathcal{Y}$, the submersion π_X, and the right inverse $\mathcal{L}$ supplied by Lemma 4.6. Given $A_0, B_0 \geq 0$, one can choose a compact set $K_{A_0,B_0} \Subset \mathcal{Y}$, an open neighborhood Ω of K_{A_0,B_0} in $\mathbb{C}^{2m}$, a holomorphic map $\widetilde{\pi}_X : \Omega \to X$ with $\widetilde{\pi}_X|_{\mathcal{Y} \cap \Omega} = \pi_X$, and constants*

$$\Gamma, D_1, D_{\widetilde{f}}, C_{\theta,\infty}, D_\theta < \infty.$$

These cap-dependent choices depend only on $(X, \omega, \iota, A_0, B_0)$ and are independent of the source radius and the individual disk; the affine model and its splitting depend only on ι. For every $R > 0$

and every map f holomorphic on a neighborhood of $\overline{\mathbb{D}}_R$ with $\mathrm{Area}_R(f) \le A_0$ and $B_R(f) \le B_0$, one can choose a frame e, a lifted disk $\widetilde{f}$, and a matrix θ compatibly so that the following properties hold:

(i) *e is a holomorphic frame of f^*T_X, smooth on the closed disk and unitary on its boundary, whose Gram matrix $G(z) := (\langle e_j(z), e_k(z)\rangle_\omega)_{j,k=1}^n$ satisfies*

$$\Gamma^{-1}\,\mathrm{Id}_n \le G \le \Gamma\,\mathrm{Id}_n, \qquad \sum_{a=1}^n \int_{\mathbb{D}_R} |\nabla_z e_a|^2 dA \le D_1;$$

(ii) *$\widetilde{f}$ is a map from a neighborhood of $\overline{\mathbb{D}}_R$ to $\mathcal{Y}$, holomorphic on that neighborhood, such that*

$$\pi_X \circ \widetilde{f} = f, \quad \widetilde{f}(\overline{\mathbb{D}}_R) \subset K_{A_0,B_0}, \quad \int_{\mathbb{D}_R} |\widetilde{f}'|^2 dA \le D_{\widetilde{f}};$$

(iii) *the frame vectors have holomorphic lifts $\theta_1, \dots, \theta_n : \mathbb{D}_R \to \mathbb{C}^{2m}$ along $\widetilde{f}$. Their column matrix and the lifting identities are*

$$\begin{aligned} &\theta := (\theta_1, \dots, \theta_n) : \mathbb{D}_R \to \mathrm{Mat}_{2m\times n}(\mathbb{C}), \\ &d(\widetilde{\pi}_X)_{\widetilde{f}(z)}\theta_a(z) = e_a(z) \qquad (1 \le a \le n). \end{aligned} \tag{47}$$

The matrix θ is smooth on the closed disk and satisfies

$$\sup_{z\in\mathbb{D}_R} \|\theta(z)\|_{\mathrm{op}} \le C_{\theta,\infty}, \qquad \int_{\mathbb{D}_R} \|\theta'\|_{\mathrm{HS}}^2\, dA \le D_\theta.$$

Proof. Fix a disk f satisfying the caps A_0, B_0.

(i) Choose $R' > R$ so that $\overline{\mathbb{D}}_{R'}$ lies in the given source neighborhood, and choose a holomorphic frame $\widetilde{e}$ of f^*T_X on $\mathbb{D}_{R'}$; such a frame exists by the Oka–Grauert principle [28, Satz 6, p. 270], since the source is contractible and Stein. Restrict its Gram matrix to $\partial\mathbb{D}_R$ and apply Lemma 4.1 to $H(\zeta) = \overline{\widetilde{G}(R\zeta)}$. For the resulting factor A, set $e(z) := \widetilde{e}(z)A(z/R)$. Conjugating the factorization identity gives its boundary Gram matrix $A(z/R)^{\mathsf{T}}\widetilde{G}(z)\overline{A(z/R)} = \mathrm{Id}_n$. Thus e is holomorphic, smooth on the closed disk, and unitary on its boundary. Proposition 4.3 gives (39) and (40), proving (i).

(ii) By Lemma 4.4, f has a homogeneous lift $\mathbf{s}$ with

$$|\mathbf{s}| \le 1, \qquad |\mathbf{s}| \ge \delta_{\mathbf{s}}, \qquad \int_{\mathbb{D}_R} |\mathbf{s}'|^2 dA \le c_\iota A_0.$$

Its coordinates satisfy the hypotheses of Theorem 4.5 with algebra-norm bound $C_{\mathbf{s}}$ and corona lower bound $\delta_{\mathbf{s}}/\sqrt{m}$. Thus (43) supplies a vector χ satisfying $\chi \cdot \mathbf{s} = 1$, and the dilation construction leading to (44) gives a vector $\widehat{\chi}$, holomorphic near the closed disk, with

$$\widehat{\chi} \cdot \mathbf{s} = 1, \qquad \|\widehat{\chi}\|_{L^\infty(\mathbb{D}_R)} \le 2C_\chi, \qquad \|\widehat{\chi}'\|_{L^2(\mathbb{D}_R)} \le D_\chi.$$

The identity $\widehat{\chi} \cdot \mathbf{s} = 1$ shows that $\widetilde{f} := (\mathbf{s}, \widehat{\chi})$ takes values in the fixed space $\mathcal{Y}$ of (45). It satisfies

$$\widetilde{f}(\overline{\mathbb{D}}_R) \subset K_{A_0,B_0} := \mathcal{Y} \cap \{|\mathbf{s}| \le 1,\ |\chi| \le 2C_\chi\}, \qquad \int_{\mathbb{D}_R} |\widetilde{f}'|^2 dA \le c_\iota A_0 + D_\chi^2 =: D_{\widetilde{f}}. \tag{48}$$

The set K_{A_0,B_0} is compact: it is closed and bounded, and $\chi \cdot \mathbf{s} = 1$ prevents $\mathbf{s}$ from approaching the cone vertex.

Since $\mathcal{Y}$ is a closed Stein submanifold of $\mathbb{C}^{2m}$, the Docquier–Grauert tubular neighborhood theorem [16, Satz 3], in the holomorphic deformation-retraction formulation [22, Theorem 3.1], gives a holomorphic retraction $\mathfrak{r} : \Omega \to \mathcal{Y} \cap \Omega$ from a neighborhood Ω of K_{A_0,B_0}, equal to the identity on $\mathcal{Y} \cap \Omega$. Its derivatives are bounded on smaller compact neighborhoods of K_{A_0,B_0}. Set $\widetilde{\pi}_X := \pi_X \circ \mathfrak{r}$. Since $\pi_X \circ \widetilde{f} = f$ and $\widetilde{\pi}_X|_{\mathcal{Y}\cap\Omega} = \pi_X$, (48) proves (ii). The choices of K_{A_0,B_0}, Ω, and $\widetilde{\pi}_X$ depend only on the fixed target data and the two caps; they are fixed before any individual lifted disk $\widetilde{f}$ is chosen. The space $\mathcal{Y}$, the projection π_X, and the splitting $\mathcal{L}$ were already fixed from the embedding alone.

(iii) Regard the splitting $\mathcal{L}$ from (46) as a bundle map from $\pi_X^* T_X$ to the ambient trivial bundle $\mathcal{Y} \times \mathbb{C}^{2m}$. In $\nabla\mathcal{L}$, use the pullback Chern connection on $\pi_X^* T_X$ and the flat connection on this ambient bundle. Let

$$L_0 \coloneqq \sup_{K_{A_0,B_0}} \|\mathcal{L}\|, \qquad L_1 \coloneqq \sup_{K_{A_0,B_0}} \|\nabla\mathcal{L}\|.$$

Set

$$\theta_a(z) \coloneqq \mathcal{L}_{\widetilde{f}(z)} e_a(z).$$

Then $d(\widetilde{\pi}_X)_{\widetilde{f}}\theta_a = e_a$. With the preceding connection convention, ambient differentiation gives

$$\theta_a' = \mathcal{L}_{\widetilde{f}} \nabla_z e_a + (\nabla_{\widetilde{f}'}\mathcal{L}) e_a. \tag{49}$$

The frame and lifted-disk bounds, with $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, give

$$\|\theta\|_{L^\infty,\mathrm{op}} \leq L_0\sqrt{\Gamma} =: C_{\theta,\infty}, \tag{50}$$

$$\int_{\mathbb{D}_R} \|\theta'\|_{\mathrm{HS}}^2 dA \leq 2L_0^2 D_1 + 2n\Gamma L_1^2 D_{\widetilde{f}} =: D_\theta. \tag{51}$$

This proves (iii). □

We now add the lifted variation fields and project to X. Radial regularization extends the fields past the closed disk while keeping the lifted map fixed. The multiplier retains its vanishing orders, and the paired trace converges to the value computed in Section 3. The parameter range and cubic remainder remain uniform throughout this approximation.

The embedding fixes $\mathcal{Y}, \pi_X, \mathcal{L}$, independently of the disks and caps. The caps A_0, B_0 then fix $K_{A_0,B_0}, \Omega, \widetilde{\pi}_X$; together with D_Φ, N_Φ, these data determine t_0, C_T. Each disk has its own compatible frame, lift, and lifted fields satisfying Theorem 4.7. The approximation parameter ρ may depend on this triple, on Φ, and on the prescribed trace error ε, whereas t_0, C_T are independent of R, f, Φ, ρ, and ε. The extension neighborhood may depend on the family, Φ, and ρ; its width need not be uniform.

Theorem 4.8. *Fix (X,ω), the projective embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$, and the bounds $A_0, B_0 \geq 0$ as in Theorem 4.7. Let $D_\Phi \geq 0$, and let $N_\Phi \geq 1$ be an integer. There are constants $t_0 > 0$ and $C_T < \infty$, depending only on $(X, \omega, \iota, A_0, B_0, D_\Phi, N_\Phi)$, with the following property.*

Let $R > 0$, and let $f : \mathbb{D}_R \to X$ extend holomorphically to a neighborhood of $\overline{\mathbb{D}}_R$ and satisfy $\mathrm{Area}_R(f) \leq A_0$ *and $B_R(f) \leq B_0$. Use the fixed K_{A_0,B_0}, Ω, and $\widetilde{\pi}_X$ from Theorem 4.7, and choose any compatible disk-dependent triple $e, \widetilde{f}, \theta$ supplied by that theorem. Suppose that $\Phi = (\Phi_1, \ldots, \Phi_{N_\Phi}) :$* $\mathbb{D}_R \to \mathrm{Mat}_{n\times N_\Phi}(\mathbb{C})$ *extends holomorphically to an open neighborhood of $\overline{\mathbb{D}}_R$ and satisfies*

$$\sup_{z\in\overline{\mathbb{D}}_R} \|\Phi(z)\|_{\mathrm{op}} \leq 1, \qquad \int_{\mathbb{D}_R} \|\Phi'\|_{\mathrm{HS}}^2 \, dA \leq D_\Phi,$$

where the column Φ_a determines the section $e\Phi_a = \sum_{j=1}^n e_j \Phi_{ja}$ of f^*T_X. The constants t_0 and C_T are independent of R, f, Φ and of this compatible triple.

For each $0 < \rho < 1$, write

$$\Xi_\rho(z) \coloneqq \theta(\rho z)\Phi(z), \qquad (\Xi_\rho)_a(z) \coloneqq \sum_{j=1}^n \theta_j(\rho z)\Phi_{ja}(z).$$

(i) Holomorphic realization. *For every $0 < \rho < 1$, each column defines the family*

$$f_{a,t}^\rho(z) \coloneqq \widetilde{\pi}_X\big(\widetilde{f}(z) + t(\Xi_\rho)_a(z)\big), \qquad t \in \mathbb{C}, \quad |t| \leq t_0, \quad 1 \leq a \leq N_\Phi.$$

All these families are jointly holomorphic on one open neighborhood of $\overline{\mathbb{D}}_R \times \{t \in \mathbb{C} : |t| \leq t_0\}$; for every $0 < \rho < 1$ that neighborhood may depend on the selected disk-dependent triple, Φ, and ρ, but not on the column a.

(ii) Uniform cubic remainder. *For every $0<\rho<1$, $1\le a\le N_\Phi$, and $\tau\in\{1,i\}$, write $t=\tau s$ with $s\in\mathbb{R}$ and set*

$$\ell^{\rho}_{a,\tau}:=\frac{d}{ds}\Big|_{s=0}\mathrm{Area}_R(f^{\rho}_{a,\tau s}),\qquad H^{\rho}_{a,\tau}:=\frac{d^2}{ds^2}\Big|_{s=0}\mathrm{Area}_R(f^{\rho}_{a,\tau s}).$$

For $|s|\le t_0$,

$$\mathrm{Area}_R(f^{\rho}_{a,\tau s})\le \mathrm{Area}_R(f)+\ell^{\rho}_{a,\tau}s+\frac{1}{2}H^{\rho}_{a,\tau}s^2+C_T|s|^3 \tag{52}$$

with the same constant C_T for both choices of τ.

(iii) Trace approximation. *If Φ is boundary coisometric, meaning $\Phi\Phi^*=\mathrm{Id}_n$ on $\partial\mathbb{D}_R$, then for every $\varepsilon>0$ there is $0<\rho_\varepsilon<1$, depending only on the individual data displayed above and on ε, such that*

$$\left|\sum_{a=1}^{N_\Phi}(H^{\rho_\varepsilon}_{a,1}+H^{\rho_\varepsilon}_{a,i})-\left(4\int_{\mathbb{D}_R}\|\Phi'\|^2_{\mathrm{HS}}dA-2\int_{\mathbb{D}_R}f^*\,\mathrm{Ric}(\omega)\right)\right|\le\varepsilon. \tag{53}$$

Only this conclusion requires choosing ρ_ε close to 1; t_0 and C_T remain independent of ε and ρ_ε.

Proof. Use the fixed cap-dependent ambient data and the selected compatible triple from Theorem 4.7. Set

$$\delta_0:=\min\{1,\mathrm{dist}(K_{A_0,B_0},\mathbb{C}^{2m}\setminus\Omega)\}>0,$$

and choose a compact $K'\Subset\Omega$ whose interior contains the closed Euclidean $3\delta_0/4$-thickening of K_{A_0,B_0}. In the standard coordinate frame of $\mathbb{C}^{2m}$, let g_0 be the positive-semidefinite Hermitian coefficient matrix of $\widetilde{\pi}_X^*\omega$ on K', so

$$v^{\mathsf{T}}g_0(w)\bar{v}=|d(\widetilde{\pi}_X)_w(v)|^2_\omega$$

for a coefficient column v. Only smooth coefficient bounds are used; the pullback form may be degenerate along the fibers. Take $C_{g,3}$ to bound the coefficients and their derivatives through order three on K'.

Take a multiplier Φ satisfying the stated bounds and, for $0<\rho<1$, set $\Xi_\rho:=\theta(\rho\,\cdot)\Phi$. Dilation gives $\int_{\mathbb{D}_R}|\rho\theta'(\rho z)|^2\,dA(z)=\int_{\mathbb{D}_{\rho R}}|\theta'(w)|^2\,dA(w)\le D_\theta$. Use this identity, $\|\Phi\|_\infty\le 1$, and $(\theta(\rho z)\Phi)'=\rho\theta'(\rho z)\Phi+\theta(\rho z)\Phi'$ to obtain

$$\sup_{z\in\mathbb{D}_R}\|\Xi_\rho(z)\|_{\mathrm{op}}\le C_{\theta,\infty},\qquad \int_{\mathbb{D}_R}\|\Xi'_\rho\|^2_{\mathrm{HS}}dA\le 2D_\theta+2C^2_{\theta,\infty}D_\Phi=:D_\Xi. \tag{54}$$

Choose

$$t_0:=\min\left\{1,\frac{\delta_0}{4\max\{1,C_{\theta,\infty}\}}\right\}.$$

For $|t|\le 2t_0$, the displacement of each column perturbation is at most $2t_0C_{\theta,\infty}\le\delta_0/2$. Thus $\widetilde{f}+t(\Xi_\rho)_a$ stays in K', uniformly in the disk and the column.

(i) Fix a column and write $\Xi:=(\Xi_\rho)_a$. The function $\theta(\rho z)$ is holomorphic for $|z|<R/\rho$; since $\widetilde{f}$ and Φ extend past the closed disk, so does $\widetilde{f}+t\Xi$. On the closed disk and for $|t|\le t_0$ its image lies in the closed $\delta_0/2$-thickening of K_{A_0,B_0}, which is compactly contained in $\mathrm{int}\,K'\Subset\Omega$. Compactness gives a neighborhood of $\overline{\mathbb{D}}_R\times\{|t|\le t_0\}$ on which the image remains in Ω, so the projected family is jointly holomorphic there. Intersect these neighborhoods over the finitely many columns.

(ii) In ambient coordinates write $w(t):=\widetilde{f}+t\Xi$ and $v(t):=\widetilde{f}'+t\Xi'$. The area density is $v(t)^{\mathsf{T}}g_0(w(t))\overline{v(t)}$. Along either real parameter axis $t=\tau s$, with $\tau\in\{1,i\}$, the functions $w(\tau s)$

and $v(\tau s)$ are affine in s, with first derivatives $\tau\Xi$ and $\tau\Xi'$. For fixed z, put $G(s) := g_0(w(\tau s))$, $V(s) := v(\tau s)$, and $Q := \tau\Xi'$. Ordinary differentiation in the real variable s gives

$$\frac{d^3}{ds^3}(V^\mathsf{T} G\bar{V}) = V^\mathsf{T} G'''\bar{V} + 3Q^\mathsf{T} G''\bar{V} + 3V^\mathsf{T} G''\bar{Q} + 6Q^\mathsf{T} G'\bar{Q}.$$

The chain rule bounds $|G^{(k)}(s)|$ by $C_{g,3}|\Xi|^k$ for $1 \le k \le 3$, up to fixed dimension factors. The bounds $\|\Xi\|_\infty \le C_{\theta,\infty}$ and $|t| \le t_0$ consequently give the integrable estimate

$$\left|\frac{d^3}{ds^3}\left(v(\tau s)^\mathsf{T} g_0(w(\tau s))\overline{v(\tau s)}\right)\right| \le C(C_{g,3}, C_{\theta,\infty}, t_0)(|\widetilde{f}'|^2 + |\Xi'|^2).$$

The right-hand side has a uniformly bounded integral by (48) and (54). Thus the third derivative of the area is uniformly bounded along both real parameter axes. Taylor's theorem gives (52) with one remainder constant C_T.

(iii) The tangent fields of the actual families satisfy, for each fixed f,

$$\xi_{\rho,a} := d(\widetilde{\pi}_X)_{\widetilde{f}}(\theta(\rho\,\cdot)\Phi_a) \longrightarrow e\Phi_a \quad \text{in } C^1(\overline{\mathbb{D}}_R) \quad (\rho \uparrow 1),$$

because $\theta(\rho\,\cdot) \to \theta$ in the same norm. Here the Kähler condition enters through Lemma 3.3. If Φ is boundary coisometric, that lemma, continuity of (29), and Proposition 3.4 therefore give

$$\sum_a \mathcal{H}_{R,f}(\xi_{\rho,a}) \longrightarrow 4\int_{\mathbb{D}_R} \|\Phi'\|_{\mathrm{HS}}^2\, dA - 2\int_{\mathbb{D}_R} f^* \operatorname{Ric}(\omega).$$

The summand is $H^\rho_{a,1} + H^\rho_{a,i}$ by Lemma 3.3. Choose ρ_ε close enough to one to obtain the prescribed error. Parts (i)–(ii) and (54) hold for every $\rho \in (0,1)$ with the same t_0, C_T. Their dependence is only on the fixed target data and A_0, B_0, D_Φ, N_Φ. ☐

4.3. **A common area and derivative estimate**

For the fixed caps, Theorem 4.7 supplies a common compact set K, ambient projection Π, and Dirichlet bound D. Each compatible lift H satisfies

$$\Pi H = f, \quad H(\overline{\mathbb{D}}_R) \subset K, \quad \int_{\mathbb{D}_R} |H'|^2 dA \le D. \tag{55}$$

Choose $\epsilon > 0$ such that the closed Euclidean ϵ-neighborhood K_ϵ of K is compactly contained in Ω. Write $g(w)$ for the Hermitian coefficient matrix of $\Pi^*\omega$ in ambient coordinates. Thus

$$a^\mathsf{T} g(w)\bar{a} = |d\Pi_w(a)|^2_\omega.$$

The form can be degenerate. On K_ϵ, fix bounds

$$\|g(w)\|_{\mathrm{op}} \le M_0, \qquad \|Dg(w)[v]\|_{\mathrm{op}} \le L_g|v|, \tag{56}$$

where Dg is the real derivative in ambient coordinates. These bounds are fixed before selecting an individual disk.

Lemma 4.9. *If $F : \mathbb{D}_R \to \mathbb{C}^N$ is holomorphic and $\int_{\mathbb{D}_R} |F'|^2 dA \le E$, then*

$$(R - |z|)|F'(z)| \le \sqrt{E/\pi} \quad (z \in \mathbb{D}_R).$$

Proof. For $0 < r < R - |z|$, the classical submean inequality [14, Ch. I, §4], applied to $|F'|^2$, gives

$$\pi r^2 |F'(z)|^2 \le \int_{\mathbb{D}_r(z)} |F'|^2 dA \le E.$$

Let $r \uparrow R - |z|$. ☐

Proposition 4.10. *Suppose H satisfies* (55)*, and let Ξ be holomorphic near $\overline{\mathbb{D}}_R$, with*

$$\|\Xi\|_\infty \le S, \qquad \int_{\mathbb{D}_R} |\Xi'|^2\, dA \le E.$$

Fix $0 < \bar{t} \le 1$ with $\bar{t}S < \epsilon$. For $|t| \le \bar{t}$, set $f_t = \Pi(H + t\Xi)$ and

$$C_A = L_g SD + 2M_0\sqrt{DE} + M_0\bar{t}E. \tag{57}$$

Then

$$|\operatorname{Area}_R(f_t) - \operatorname{Area}_R(f)| \le C_A|t|, \tag{58}$$

$$|B_R(f_t)^2 - B_R(f)^2| \le \frac{C_A}{\pi}|t|. \tag{59}$$

In particular, both an upper and a lower bound for the new B follow:

$$\sqrt{\max\{0, B_R(f)^2 - C_A|t|/\pi\}} \le B_R(f_t) \le \sqrt{B_R(f)^2 + C_A|t|/\pi}. \tag{60}$$

The same family has the pointwise displacement bound

$$\operatorname{dist}_\omega(f_t(z), f(z)) \le \sqrt{M_0}\, S|t| \qquad (z \in \mathbb{D}_R). \tag{61}$$

Proof. The entire segment $H(z) + st\Xi(z)$, $0 \le s \le 1$, lies in K_ϵ, so (56) applies along it. Put $a = H'(z)$, $b = \Xi'(z)$, and $g_t = g(H(z) + t\Xi(z))$. The change in area density has three contributions: the change in the metric coefficients, a mixed derivative term, and a quadratic derivative term. More precisely,

$$(a+tb)^\mathsf{T} g_t \overline{a+tb} - a^\mathsf{T} g_0 \bar{a} = a^\mathsf{T}(g_t - g_0)\bar{a} + 2\operatorname{Re}(tb^\mathsf{T} g_t \bar{a}) + |t|^2 b^\mathsf{T} g_t \bar{b}. \tag{62}$$

Its absolute value is at most

$$|t| L_g S|a|^2 + 2M_0|t||a||b| + M_0|t|^2|b|^2.$$

Integrating, and applying Cauchy–Schwarz to the middle term, proves (58).

For the second estimate, put $d = R - |z|$. Lemma 4.9 gives $d|a| \le \sqrt{D/\pi}$ and $d|b| \le \sqrt{E/\pi}$. Multiplying the same pointwise estimate by d^2 therefore bounds the difference of the two squared weighted derivatives by $C_A|t|/\pi$. Use $|\sup p - \sup q| \le \sup|p - q|$ and $B_R(f)^2 = \sup d^2|f_z|^2_\omega$ to obtain (59). Finally, the path $s \mapsto \Pi(H(z) + st\Xi(z))$, $0 \le s \le 1$, has speed at most $\sqrt{M_0}|t||\Xi(z)|$. Its length proves (61). Only bounds on the possibly degenerate metric coefficients have been used. ☐

4.4. **Uniform descent and enlargement**

When $\operatorname{Ric}(\omega) \ge \kappa\omega$ and $\operatorname{Area}_R(f) > A_*$, the trace is negative. Selecting one real direction and the sign of its parameter gives descent; the common parameter radius and Taylor remainder give a uniform step size under fixed area and weighted-derivative bounds.

Proof of Proposition 2.1. Put $x_0 = f(0)$ and $\lambda = \lambda^F_{x_0}$. We allow the scalar $k = \lambda(f_z(0))$ to be arbitrary; its value is preserved below.

For the multiplier (32), one has $N_\Phi = n$, $D_\Phi = \pi(n+1)$, and, by (34),

$$4\pi(n+1) - 2\kappa \operatorname{Area}_R(f) \le -2\kappa\delta.$$

Apply Theorem 4.8 with trace error $\varepsilon = \kappa\delta$, choose the value ρ_ε supplied by part (iii), and abbreviate $f_{a,t} = f^{\rho_\varepsilon}_{a,t}$, $\ell_{a,\tau} = \ell^{\rho_\varepsilon}_{a,\tau}$, and $H_{a,\tau} = H^{\rho_\varepsilon}_{a,\tau}$. For every column a and every $|t| \le t_0$, the identities $\Phi_a(0) = 0$ and $d(\widetilde{\pi}_X)_{\widetilde{f}(0)}\theta(0) = e(0)$ give the exact interpolation formula

$$\begin{aligned} f_{a,t}(0) &= f(0),\\ (f_{a,t})_z(0) &= f_z(0) + t\, e(0)\Phi_a'(0),\\ \lambda((f_{a,t})_z(0)) &= \lambda(f_z(0)) + t\lambda(e(0)\Phi_a'(0)) = k. \end{aligned} \tag{63}$$

Indeed, the ambient perturbation at $z = 0$ equals $\widetilde{f}(0)$ for every t, so the chain rule always evaluates $d\widetilde{\pi}_X$ at the same point $\widetilde{f}(0)$. Moreover, the product rule gives $(\Xi_{\rho_\varepsilon})'_a(0) = \theta(0)\Phi'_a(0)$, so radial regularization does not alter the derivative of the increment at the origin. Formula (63) fixes the displayed scalar component, not the entire first jet. For this choice, part (iii) and the preceding trace bound give $\sum_a(H_{a,1} + H_{a,i}) \le -\kappa\delta$. One of the $2n$ real axes therefore has second area derivative $H_{a,\tau} \le -\kappa\delta/(2n)$. Choose the sign of s so that $\ell_{a,\tau}s \le 0$. Then (52) gives

$$\mathrm{Area}_R(f_{a,\tau s}) - \mathrm{Area}_R(f) \le -\frac{\kappa\delta}{4n}s^2 + C_T|s|^3.$$

Before selecting the step, decrease the engine parameter radius to $\min\{t_0, 1, \epsilon/(2\max\{1, C_{\theta,\infty}\})\}$ if necessary, so that the selected perturbation segment lies in the compact tube used in Proposition 4.10. Set

$$t_* := \min\left\{\frac{t_0}{2}, \frac{\kappa\delta}{8n\max\{C_T, 1\}}\right\}, \qquad \sigma := \frac{\kappa\delta}{8n}t_*^2. \tag{64}$$

Take $s \in \{t_*, -t_*\}$ with the chosen sign and set $g = f_{a,\tau s}$. The Taylor estimate gives $\mathrm{Area}_R(g) \le \mathrm{Area}_R(f) - \sigma$. The selected increment has the bounds in (54). Proposition 4.10 gives the remaining estimate in (7) with

$$\beta_I = C_{A,I}t_*/\pi,$$

where $C_{A,I}$ is (57) for this increment. The parameter choice keeps its entire segment in the common tube. The constants $t_0, C_T, C_{A,I}$ depend only on the fixed caps and target data. The multiplier has the same cost for every nonzero covector $\lambda^F_{x_0}$ and every scalar k. Thus t_*, σ, β_I are independent of the radius, the disk, the center in F, and the scalar. □

For scalar enlargement, it suffices to lift one finite-area disk and perturb its lift in one constant ambient direction.

Proof of Lemma 2.2. Choose $s_j \uparrow R$. Each restriction $h|_{\mathbb{D}_{s_j}}$ extends holomorphically past its closed disk and has the common bounds

$$\mathrm{Area}_{s_j}(h) \le A_0, \qquad B_{s_j}(h) \le B_0.$$

Apply Theorem 4.7(ii) with these caps. It supplies one compact set $K = K_{A_0,B_0} \Subset \mathcal{Y}$, one ambient neighborhood $\Omega \subset \mathbb{C}^{2m}$ of K, and one holomorphic projection $\Pi = \widetilde{\pi}_X : \Omega \to X$ satisfying $\Pi|_{\mathcal{Y}\cap\Omega} = \pi_X$. The corresponding lifts H_j obey

$$H_j(\overline{\mathbb{D}}_{s_j}) \subset K, \qquad \Pi \circ H_j = h \quad \text{on } \mathbb{D}_{s_j}, \qquad \int_{\mathbb{D}_{s_j}} |H'_j|^2\, dA \le D_{\widetilde{f}},$$

where the last constant is independent of j.

The compact range makes the coordinate functions of H_j a normal family on every compact subdisk of $\mathbb{D}_R$. A diagonal subsequence therefore converges locally uniformly, together with its derivatives, to a holomorphic map $H : \mathbb{D}_R \to \mathbb{C}^{2m}$. Since K is closed, $H(\mathbb{D}_R) \subset K$, and continuity gives $\Pi \circ H = h$. For every $s < R$, convergence of derivatives on $\overline{\mathbb{D}}_s$ gives $\int_{\mathbb{D}_s} |H'|^2\, dA \le D_{\widetilde{f}}$. Exhausting $\mathbb{D}_R$ yields

$$H(\mathbb{D}_R) \subset K, \qquad \Pi \circ H = h, \qquad \int_{\mathbb{D}_R} |H'|^2\, dA \le D_{\widetilde{f}}. \tag{65}$$

No compatibility between the separately chosen lifts is required.

Enlarge $D = D_{\widetilde{f}}$ to at least 1 and put

$$Q = \sqrt{D/\pi}, \qquad \bar{t} = \min\{1, \epsilon/(2\max\{1, Q\})\}.$$

Here ϵ is the common tube radius used in Section 4.3. For $v = H'(0)$, Lemma 4.9 gives

$$R|v| \leq Q, \qquad \|zv\|_\infty \leq Q, \qquad \int_{\mathbb{D}_R} |(zv)'|^2 dA = \pi R^2 |v|^2 \leq D.$$

Thus the affine formula (8) stays in the common tube. Its derivative is $(h_t)_z(0) = (1+t)h_z(0)$. Proposition 4.10, valid on the open disk by the same pointwise proof and exhaustion, gives the area and B^2 bounds with

$$C_{II} = D(L_g Q + 3M_0). \tag{66}$$

Source dilation satisfies the exact identities

$$\mathrm{Area}_{(1+t)R}(G_t) = \mathrm{Area}_R(h_t), \qquad B_{(1+t)R}(G_t) = B_R(h_t).$$

It also restores the entire derivative at zero. The displacement estimate refers to corresponding source points:

$$\mathrm{dist}_\omega(G_t((1+t)z), h(z)) \leq \sqrt{M_0}\, Qt.$$

After restricting to $\mathbb{D}_{R'}$, this comparison applies only where $|(1+t)z| < R'$. No closeness estimate for a principal recovery follows from this finite deformation bound. This proves the full-domain assertions. If h extends past the closed disk, its lift can be chosen with that property directly from Theorem 4.7, so both families have the corresponding extension. For an open disk, restrict at $R' = (1+t/2)R$. This lies strictly inside the new open domain. Restriction decreases both area and B, proving (12). $\square$

4.5. **Scalar repair by an absolute radius increment**

A second explicit enlargement repairs only the scalar derivative. Fix the same area and derivative caps A_0, B_0 as in the two operations. Take $f \in \mathcal{A}_r^{F,\lambda^F}$ with $\mathrm{Area}_r(f) \leq A_0$ and $B_r(f) \leq B_0$; in particular, f extends holomorphically past $\overline{\mathbb{D}}_r$ and satisfies the scalar normalization. Set $x = f(0)$ and $\lambda = \lambda_x^F$. Choose a positive increment Δ. Put $R = r + \Delta$ and $\alpha = r/R$. Start on the larger disk with $f^{(\alpha)}(z) = f(\alpha z)$. Then

$$\mathrm{Area}_R(f^{(\alpha)}) = \mathrm{Area}_r(f), \quad B_R(f^{(\alpha)}) = B_r(f), \quad \lambda((f^{(\alpha)})_z(0)) = \alpha.$$

Apply Theorem 4.7 to $f^{(\alpha)}$, and denote its compatible lift, frame, and lifted frame matrix by H, e, and θ, respectively. Set

$$L_f = \lambda \circ e(0), \qquad c_f = \frac{L_f^*}{L_f L_f^*}, \qquad \lambda_{\min} = \min_{y \in F} \|\lambda_y^F\|_{\omega,*}, \qquad c_0 = \max\{1, \sqrt{\Gamma}/\lambda_{\min}\}.$$

The minimum is positive by compactness of F. The frame bound therefore gives $|c_f| \leq c_0$ and $L_f c_f = 1$, with c_0 uniform in the center. Thus

$$\widehat{\phi}(z) = \frac{z}{R}\frac{c_f}{c_0}, \qquad \widehat{\Xi}(z) = \theta(\rho z)\widehat{\phi}(z)$$

has uniform supremum and Dirichlet bounds; any fixed $0 < \rho < 1$ is allowed. Here the engine is applied with $D_\Phi = \pi$ and $N_\Phi = 1$, so denote its parameter radius by $\widehat{t}_{0,\mathrm{abs}}$. With $S_{\mathrm{abs}} = C_{\theta,\infty}$, set

$$t_{0,\mathrm{abs}} = \min\{\widehat{t}_{0,\mathrm{abs}}, 1, \epsilon/(2\max\{1, S_{\mathrm{abs}}\})\}.$$

This parameter is chosen separately from the descent parameter and ensures the same strict tube margin. The family $\widehat{f}_t = \Pi(H + t\widehat{\Xi})$ satisfies

$$\widehat{f}_t(0) = x, \qquad \lambda((\widehat{f}_t)_z(0)) = \alpha + \frac{t}{c_0 R}.$$

The explicit choice is

$$t = c_0 R(1-\alpha) = c_0 \Delta. \tag{67}$$

Provided $c_0\Delta \le t_{0,\mathrm{abs}}/2$, the new disk $g = \widehat{f}_{c_0\Delta}$ is admissible on the larger closed disk. Proposition 4.10 supplies a constant $C_{A,\mathrm{abs}}$ independent of r, R such that

$$\begin{aligned} R_{\mathrm{new}} &= r + \Delta, \\ |\operatorname{Area}_R(g) - \operatorname{Area}_r(f)| &\le c_0 C_{A,\mathrm{abs}}\Delta, \\ |B_R(g)^2 - B_r(f)^2| &\le c_0 C_{A,\mathrm{abs}}\Delta/\pi. \end{aligned} \tag{68}$$

The coefficient in (68) is independent of both radii. This variant may replace the affine enlargement in the finite stages by choosing a common positive Δ with area cost at most η_0; the radius counter then uses $\lceil(\ell_F M - R_s)/\Delta\rceil$. The enlargement in Lemma 2.2 has the additional property of preserving the entire first jet.

5. Compactness, marked values, and finite-area disks

The main proof uses compactness to obtain a principal map in the original source coordinate and a rational chain retaining two marked values. We obtain both from one fixed-mark exhaustion statement, with a joint area bound for the principal map and the attached spheres. Point selection then gives the weighted derivative bound for an individual finite-area disk. Throughout this section, X is a compact complex manifold equipped with a smooth Hermitian metric ω.

5.1. **Finite-area removal**

Finite area fills the punctures of a principal limit, including infinity when its source is $\mathbb{C}$. We first prove removal so that the compactness statement can include the extended principal map.

With $g_\omega(v, w) = \omega(v, Jw)$, the energy convention in the compactness and removal results used below is

$$\|df\|_{L^2}^2 = \int (|f_x|_{g_\omega}^2 + |f_y|_{g_\omega}^2)\, dA = 2\int f^*\omega = 2\operatorname{Area}(f), \tag{69}$$

because $f_y = Jf_x$. Thus their energy hypotheses are equivalent to our area hypotheses, with the fixed factor 2.

Proof of Theorem 2.4. Finite area gives $\int_{0<|z|<r} f^*\omega \to 0$ as $r \downarrow 0$, so every sufficiently small concentric annulus has small energy by (69). Compactness of X supplies completeness of the target metric and uniform continuity of its complex structure. The removal theorem of Ivashkovich and Shevchishin [36, Corollary 3.6] therefore gives a continuous extension at the puncture. In a target chart about the limiting value, the coordinate functions extend holomorphically by the Riemann removable singularity theorem. For an entire map, $\zeta = 1/z$ identifies infinity with a puncture and preserves area, so the same argument applies there. See also [45, Theorem 4.2.1]. □

5.2. **Marked compactness on an exhaustion**

Spheres obtained by rescaling near energy concentration points go back to Sacks and Uhlenbeck [52, pp. 2–3 and Section 4]; Gromov's compactness theory records how such components attach [29]. We use the theorem of Ivashkovich and Shevchishin [37]. For precise locators, we refer to its preprint formulation, whose title and numbering differ from the published version: Definition 1.6, Theorem 1.1, and the marked-point construction in Remark 2 following that theorem [36].

The *principal map* is the limit in the original source coordinate and may be constant. A *bubble* is a sphere obtained by rescaling; the finite-area extension of an entire principal map is also a sphere, but need not be a bubble.

A *ghost sphere* is a constant sphere component retaining marks or separating nodes. It has at least three special points, namely marks or nodes; the fixed-coordinate root disk has no such stability requirement. The dual graph records the attachments, with one vertex for each component and one edge for each node. Here it is a rooted tree.

The following proposition combines the principal limit used for recovery with the marked information needed for incidence. Fixed distinct source marks suffice for both applications. Constant components inside concentration trees must still be retained, since they can carry a marked value or join nonconstant components.

Proposition 5.1. *Let $R_j > 0$ and let $f_j : U_j \to X$ be holomorphic, where $U_j \subset \mathbb{C}$ is open and $\overline{\mathbb{D}}_{R_j} \Subset U_j$. Suppose $\sup_j \mathrm{Area}_{R_j}(f_j) < \infty$, and set $A_\infty = \liminf_j \mathrm{Area}_{R_j}(f_j)$. Choose $R_\infty \in (0, \infty]$ with $R_\infty \le \liminf_j R_j$, interpreting $\mathbb{D}_\infty$ as $\mathbb{C}$. Fix finitely many pairwise distinct points $a_1, \dots, a_m \in \mathbb{D}_{R_\infty}$, allowing $m = 0$. After first passing to a subsequence realizing A_∞ and then to a further subsequence, the following data exist simultaneously.*

(i) *A finite set $\Sigma_\infty \subset \mathbb{D}_{R_\infty}$ and a holomorphic principal map $f_\infty : \mathbb{D}_{R_\infty} \to X$ satisfy*

$$f_j \longrightarrow f_\infty \quad \text{smoothly on compact subsets of } \mathbb{D}_{R_\infty} \setminus \Sigma_\infty$$

in the original source coordinate.

(ii) *Finitely many holomorphic sphere components $u_\alpha : \mathbb{P}^1 \to X$, possibly constant, form trees attached to the principal root at points of Σ_∞. Each attached tree contains a nonconstant component; each constant sphere is stable. Let N_α be the set of nodes on the component, and let z_α be the root attachment point of the tree containing it. There are source reparametrizations $\varphi_{j,\alpha}$, defined near every compact subset of $\mathbb{P}^1 \setminus N_\alpha$ for all sufficiently large j, such that*

$$f_j \circ \varphi_{j,\alpha} \longrightarrow u_\alpha \quad \text{smoothly}, \qquad \varphi_{j,\alpha} \longrightarrow z_\alpha \quad \text{locally uniformly}$$

on $\mathbb{P}^1 \setminus N_\alpha$.

(iii) *Root, spheres, nodes and marks form one nodal map. Writing u_v for its component maps, p_{vw}, p_{wv} for the two points of a node, and p_ℓ on v_ℓ for the limiting ℓth mark, one has*

$$u_v(p_{vw}) = u_w(p_{wv}), \qquad f_j(a_\ell) \longrightarrow u_{v_\ell}(p_\ell).$$

A marked value therefore propagates along any path of constant components to its first nonconstant component.

(iv) *The areas, counted with the multiplicities of the maps, satisfy*

$$\mathrm{Area}_{R_\infty}(f_\infty) + \sum_\alpha \mathrm{Area}(u_\alpha) \le A_\infty. \tag{70}$$

Proof. Retain a subsequence whose total areas converge to A_∞. The proof first selects the principal map on the whole exhaustion, then makes one marked extraction on a fixed disk, and finally counts their areas together.

The original-coordinate principal map.

A diagonal weak selection of the area measures on compact subdisks gives a Radon measure μ on $\mathbb{D}_{R_\infty}$. For every continuous compactly supported χ with $0 \le \chi \le 1$,

$$\int \chi \, d\mu = \lim_j \int \chi \, f_j^* \omega \le A_\infty,$$

so $\mu(\mathbb{D}_{R_\infty}) \le A_\infty$ by exhaustion. Fix $p > 2$ and an area threshold $\epsilon_0 > 0$ small enough for the local compactness result [36, Lemma 3.1 and Corollary 3.3], using (69), and set

$$\Sigma_\infty = \{z : \mu(\{z\}) \ge \epsilon_0\}.$$

The mass bound makes this set finite. For $z \notin \Sigma_\infty$, choose a source disk with μ-null boundary and μ-mass less than ϵ_0. Weak convergence puts its area below that threshold for all large j. The cited result gives strong local $W^{1,p}$ convergence on a smaller disk; Sobolev embedding gives uniform convergence, and target charts and Cauchy estimates give smooth convergence. A diagonal choice on a countable cover gives f_∞ on the complement of Σ_∞ in the original coordinate. Its local definitions agree because they are limits of the same subsequence. Smooth convergence and exhaustion give $\mathrm{Area}(f_\infty) \le A_\infty$. Theorem 2.4 fills the finitely many punctures without changing area.

One marked extraction and its root.

Choose a fixed disk $\mathbb{D}_r$ containing all concentration points and source marks, and then choose $r < \rho < r' < R_\infty$. Apply marked compactness to the already selected subsequence on $\mathbb{D}_{r'}$, and retain the limit over $\mathbb{D}_\rho$, whose interior contains all the concentration points and source marks. The standard complex structure is fixed on this source disk, so the boundary-annulus nondegeneration condition is automatic [36, Definition 1.7 and Theorem 1.1]. The positive collar between $\overline{\mathbb{D}}_\rho$ and $\partial\mathbb{D}_{r'}$ keeps this application strictly inside the available domains. The target is compact, and the energy bound follows from (69).

The fixed-domain construction uses the original coordinate away from concentration points and creates sphere components by rescaling shrinking regions at those points [36, proof of Theorem 1.1, Case 1, and the remark following its proof]. Its root coordinate is identified with the original disk: the holomorphic identification agrees with the identity on an outer open set and hence everywhere. Uniqueness of the already selected limit then identifies the root map with f_∞ off Σ_∞, and holomorphic extension identifies it throughout $\mathbb{D}_\rho$. We extend this root by the global map f_∞.

The marked construction in Remark 2 following the same theorem retains the marked evaluations, inserting stable constant components when a mark approaches a node in a rescaled coordinate. Nonconstant components can attach only in Σ_∞, since convergence is smooth elsewhere. Nor can a tree of constant components attach to the root. If it had $g \ge 1$ vertices and m_z marks, its $g-1$ internal edges and one edge to the root would contribute $2(g-1)+1$ nodal special points. Stability would therefore give

$$3g \le 2(g-1) + 1 + m_z, \qquad m_z \ge g+1 \ge 2.$$

All these marks would have the same original-coordinate limit, contrary to the fixed distinct marks. Thus all attached trees lie over Σ_∞ and contain a nonconstant sphere. The shrinking-region construction gives the reparametrizations and their collapse asserted in (ii), including for internal ghosts. The compactness theorem gives finiteness of the marked tree.

By Definition 1.6 of the cited preprint, all components belong to one continuous nodal limit. Its two branches therefore have equal values at every node. Marked convergence gives the second identity in (iii); repeatedly applying the node identity propagates a marked value through constant components. No derivative at a marked point is asserted to survive.

Joint global area accounting.

Choose a compact subset $Q \subset \mathbb{D}_{R_\infty} \setminus \Sigma_\infty$ and compact subsets $Q_\alpha \subset \mathbb{P}^1 \setminus N_\alpha$ on the sphere components. The fixed extraction represents the Q_α in mutually disjoint shrinking source regions at separated scales. For all large j these regions also avoid Q, even when Q extends outside the retained disk $\mathbb{D}_\rho$, because Q stays a positive distance from Σ_∞. Taking the sequence limit first gives

$$\int_Q f_\infty^*\omega + \sum_\alpha \int_{Q_\alpha} u_\alpha^*\omega = \lim_j \left(\int_Q f_j^*\omega + \sum_\alpha \int_{\varphi_{j,\alpha}(Q_\alpha)} f_j^*\omega \right) \le A_\infty.$$

Now exhaust the global principal domain minus Σ_∞ and each sphere minus its nodes. The omitted points have zero area, and all omitted source areas are nonnegative. This proves (70). Constant components contribute zero; multiplicities are retained because we integrate pullbacks by the component maps. □

Taking no marks gives exactly the principal compactness statement used in the continuation argument.

Proof of Proposition 2.3. Apply Proposition 5.1 with $m = 0$ and omit the nonnegative sphere areas in (70). Its principal map and convergence have the required properties. □

5.3. **Two marked values and incidence**

Fixing the complete first jet at 0 does not ensure that a nonconstant bubble retains the prescribed tangent direction. In the following example, the marked value lies on the bubble, while that direction remains on the principal component. On the unit disk, with the product Fubini–Study metric, consider

$$\widehat{f}_j(z) \coloneqq \big([jz^2 : 1], [z : 1]\big) : \mathbb{D} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

The areas are uniformly bounded, and in the affine coordinates $([w : 1], [v : 1])$ the complete first jet is fixed:

$$\widehat{f}_j(0) = ([0 : 1], [0 : 1]), \qquad (\widehat{f}_j)_z(0) = (0, 1).$$

Away from zero, the principal limit is

$$\widehat{f}_\infty(z) = ([1 : 0], [z : 1]).$$

Its extension has value $([1 : 0], [0 : 1])$ at zero, different from the marked value. Rescaling by $z = \zeta/\sqrt{j}$ instead gives

$$\widehat{f}_j(\zeta/\sqrt{j}) = \big([\zeta^2 : 1], [\zeta/\sqrt{j} : 1]\big) \longrightarrow \big([\zeta^2 : 1], [0 : 1]\big).$$

The prescribed second-factor direction remains on the residual principal component. Correspondingly, $B_1(\widehat{f}_j)/|(\widehat{f}_j)_z(0)|_\omega \to \infty$.

Marked and nodal values retain incidence. Lemma 2.5 gives a chain joining two compact target sets with controlled total area. It supplies the compactness alternative in Section 2 and the final passage to the sharp area bound.

Proof of Lemma 2.5. Pass first to a subsequence realizing the original lower limit $A_\infty \coloneqq \liminf_j \mathrm{Area}_{R_j}(f_j)$. Apply Proposition 5.1 with $R_\infty = \infty$ and the fixed marks $0, 1$. It supplies an entire principal map f_∞ and one finite marked tree, with the joint area bound (70). Theorem 2.4 fills infinity, producing the principal sphere. Replace the principal root of the marked limit by this extension and retain its attached sphere trees. Each mark lies either on the principal sphere or on one of these trees. Its marked value lies in K_0 or K_1 because these sets are closed. On the resulting finite tree, take the path between the marked components and omit constant components. Nodal equality gives nonconstant sphere maps $u_1, \ldots, u_\ell$ with

$$u_1(\mathbb{P}^1) \cap K_0 \neq \varnothing, \qquad u_\ell(\mathbb{P}^1) \cap K_1 \neq \varnothing, \qquad u_i(\mathbb{P}^1) \cap u_{i+1}(\mathbb{P}^1) \neq \varnothing \quad (1 \leq i < \ell).$$

There is at least one such component since $K_0 \cap K_1 = \varnothing$. The components on the path are selected from the global principal sphere and the sphere components of the fixed rooted limit. The joint area estimate (70), after omitting unused components of nonnegative area, gives

$$\sum_{i=1}^{\ell} \int_{\mathbb{P}^1} u_i^* \omega \leq A_\infty.$$

The path belongs to the single marked limit supplied by the proposition. Components arising near the moving outer boundary are not needed for its incidence or area estimate. ☐

5.4. **Rescaling and finite-area weighted derivatives**

Brody reparametrization [6, Lemma 2.1], in the affine form described by Duval [20, Section 1], turns divergence of B_R into a nonconstant entire limit. For a single finite-area disk, the positive rescaled area cannot persist in shrinking boundary collars; this gives Lemma 2.7.

Lemma 5.2. *Let $f_j : U_j \to X$ be holomorphic maps, where $U_j \subset \mathbb{C}$ is open and $\overline{\mathbb{D}}_{R_j} \Subset U_j$, and suppose $B_{R_j}(f_j) \to \infty$. Then after affine source rescaling, a subsequence converges locally smoothly on $\mathbb{C}$ to a nonconstant entire map $u : \mathbb{C} \to X$. If in addition $\sup_j \mathrm{Area}_{R_j}(f_j) < \infty$, then*

$$\mathrm{Area}(u) \leq \liminf_j \mathrm{Area}_{R_j}(f_j).$$

Proof. Under the additional area hypothesis, first pass to a subsequence for which $\mathrm{Area}_{R_j}(f_j)$ converges to the lower limit of the original area sequence. In either case, choose w_j maximizing $(R_j-|z|)|(f_j)_z(z)|$, and set $d_j := R_j - |w_j|$, $\Lambda_j := |(f_j)_z(w_j)|$. Then

$$d_j\Lambda_j = B_{R_j}(f_j) \longrightarrow \infty, \qquad |(f_j)_z(z)| \le 2\Lambda_j \quad (|z-w_j| < d_j/2).$$

Indeed, $|z-w_j| < d_j/2$ implies $R_j - |z| > d_j/2$. Maximality then gives $(d_j/2)|(f_j)_z(z)| \le d_j\Lambda_j$, which is the second bound. The rescaled map

$$u_j(\zeta) := f_j(w_j + \zeta/\Lambda_j), \qquad |\zeta| < \frac{1}{2}d_j\Lambda_j,$$

satisfies $|(u_j)_\zeta|_\omega \le 2$ and $|(u_j)_\zeta(0)|_\omega = 1$. The source radii $d_j\Lambda_j/2$ tend to infinity. On every fixed compact source disk, the derivative bound gives equicontinuity, and compactness of X gives a uniformly convergent subsequence. A diagonal selection and Cauchy estimates in local target charts give a smooth limit $u : \mathbb{C} \to X$ with $|u_\zeta(0)|_\omega = 1$; hence u is nonconstant. Under the additional area hypothesis, conformal invariance gives, for each fixed $S > 0$,

$$\mathrm{Area}_S(u) = \lim_j \mathrm{Area}_S(u_j) \le \lim_j \mathrm{Area}_{R_j}(f_j) = \liminf_j \mathrm{Area}_{R_j}(f_j).$$

Letting $S \to \infty$ proves the area bound. □

Proof of Lemma 2.7. Suppose $B_R(h) = \infty$. As $s \uparrow R$, the quantities $B_s(h)$ increase to $B_R(h)$: for each fixed $z \in \mathbb{D}_R$, $(s-|z|)|h_z(z)|_\omega$ tends to $(R-|z|)|h_z(z)|_\omega$ once $s > |z|$, and the suprema increase. Choose $s_j \uparrow R$ with $B_{s_j}(h) \to \infty$, and apply the point-selection construction of Lemma 5.2 to $h|_{\mathbb{D}_{s_j}}$. These restrictions extend holomorphically past their closed disks. The construction gives centers $w_j \in \mathbb{D}_{s_j}$ and numbers $\Lambda_j = |h_z(w_j)|_\omega$ such that

$$(s_j - |w_j|)\Lambda_j \longrightarrow \infty, \qquad u_j(\zeta) := h(w_j + \zeta/\Lambda_j) \longrightarrow u(\zeta)$$

smoothly on compact subsets of $\mathbb{C}$, where $|u_\zeta(0)|_\omega = 1$. Because $s_j \le R$, we have $\Lambda_j \to \infty$. The derivative of the fixed map h is bounded on each compact subdisk, so $|w_j| \to R$.

Choose $S > 0$ with $\mathrm{Area}_S(u) > 0$. For all large j, the disk $\mathbb{D}_{S/\Lambda_j}(w_j)$ lies in $\mathbb{D}_{s_j}$, and conformal invariance and smooth convergence give

$$\int_{\mathbb{D}_{S/\Lambda_j}(w_j)} h^*\omega = \mathrm{Area}_S(u_j) \longrightarrow \mathrm{Area}_S(u) > 0.$$

On the other hand, $|w_j| \to R$ and $S/\Lambda_j \to 0$ place these disks in arbitrarily thin boundary collars of $\mathbb{D}_R$. Since $\mathrm{Area}_R(h) < \infty$, $\int_{\mathbb{D}_R \setminus \overline{\mathbb{D}}_s} h^*\omega \to 0$ as $s \uparrow R$. This contradicts the preceding positive limit. □

6. A finite catalogue and a finite-loop alternative

We return to the compact Kähler manifold with $\mathrm{Ric}(\omega) \ge \kappa\omega$, $\kappa > 0$, and the fiber data fixed in Section 2. Here we keep the source radius fixed and increase the scalar derivative. Recovery replaces a high-B disk by a member of a finite catalogue, preserving its scalar derivative while allowing its center to move in F. Compactness first yields the required sphere or, by a finite-cover selection, such a catalogue. In the latter case, the loop reaches each prescribed scalar target in finitely many operations. This gives an alternative to the successive limits of recovered maps in Section 2.3.

Fix a source radius $R_0 > 0$, an area cap C, and a real scalar window $0 < k_- \le k \le k_+ < \infty$. The disks in this section satisfy

$$f(0) \in F, \qquad \lambda^F_{f(0)}(f_z(0)) = k. \tag{71}$$

The real positive window suffices for the iteration below.

Proposition 6.1. *One can select either a sphere meeting and escaping F of area at most C, or finitely many maps G_i holomorphic near $\overline{\mathbb{D}}_{L_i}$ for some $L_i > R_0$, with centers in F and scalar derivatives $k_i > 0$, together with relatively open parameter neighborhoods $W_i \subset [k_-, k_+] \times [0, C]$ and a cutoff $M < \infty$, with the following property. Every map f holomorphic near $\overline{\mathbb{D}}_{R_0}$, satisfying* (71), $\mathrm{Area}_{R_0}(f) \le C$, *and* $B_{R_0}(f) > M$, *has* $(k, \mathrm{Area}_{R_0}(f)) \in W_i$ *for at least one* i. *Select the first such index and define*

$$g(z) = G_i((k/k_i)z), \qquad z \in \mathbb{D}_{R_0}. \tag{72}$$

This map extends across a neighborhood of the closed source disk, satisfies (71) *with the same* k, *and obeys*

$$\mathrm{Area}_{R_0}(g) \le \mathrm{Area}_{R_0}(f) - e_X/2, \qquad R_0 k/\ell_F \le B_{R_0}(g) \le M. \tag{73}$$

The source extension neighborhood can be chosen uniformly for the members of each catalogue family in (72).

Proof. Let $Z = [k_-, k_+] \times [0, C]$ be the rectangle of scalar and area parameters. For each positive integer m, let K_m be the closure in Z of the pairs $(k, \mathrm{Area}_{R_0}(f))$ arising from admissible disks with $B_{R_0}(f) > m$. These compact sets decrease with m. Their intersection $P = \bigcap_{m \ge 1} K_m$ records the limiting parameter pairs as B tends to infinity.

For $p = (k_p, E_p) \in P$, choose such disks f_j with $B_{R_0}(f_j) > j$ and parameter pairs tending to p. Apply relative compactness and the area-loss argument of Lemma 2.8. These apply to the varying scalar: the maps $z \mapsto f_j(z/k_j)$ have scalar 1, radius $k_j R_0$ tending to $k_p R_0 > 0$, and unchanged area and B. Rescaling the principal map back by $z \mapsto k_p z$ gives an open-disk map $h_p : \mathbb{D}_{R_0} \to X$ with scalar k_p, center in F, and

$$\mathrm{Area}_{R_0}(h_p) \le E_p - e_X.$$

If the alternative supplies an escaping sphere at any such choice, retain it as the first result of this proposition.

If a principal map is returned, apply the affine enlargement of Lemma 2.2. After source dilation, the enlargement preserves the full derivative, so the same construction also applies when $k_p \ne 1$. Choose a positive parameter small enough that the area cost is less than $e_X/8$. Restrict at a radius $L_p > R_0$ strictly inside the enlarged open disk. This gives a map G_p holomorphic near $\overline{\mathbb{D}}_{L_p}$, with scalar k_p and

$$\mathrm{Area}_{L_p}(G_p) \le E_p - 7e_X/8.$$

Choose $R_0 < L'_p < L_p$. In a sufficiently small relative open neighborhood W_p of p in Z, every (k', a') satisfies

$$(k'/k_p) R_0 < L'_p, \qquad |a' - E_p| < e_X/8.$$

Shrink W_p further so that k'/k_p has a uniform upper bound strictly smaller than L'_p/R_0. The maps $G_p((k'/k_p)z)$ are then defined on one common neighborhood of $\overline{\mathbb{D}}_{R_0}$. Their areas are at most $E_p - 7e_X/8 \le a' - 3e_X/4$, and their weighted derivatives satisfy

$$B_{R_0}(G_p((k'/k_p)\cdot)) = B_{(k'/k_p)R_0}(G_p) \le B_{L_p}(G_p) < \infty.$$

Their centers lie in F and their scalars are exactly k'.

Compactness of P selects finitely many of these neighborhoods, say $W_1, \dots, W_s$, covering P. If P is empty, take this collection to be empty. The sets

$$W_1, \dots, W_s, \quad Z \setminus K_1, Z \setminus K_2, \dots$$

form an open cover of Z. A finite subcover and the decreasing property of K_m give an integer m_0 with $K_{m_0} \subset \bigcup_{i=1}^{s} W_i$; if no complementary set was needed, take $m_0 = 1$. Choose

$$M \ge \max\{1, m_0, B_{L_1}(G_1), \dots, B_{L_s}(G_s)\}.$$

For every disk with $B > M$, its parameter pair belongs to K_{m_0}, so the first applicable catalogue map has all the stated properties, including the weaker decrease $e_X/2$ in (73). In the empty-catalogue case there are no such disks, and the assertion about replacement is vacuous. $\square$

The catalogue is chosen once for the fixed parameter rectangle. A later disk is replaced by rescaling the first applicable member; its center may change within F.

Proposition 6.2. *Fix $\delta > 0$ and a normalized closed coordinate seed on $\mathbb{D}_{R_0}$ with scalar 1 and area $A_s < A_*$. For every $K > 1$, the catalogue alternative followed by the following finite loop gives either an escaping sphere of area at most $C = A_* + 2\delta$, or a closed disk on $\mathbb{D}_{R_0}$ with scalar at least K and area less than C.*

Proof. Prepare Proposition 6.1 for the scalar window $[1, 2K]$ and cap C. Retain a sphere if it is supplied. Otherwise enlarge M to include the seed's B; the same catalogue remains valid for this larger cutoff. Use this one B cap for both surgeries. Surgery I preserves every scalar k, not just 1: the vanishing and kernel conditions on its multiplier are unchanged. Its decrease $\sigma > 0$ is therefore uniform throughout the window.

In Surgery II retain $h_t(z) = \Pi(H(z) + tzH'(0))$ on the *same* radius R_0, without the source dilation. Its scalar becomes $(1+t)k$. Its area and B^2 changes obey the same estimates as in Section 4.4. Fix

$$\tau = \min\{\bar{t}/2, \eta_0/\max\{1, C_{II}\}\} > 0, \qquad L = A_* + \delta, \quad \eta_0 = \delta/2.$$

In particular $\tau \leq 1$. At the start of each loop $B \leq M$. If $k \geq K$, stop. Otherwise do Surgery I when $A \geq L$, and the same-radius Surgery II with $t = \tau$ when $A < L$. If the resulting B exceeds M, immediately apply (72) once. This restores $B \leq M$, lowers area by at least $e_X/2$, and keeps the scalar just obtained.

Area stays below C at all times. While enlargement is needed the old scalar is less than K, so the new scalar is less than $2K$. All catalogue and surgery calls are therefore within their declared caps. After

$$N = \left\lceil \frac{\log K}{\log(1+\tau)} \right\rceil$$

enlargements, the scalar has reached K. After any completed part of the loop, let N_I, N_{II}, and N_{III} count Surgery I steps, Surgery II steps, and catalogue replacements, respectively. The area estimates and the stopping rule give

$$N_I\sigma + N_{III}e_X/2 \leq A_s + N_{II}\eta_0, \qquad N_{II} \leq N. \tag{74}$$

Thus one can run a bounded loop with at most

$$N + \left\lfloor \frac{A_s + N\eta_0}{\min\{\sigma, e_X/2\}} \right\rfloor$$

individual surgery and catalogue operations. The stated scalar test has terminated it by that bound. All tests concern the actual current map, and a possible crossing of the B cap is repaired before the next surgery using that cap. $\square$

To turn these scalar targets into the relative sphere, choose a positive upper bound H for $|\eta \circ \psi|$ on V. For a disk f with scalar k, if $f(\mathbb{D}_r) \subset V$ for some $0 < r < R_0$, the Cauchy estimate at the source center gives $k \leq H/r$. Hence when $2H/k < R_0$ there is a first-exit point z with

$$0 < |z| \leq 2H/k, \qquad f(z) \in \partial V.$$

Indeed a disk of radius $2H/k$ whose entire image lies in V would have derivative at most $k/2$; restriction to the first exit along a radial segment supplies the stated point. Applying Proposition 6.2 with $K \to \infty$ therefore gives, unless a sphere has already been returned, maps $\zeta \mapsto f(z\zeta)$ on radii tending to infinity, with source marks $0, 1$ mapped into the disjoint compact sets $F, \partial V$. Their areas stay at most C. Select a subsequence whose areas converge. The two-marked chain theorem yields the required escaping sphere, and the last paragraph of the proof of Theorem 1.1 gives the sharp bound.

The retained transverse information is the base scalar and the two separated marked values. A positive amount of total area inside V alone could belong to a component contained in F. In the principal branch it is convergence of the bounded base compositions, and thus of their nonzero derivatives, that retains transverse information. The catalogue loop supplies this normalization explicitly; it does not rely on a localization conclusion from unmarked total area alone.

7. Applications

The relative theorem recovers the rational connectedness of Fano manifolds [7, 43]: two general points can be joined by a rational curve. Indeed, choose a positive-Ricci metric as in Proposition 3.1. For any quasifibration $q : X \dashrightarrow Y$ with $\dim Y > 0$, restrict to dense open subsets on which q is proper with connected fibers. Generic smoothness [34, III, Corollary 10.7] makes its general full fiber F smooth and compact. Theorem 1.1 gives a rational curve meeting F but not contained in it. Campana's criterion [7, Proposition 2.7] then gives rational chain connectedness, and smoothing chains in characteristic zero gives rational connectedness [2].

The pointwise construction also recovers Mori's characterization of projective space by an ample tangent bundle [47]; for an earlier characteristic-zero proof, see [51]. Let X be a connected smooth complex projective n-fold with T_X ample. The case $n = 1$ is immediate. For $n \geq 2$, the determinant $K_X^{-1} = \det T_X$ is ample [32], so X is Fano and Theorem 1.2 supplies a rational curve. The classical two-point degree reduction gives one with $-K_X \cdot C \leq n + 1$ [15, Section 5, Theorem 3, first step]. Ampleness then forces equality and the minimal-curve conditions of [15, Section 5, Corollary (a),(c)]. Mori's classification argument identifies X with $\mathbb{P}^n$ [15, Section 6]. These remaining algebraic steps take place over $\mathbb{C}$.

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