

ANALYTIC CONSTRUCTION OF RATIONAL CURVES ON FANO MANIFOLDS

YUN-HENG DU, BIN GUO, AND SONG-YAN XIE

ABSTRACT. Inspired by constructions of entire curves in Oka geometry, we construct rational curves on complex Fano manifolds by holomorphic disk surgeries. Positive Ricci curvature guides an area-decreasing deformation, and an affine-lift perturbation enlarges the source. Continuation under a uniform area bound, together with marked compactness, gives a sphere meeting a prescribed compact fiber without being contained in it, with a sharp metric area bound. The construction uses analytic compactness to pass from the disks to a sphere.

CONTENTS

1. Introduction and main results	1
2. Proof of the relative sphere theorem	6
3. Ricci traces and scalar-constrained variations	13
4. Uniform holomorphic realization and disk operations	16
5. Compactness, marked values, and finite-area disks	28
6. Applications	33
Acknowledgments	33
AI use disclosure	33
References	33

1. INTRODUCTION AND MAIN RESULTS

1.1. The analytic construction problem

We construct rational curves on complex Fano manifolds by analytic methods. A Fano manifold X is connected, smooth, and compact, with ample anticanonical line bundle K_X^{-1} .

Rational curves are central to Mori's solution [47] of Hartshorne's conjecture on ample tangent bundles. In his monograph *Ample subvarieties of algebraic varieties*, Hartshorne [33] asked whether a smooth complex projective variety with ample tangent bundle must be projective space. Mori's construction combines deformation theory and bend-and-break, which also apply in characteristic zero, with Frobenius in positive characteristic and specialization back to $\mathbb{C}$.

On the analytic side, Siu and Yau [55] used energy-minimizing harmonic spheres in their 1980 proof of the Frankel conjecture. Their argument uses positive holomorphic bisectional curvature, a stronger condition than the Ricci positivity that Yau's prescribed-Ricci theorem provides on every Fano manifold [61]. In 1982, Yau asked whether any two points of a compact Kähler manifold with positive Ricci curvature could be joined by a chain of rational curves [62, p. 44]. Rational connectedness of Fano manifolds was subsequently established by Campana and by Kollár, Miyaoka, and Mori [7, 43]. An analytic construction remained a separate challenge, which Siu later proposed to approach through complex Monge–Ampère equations, without reduction to positive characteristic [54].

2020 *Mathematics Subject Classification*. Primary 14J45, 32Q15; Secondary 14M22, 32Q10, 32Q45.

Key words and phrases. Fano manifold, rational curve, rational connectedness, holomorphic disk, Ricci curvature.

Our approach to this problem draws on methods for constructing entire curves in Oka geometry, in particular those of Xie [58] (see also [10, 35, 57]). These methods build entire curves by successively modifying holomorphic disks and enlarging their domains. Guo and Xie [31, Section 5] suggested pursuing such constructions on Fano manifolds with a slow-growth condition, with the aim of producing both entire and rational curves analytically. The construction developed here gives a concrete realization of this vision.

A nonconstant entire curve of finite area gives a rational curve by removal of the singularity at infinity (Theorem 2.4). We therefore try to construct such an entire curve as a limit of holomorphic disks with radii tending to infinity. Brody’s lemma [6] suggests the basic strategy: normalize the derivative at the origin and obtain convergence from derivative bounds on compact subsets. Here we must also keep the total area uniformly bounded. Positive Ricci curvature provides the area-decreasing deformations used for this purpose. Derivative concentration may prevent direct convergence, so we also analyze the spheres arising through bubbling and retain their required incidence properties.

To carry out these deformations, we need room to vary holomorphic disks in the target. In the Fano setting, the following folklore conjecture in Oka geometry gives reason to expect such flexibility; see [25, p. 660].

Conjecture A. *Every Fano manifold is Oka.*

Our recent theorem that rationally connected smooth projective manifolds are Oka-1 [17, Corollary 1.8] gives a first step toward the conjecture, since it includes every Fano manifold. This result encourages us to seek considerable freedom in deforming holomorphic disks. Its rational-connectedness hypothesis means that it provides motivation here, rather than an input to the rational-curve construction or its applications. Known cases of the Oka conjecture include smooth quadric [28], cubic [39], and quartic [18] Fano hypersurfaces; see also [59]. For background on Oka and Oka-1 theory, we refer to [24, 1].

This suggests an analogy with making a baozi, a Chinese steamed bun. The dough is repeatedly stretched and gathered around its filling, just as we deform holomorphic maps on successively larger disks. Sealing the bun corresponds to compactifying the resulting finite-area entire curve at infinity.

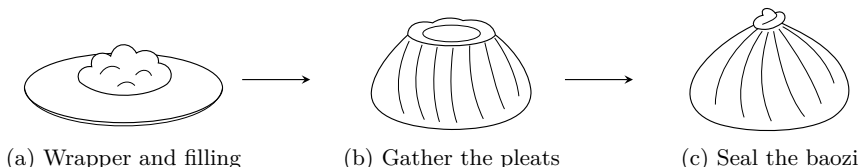


FIGURE 1. Shaping a baozi by gathering a dough wrapper around a filling and sealing the pleats. This traditional Chinese craft suggests an analogy for closure. In the proof, a finite-area entire principal map extends across infinity; bubbling is treated separately by marked compactness.

To carry out this construction, two difficulties must be resolved. The first is area control: holomorphic approximation alone supplies no uniform bound as the source radii tend to infinity.

The second difficulty is to retain location in the limit. Classical Brody reparametrization moves the source centers [6], so even disks through a fixed point need not yield an entire curve through that point. Duval addressed this loss of location by replacing derivative selection with control of where area accumulates [19]; see also [20]. For a compact set charged by an Ahlfors current, his theorem produces an entire curve whose intersection with that set has positive area. Duval’s localization viewpoint also guides our construction. In our finite-area setting, we retain position through a base constraint and two marked values, rather than invoke the Ahlfors-current theorem itself.

Even when the source center is fixed, bubbling can separate its prescribed local data from the nonconstant sphere components. The analysis of bubbling goes back to Sacks and Uhlenbeck's work on harmonic maps [52] and Gromov's theory of pseudoholomorphic curves [29]. Area may also concentrate near the source boundary. We must therefore obtain a nonconstant sphere satisfying both the required incidence conditions and the area bound.

We use Ricci positivity to deform disks so that their area decreases above an explicit threshold. The key is a Ricci-trace estimate, realized through approximation and interpolation by holomorphic families that preserve a scalar derivative condition and satisfy uniform estimates. To control where the resulting curve lies, we work relative to a compact fiber F of a local holomorphic projection q . We allow the center of a disk f to move within F , while fixing one nonzero scalar component of the derivative of $q \circ f$. This keeps the initial tangent vector outside the tangent space of F without fixing the entire vector. Together with marked compactness, this normalization produces a sphere that meets F but is not contained in it, with a sharp area bound. Taking F to be a single point gives a rational curve through that point. Area descent and enlargement continue normalized disks under a strict area bound, while marked compactness retains the required incidence when bubbling occurs. The curves are constructed analytically over $\mathbb{C}$.

1.2. Main results

For rational connectedness, we need a rational curve that meets a fiber of the rational quotient without remaining inside it. The main theorem therefore treats a compact fiber of a holomorphic map q defined on an open neighborhood N . Here $F = q^{-1}(y_0)$ denotes the entire fiber over y_0 in N , including all its components. We use the Ricci and area conventions in (1).

Theorem 1.1. *Let (X, ω) be a compact Kähler n -fold satisfying*

$$\text{Ric}(\omega) \geq \kappa \omega, \quad \kappa > 0.$$

For an open subset $N \subset X$ and a complex manifold Y of positive dimension, consider a holomorphic map $q : N \rightarrow Y$ and a point $y_0 \in Y$. Suppose that the full fiber $F := q^{-1}(y_0)$ is nonempty and compact and that q is a submersion on a neighborhood of F in N . Then there is a nonconstant holomorphic map $u : \mathbb{P}^1 \rightarrow X$ such that

$$u(\mathbb{P}^1) \cap F \neq \emptyset, \quad u(\mathbb{P}^1) \not\subset F, \quad \int_{\mathbb{P}^1} u^* \omega \leq \frac{2\pi(n+1)}{\kappa}.$$

Every Fano manifold admits a metric with this Ricci lower bound, by Yau's prescribed-Ricci theorem [61]; a Kähler–Einstein metric is not required. See Proposition 3.1. Conversely, the lower bound gives positive curvature on K_X^{-1} , hence ampleness and projectivity; see Remark 3.2. Thus the reduced image of the sphere in the theorem is a rational curve.

The projective-space example in Remark 2.8 shows that the coefficient $2\pi(n+1)$ cannot be improved in a universal metric area bound.

The relative viewpoint has a classical antecedent in Campana's generic-fiber escape criterion and his Fano theorem [7, Proposition 2.7 and Theorem 3.1]. Here we work near a specified compact full fiber of a holomorphic submersion and obtain a metric area bound.

During the construction, the center is allowed to move within F . If compactness has not already produced a sphere meeting and leaving F , this freedom allows the normalized base derivative to pass to the principal map. The final limit used to obtain the sharp area bound retains the two incidence properties, but need not retain the derivative normalization. Thus the theorem prescribes neither a tangent direction nor a contact order.

Taking q to be a holomorphic coordinate map near x gives the fiber $F = \{x\}$, so the relative theorem immediately yields the following pointwise form.

Theorem 1.2. *Let (X, ω) be a positive-dimensional compact Kähler manifold of complex dimension n , and suppose $\text{Ric}(\omega) \geq \kappa \omega$ for some $\kappa > 0$. Then, for every $x \in X$, there is a nonconstant holomorphic map $u_x : \mathbb{P}^1 \rightarrow X$ such that*

$$x \in u_x(\mathbb{P}^1), \quad \int_{\mathbb{P}^1} u_x^* \omega \leq \frac{2\pi(n+1)}{\kappa}.$$

Classical bend-and-break produces a rational curve C through a prescribed point of a Fano n -fold with

$$0 < -K_X \cdot C \leq n + 1.$$

Mori's account [48, p. 749] attributes this form of the result to Kollár, by a technique similar to that of [47]. Indeed, integration over C gives

$$\kappa \int_C \omega \leq \int_C \text{Ric}(\omega) = 2\pi(-K_X \cdot C).$$

The classical result controls the anticanonical degree, giving more information than the area bound for a fixed metric. Our estimate has the same order of growth in n . Our contribution is the analytic construction: uniform area descent and finite-area enlargement permit continuation under a fixed area cap. For the classical argument, see [13, Chapter 3].

1.3. From expanding disks to rational curves

The construction uses two disk surgeries. The first decreases area while preserving a scalar derivative normalization. The second uses the resulting margin below an area cap to enlarge the source and restore the normalization.

For $R > 0$, set $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$. For a holomorphic map $f : \mathbb{D}_R \rightarrow X$, write $f_z := df(\partial/\partial z)$ and

$$\text{Area}_R(f) := \int_{\mathbb{D}_R} f^* \omega, \quad B_R(f) := \sup_{z \in \mathbb{D}_R} (R - |z|) |f_z(z)|_\omega.$$

The quantity B_R measures derivative concentration on the scale of the distance to the source boundary; it is the scale used in Brody's reparametrization argument [6].

For area descent, we first consider the variational problem with $f(0) = x$ and $\lambda(f_z(0)) = 1$ for a nonzero $\lambda \in T_x^* X$. The relative continuation uses the same pointwise calculation, but allows x to vary in F and takes λ to be the pullback of a fixed covector on the base. As in Brody's normalization, the nonzero scalar derivative prevents a smooth limit near the center from being constant; its transfer through bubbling is a separate compactness issue. In an adapted holomorphic frame with unitary boundary values, preserving the center requires every multiplier component to vanish at zero. Preserving one scalar derivative requires one component to vanish to second order. Thus one quadratic and $n - 1$ linear components satisfy the constraints at total Dirichlet cost $\pi(n + 1)$. The corresponding paired second-variation trace is

$$4\pi(n + 1) - 2 \int_{\mathbb{D}_R} f^* \text{Ric}(\omega) \leq 4\pi(n + 1) - 2\kappa \text{Area}_R(f).$$

Thus the threshold

$$A_* := \frac{2\pi(n + 1)}{\kappa}$$

comes from the cost of scalar interpolation. The multiplier attains the sharp cost for its boundary and interpolation conditions (see the calculation after (18)); taking the trace makes positivity of the Ricci form sufficient. A negative infinitesimal trace alone does not provide the uniform area decrease needed for continuation. Theorem 4.8 realizes the calculation by actual holomorphic families whose paired second derivatives approximate the trace, with uniform parameter radius and cubic area remainder under fixed bounds on Area_R and B_R . A normalized projective lift, a corona theorem, and a holomorphic retraction provide this uniformity. This realization uses projectivity but does not require the Ricci inequality.

Fix an area cap $A > A_*$. For continuation, the center may move in the compact fiber F , and one derivative of the local projection to the base is normalized. Compactness gives a direct alternative: first-exit rescaling produces a sphere of area at most A meeting but not contained in F , or the normalization passes to the principal limit in the original source coordinate. At a finite limiting radius, area descent on this principal map leaves a margin below A . A first-jet perturbation of its affine lift then increases the scalar derivative, and source dilation restores its value to 1 on a larger disk. Continuity of total area keeps this second surgery below the cap. Thus a finite endpoint either yields the sphere or permits continuation. The precise relation between the two surgeries is explained after Lemma 2.2; successive disks need not extend one another.

On expanding sources, the same compactness alternative either yields the required sphere or gives a finite-area entire principal map. In the latter case, the retained base derivative makes the map nonconstant, and removal at infinity gives a sphere. The sharp bound requires a final limit $A \downarrow A_*$. At this stage we retain incidence rather than a derivative on the selected sphere. Two marked target values retain a rational chain from F to the boundary of a fixed neighborhood, with total area at most A_* . Its first component leaving F meets F and has area at most A_* . The compactness and removal theorems are supplied by Ivashkovich and Shevchishin [37]; their fixed-coordinate and marked consequences are derived in Section 5.

Figure 2 summarizes the two limiting stages for a point fiber.

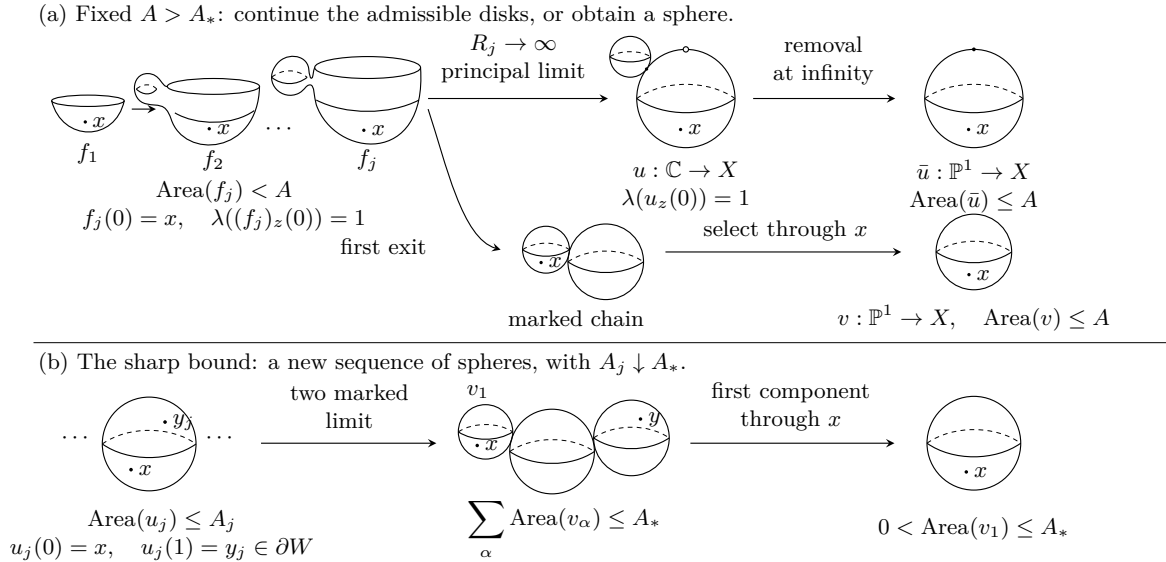


FIGURE 2. The two limiting stages, pictured for $F = \{x\}$. (a) Under a fixed cap, the compactness alternative supplies a marked chain and a sphere through x , or a normalized principal map. A finite normalized endpoint permits further continuation by the disk surgeries; on expanding sources, finite-area removal closes the entire principal map at infinity. The bulge and neck illustrate possible concentration; a sphere at the node represents a bubble obtained after reparametrization. (b) As the area caps A_j decrease to A_* , two marked values retain a chain from x to $y \in \partial W$, where W is a relatively compact neighborhood of x with smooth boundary inside a coordinate chart. Its first component has area at most A_* ; it need not contain y or retain the derivative normalization. All contours are schematic.

The controlled holomorphic deformations developed here also suggest an approach to Conjecture A. We hope that further developments will lead from deformations of disks to dominating holomorphic sprays on Fano manifolds.

Plan of the paper.

Section 2 proves the relative sphere theorem and records the metric sharpness example. The Ricci-trace calculation, uniform holomorphic realization, and marked compactness are established in Sections 3–5. The final section briefly recalls how the construction combines with classical algebraic arguments to recover rational connectedness and Mori’s ample-tangent characterization of projective space.

2. PROOF OF THE RELATIVE SPHERE THEOREM

We prove Theorem 1.1 by continuing normalized holomorphic disks under a strict area bound. At each limiting stage, compactness either produces a sphere meeting but not contained in the fiber, or retains the scalar base derivative on a principal map to which the two surgeries apply. We state the disk operations and compactness results here and give their proofs in Sections 4 and 5.

2.1. Normalized disks and the two disk operations

We fix the metric normalization and recall the area and weighted derivative quantities from the Introduction.

Unless otherwise specified, (X, ω) is a compact Kähler manifold of complex dimension $n \geq 1$. In holomorphic coordinates $w^1, \dots, w^n$, let $h = (h_{\alpha\bar{\beta}})$ be the Gram matrix of the coordinate frame. Our Hermitian inner products are complex-linear in the first variable and conjugate-linear in the second [14, Ch. V, §12]. Thus $h_{\alpha\bar{\beta}} = \langle \partial/\partial w^\alpha, \partial/\partial w^\beta \rangle_\omega$, and coefficient columns a, b pair as $a^\top h \bar{b}$, where $\top$ denotes ordinary transpose. For a holomorphic source map f , write $f_z = df(\partial/\partial z)$. With repeated coordinate indices summed and dA denoting Euclidean area measure, our normalizations are

$$\omega = \frac{i}{2} h_{\alpha\bar{\beta}} dw^\alpha \wedge d\bar{w}^\beta, \quad f^*\omega = |f_z|_h^2 dA, \quad \text{Ric}(\omega) = -i\partial\bar{\partial} \log \det h. \quad (1)$$

For a holomorphic map $f : S \rightarrow X$ from a Riemann surface, set $\text{Area}(f) := \int_S f^*\omega$; this counts geometric area with the multiplicity of the map. A holomorphic sphere is a map $u : \mathbb{P}^1 \rightarrow X$. When u is nonconstant, we call its reduced image a rational curve, while $\text{Area}(u)$ continues to include the mapping multiplicity. For $a \in \mathbb{C}$ and $r > 0$, write $\mathbb{D}_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$, and abbreviate $\mathbb{D}_R := \mathbb{D}_R(0)$ and $\mathbb{D} := \mathbb{D}_1(0)$; bars denote closure, and $\partial\mathbb{D}_R$ is the boundary circle. We use the area on $\mathbb{D}_R$ and the boundary-distance-weighted derivative bound

$$\text{Area}_R(f) := \int_{\mathbb{D}_R} f^*\omega, \quad B_R(f) := \sup_{z \in \mathbb{D}_R} (R - |z|) |f_z(z)|_\omega. \quad (2)$$

Thus $B_R(f)$ weights the derivative by its distance to the source boundary. Write $\mathcal{O}(V, X)$ for the holomorphic maps from an open set $V \subset \mathbb{C}$ to X , and $K \Subset U$ when the closure of K is compact and contained in U . Smooth convergence means uniform convergence of all derivatives on compact source subsets.

Under the Ricci bound $\text{Ric}(\omega) \geq \kappa\omega$, $\kappa > 0$, set

$$A_* = \frac{2\pi(n+1)}{\kappa}. \quad (3)$$

Positive Ricci curvature makes the anticanonical bundle positive, so Kodaira’s theorem gives a projective embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$; see Remark 3.2. We fix one such embedding to carry out the two disk operations. It remains fixed as the map and source radius vary, and the constants may depend on it. All areas and target derivative norms are still measured using ω .

For the relative constraint, we work in the proper setting obtained by the local reduction at the start of the proof of Theorem 1.1. Thus let $N \subset X$ be open and $q : N \rightarrow Y$ a proper holomorphic submersion to a complex manifold Y . Fix $y_0 \in q(N)$ and let $F = q^{-1}(y_0)$. After restricting Y to a

coordinate ball about y_0 and N to its inverse image, write $d_Y = \dim Y$ and let $\psi : N \rightarrow \mathbb{C}^{d_Y}$ be the coordinate expression of q , translated so that $\psi^{-1}(0) = F$. Fix $0 \neq \eta \in (\mathbb{C}^{d_Y})^*$ and set

$$\lambda^F = (\lambda_x^F)_{x \in F}, \quad \lambda_x^F := \eta \circ d\psi_x \in T_x^* X.$$

The covectors λ_x^F are the differentials of the same scalar function $\eta \circ \psi$ at points $x \in F$. Thus, even when $f(0)$ moves in F , normalization always means $(\eta \circ \psi \circ f)_z(0) = 1$, with the composition defined near zero. In particular, $d\psi_{f(0)}(f_z(0)) \neq 0$, so the initial tangent vector is not tangent to F . The submersion property makes λ_x^F nonzero for every $x \in F$. The classes below consist of maps on $\mathbb{D}_R$ that extend holomorphically to some open neighborhood of $\overline{\mathbb{D}}_R$. The neighborhood may depend on the map. Set

$$\mathcal{A}_R^{F, \lambda^F} := \left\{ f \in \mathcal{O}(\mathbb{D}_R, X) : f \text{ extends holomorphically near } \overline{\mathbb{D}}_R, \right. \\ \left. f(0) \in F, \quad \lambda_{f(0)}^F(f_z(0)) = 1 \right\}, \quad (4)$$

$$a_{F, \lambda^F}(R) := \inf \{ \text{Area}_R(f) : f \in \mathcal{A}_R^{F, \lambda^F} \}.$$

We call a holomorphic disk *normalized* if its center lies in F and its base scalar derivative equals 1. Membership in $\mathcal{A}_R^{F, \lambda^F}$ requires, in addition, holomorphic extension across the source boundary. We use the convention $\inf \emptyset = +\infty$. Restricting a competitor from $\mathbb{D}_t$ to $\mathbb{D}_s$, with $0 < s < t$, preserves admissibility and cannot increase its area. Taking infima gives $a_{F, \lambda^F}(s) \leq a_{F, \lambda^F}(t)$. Thus the area profile is nondecreasing, with left limit at each finite $R > 0$ given by

$$a_{F, \lambda^F}(R^-) := \sup_{0 < s < R} a_{F, \lambda^F}(s).$$

We use almost minimizers without assuming that the infimum is attained: an ε -almost minimizer at radius R is a map in $\mathcal{A}_R^{F, \lambda^F}$ with area less than $a_{F, \lambda^F}(R) + \varepsilon$. Choose $x_0 \in F$ and a holomorphic coordinate chart χ with $\chi(x_0) = 0$. Since $\lambda_{x_0}^F$ is nonzero, choose $v \in \mathbb{C}^n$ such that $\lambda_{x_0}^F(d(\chi^{-1})_0(v)) = 1$. For sufficiently small R , the map $f_{\text{seed}}(z) = \chi^{-1}(zv)$ is holomorphic near $\overline{\mathbb{D}}_R$ and belongs to $\mathcal{A}_R^{F, \lambda^F}$. Its area tends to zero as R tends to zero, so we choose R with $\text{Area}_R(f_{\text{seed}}) < A_*$. Since this seed has arbitrarily small area,

$$\lim_{R \downarrow 0} a_{F, \lambda^F}(R) = 0. \quad (5)$$

Continuation compares admissible disks at different radii; the disks need not extend or approximate a common initial map. Area descent above A_* creates a strict margin, and scalar enlargement uses that margin to increase the source radius.

The first operation lowers area by a uniform positive amount whenever the area exceeds A_* by a fixed amount and both area and weighted derivative have fixed upper bounds.

Proposition 2.1. *Assume that ω is Kähler and $\text{Ric}(\omega) \geq \kappa\omega$ with $\kappa > 0$, and keep the fixed embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$. For every $A_0, B_0, \delta > 0$, there is $\sigma > 0$ with the following property: for every $R > 0$, each $f \in \mathcal{A}_R^{F, \lambda^F}$ satisfying*

$$A_* + \delta \leq \text{Area}_R(f) \leq A_0, \quad B_R(f) \leq B_0,$$

admits a competitor $g \in \mathcal{A}_R^{F, \lambda^F}$ with $\text{Area}_R(g) \leq \text{Area}_R(f) - \sigma$.

The second operation starts with one finite-area map on an open disk. A holomorphic perturbation fixes its center and increases its scalar derivative. Source dilation then restores the normalization and enlarges the disk. Only continuity of the total area of this perturbation is needed.

Lemma 2.2. *Keep the projective embedding of X and the relative constraint above. Let $0 < R < \infty$ and let $h : \mathbb{D}_R \rightarrow X$ be holomorphic, with*

$$\text{Area}_R(h) < \infty, \quad h(0) \in F, \quad \lambda_{h(0)}^F(h_z(0)) = 1.$$

There are $t_0 > 0$ and holomorphic maps $h_t : \mathbb{D}_R \rightarrow X$, $0 \leq t < t_0$, such that $h_0 = h$ and

$$h_t(0) = h(0), \quad \lambda_{h(0)}^F((h_t)_z(0)) = 1 + t, \quad \lim_{t \downarrow 0} \text{Area}_R(h_t) = \text{Area}_R(h).$$

Consequently,

$$G_t(z) := h_t(z/(1+t)), \quad |z| < (1+t)R,$$

has center in F , scalar derivative 1, and total area $\text{Area}_R(h_t)$. For every $A > \text{Area}_R(h)$, some $R' > R$ admits a map $g \in \mathcal{A}_{R'}^{F, \lambda^F}$ with $\text{Area}_{R'}(g) < A$.

The maps h and h_t need not extend past $\partial\mathbb{D}_R$. For the last assertion, take $t > 0$ with $\text{Area}_R(h_t) < A$ and restrict G_t to any R' strictly between R and $(1+t)R$. This restriction extends holomorphically past its closed source disk. The proof in Section 4.3 constructs the perturbation by adding $tzH'(0)$ to an affine lift H of h .

The two operations require different forms of area control. When compactness retains a normalized principal limit h of almost minimizers at a finite endpoint R , the argument below proves

$$\text{Area}_R(h) = L := a_{F, \lambda^F}(R^-), \quad B_R(h) < \infty,$$

the second assertion being Lemma 2.7. Restrictions of this one map share the bounds L and $B_R(h)$. If $L > A_*$, Proposition 2.1 supplies a fixed area drop on restrictions close to R , contradicting the limiting infimum L . Thus $L \leq A_*$. For a cap $A > A_*$, the remaining margin permits Lemma 2.2 to enlarge this individual disk by continuity of total area. No common bound for the weighted derivatives of the original almost minimizers is needed. The enlargement must still respect the cap: an uncontrolled area increase would not permit continuation.

Figure 3 separates the holomorphic perturbation from the source dilation.

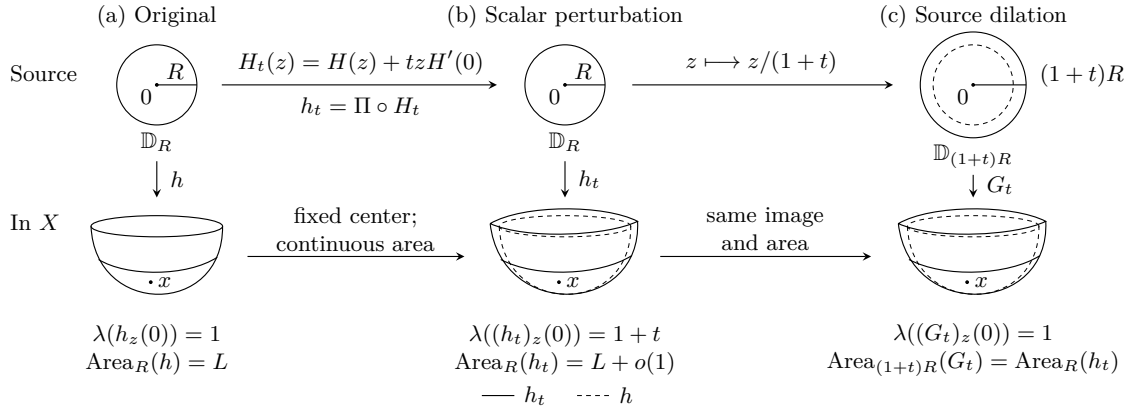


FIGURE 3. Enlargement of one finite-area disk at a fixed center $x \in F$, with $\lambda = \lambda_x^F$. The affine-lift perturbation $h_t = \Pi(H + tzH'(0))$ acts on the same disk and increases the scalar derivative from 1 to $1 + t$. Dilation gives $G_t(z) = h_t(z/(1+t))$ on $\mathbb{D}_{(1+t)R}$, with scalar derivative 1 and unchanged total area. Here $o(1) \rightarrow 0$ as $t \downarrow 0$ for this fixed h . The outlines are schematic; dashed target outlines show h .

2.2. Principal limits and marked values

We need two outputs from compactness: a principal map in the original source coordinate, and a chain retaining two marked target values when bubbling occurs. If the principal map is entire and has finite area, removal at infinity extends it to a sphere. We derive these statements from the compactness and removal theorems of Ivashkovich and Shevchishin [36, 37]. The proof locations are specified below.

Proposition 2.3. *Let (X, ω) be a compact Hermitian manifold. For each j , let $R_j > 0$ and let $f_j : U_j \rightarrow X$ be holomorphic, where $U_j \subset \mathbb{C}$ is open and $\overline{\mathbb{D}}_{R_j} \Subset U_j$. Assume that $\sup_j \text{Area}_{R_j}(f_j) < \infty$, and set $A_\infty = \liminf_j \text{Area}_{R_j}(f_j)$. Choose $R_\infty \in (0, \infty]$ with $R_\infty \leq \liminf_j R_j$, interpreting $\mathbb{D}_\infty$ as $\mathbb{C}$. After passing to a subsequence realizing A_∞ and then to a further subsequence, there are a finite set $\Sigma_\infty \subset \mathbb{D}_{R_\infty}$ and a holomorphic map $f_\infty : \mathbb{D}_{R_\infty} \rightarrow X$ such that, in the original source coordinate,*

$$f_j \longrightarrow f_\infty \quad \text{smoothly on compact subsets of } \mathbb{D}_{R_\infty} \setminus \Sigma_\infty, \quad \text{Area}_{R_\infty}(f_\infty) \leq A_\infty.$$

The possibly constant map f_∞ is holomorphic across Σ_∞ . This extension does not assert convergence of f_j at the concentration points.

The proof of Proposition 2.3 is in Section 5.2.

Theorem 2.4. *Let (X, ω) be a compact Hermitian manifold. A holomorphic map from a punctured disk to X with finite ω -area extends holomorphically over the puncture. Consequently, a finite-area holomorphic map $u : \mathbb{C} \rightarrow X$ extends to a holomorphic map $\mathbb{P}^1 \rightarrow X$.*

The proof of Theorem 2.4 is in Section 5.1.

Lemma 2.5. *Let (X, ω) be a compact Hermitian manifold, and let $K_0, K_1 \subset X$ be disjoint compact subsets. Suppose $R_j > 1$, $R_j \rightarrow \infty$, and the holomorphic maps $f_j : U_j \rightarrow X$, where $U_j \subset \mathbb{C}$ is open and $\overline{\mathbb{D}}_{R_j} \Subset U_j$, satisfy*

$$f_j(0) \in K_0, \quad f_j(1) \in K_1, \quad \sup_j \text{Area}_{R_j}(f_j) < \infty.$$

Then there are nonconstant holomorphic maps $u_1, \dots, u_\ell : \mathbb{P}^1 \rightarrow X$ whose images form a chain: the first meets K_0 , the last meets K_1 , and consecutive images intersect. Their total area, counted with the multiplicities of these maps, satisfies

$$\sum_{i=1}^{\ell} \text{Area}(u_i) \leq \liminf_j \text{Area}_{R_j}(f_j).$$

The proof of Lemma 2.5 is in Section 5.3.

The fixed source marks and the disjoint target sets play different roles. Since the marks remain at 0 and 1, stabilization cannot produce an attached tree made entirely of constant components: such a stable tree would need at least two marks converging to the same source point. Since K_0 and K_1 are disjoint, the nodal path between the marked values must contain a nonconstant component.

The scalar derivative is a derivative of the bounded local base map. If first-exit points from a fixed neighborhood of F approach zero, rescaling and marked compactness give a sphere meeting and leaving F . Otherwise, a subsequence stays in that neighborhood on a fixed source disk, allowing the scalar normalization to pass to the principal map.

Lemma 2.6. *Let (X, ω) be a compact Hermitian manifold, and retain F , N , ψ , and λ^F from (4). Suppose $R_j \rightarrow R_\infty \in (0, \infty]$ and $f_j \in \mathcal{A}_{R_j}^{F, \lambda^F}$, with*

$$\text{Area}_{R_j}(f_j) \longrightarrow E < \infty.$$

Then at least one of the following conclusions holds:

- (i) *There is a nonconstant sphere $v : \mathbb{P}^1 \rightarrow X$ such that*

$$v(\mathbb{P}^1) \cap F \neq \emptyset, \quad v(\mathbb{P}^1) \not\subset F, \quad \text{Area}(v) \leq E.$$

- (ii) *A subsequence has a principal limit $h : \mathbb{D}_{R_\infty} \rightarrow X$ in the original source coordinate, as in Proposition 2.3, satisfying*

$$\text{Area}_{R_\infty}(h) \leq E, \quad h(0) \in F, \quad \lambda_{h(0)}^F(h_z(0)) = 1.$$

On some fixed disk about zero, the base compositions $\psi \circ f_j$ converge to $\psi \circ h$ smoothly on compact subsets. Here $\mathbb{D}_\infty = \mathbb{C}$.

Proof. Choose $F \Subset V \Subset N$ with smooth boundary, and fix $r > 0$ such that $R_j > r$ for all sufficiently large j . There are two possibilities after passage to a subsequence.

First, suppose there are $z_j \rightarrow 0$ with $z_j \neq 0$ and $f_j(z_j) \in \partial V$. Such points can be chosen as first exits along segments from zero whenever no fixed small closed source disk has image in V along an infinite subsequence. The rescaled maps

$$v_j(\zeta) := f_j(z_j \zeta), \quad |\zeta| < R_j/|z_j|,$$

extend past their closed source disks and satisfy

$$v_j(0) \in F, \quad v_j(1) \in \partial V, \quad \text{Area}_{R_j/|z_j|}(v_j) = \text{Area}_{R_j}(f_j) \rightarrow E.$$

Their source radii tend to infinity. Lemma 2.5 gives a chain of nonconstant spheres from F to ∂V , of total area at most E . Take its first component not contained in F . It meets F , either at the first marked value or at its intersection with the preceding component, and has area at most E . This proves (i).

Otherwise, for some $0 < \rho < r$, an infinite subsequence satisfies $f_j(\overline{\mathbb{D}_\rho}) \subset V$. Apply Proposition 2.3 on $\mathbb{D}_{R_\infty}$ to this subsequence. Its principal map h takes values in $\overline{V}$ on $\mathbb{D}_\rho$: this holds away from the finite concentration set by convergence, and at that set by continuity. In particular, $\psi \circ h$ is defined there. The bounded holomorphic maps $\psi \circ f_j$ have a further locally uniformly convergent subsequence. Principal convergence identifies its limit with $\psi \circ h$ away from the concentration set; the identity theorem identifies them throughout $\mathbb{D}_\rho$. Cauchy estimates give smooth convergence on compact subdisks. Passing the value and scalar derivative at zero to the limit yields

$$\psi(h(0)) = 0, \quad (\eta \circ \psi \circ h)_z(0) = \lim_j (\eta \circ \psi \circ f_j)_z(0) = 1.$$

The area bound follows from principal compactness, proving (ii). $\square$

The principal map in conclusion (ii) retains a definite amount of area in the scalar base coordinate. On the fixed disk $\mathbb{D}_\rho$ from the proof, set $b_j = \eta \circ \psi \circ f_j$ and $b = \eta \circ \psi \circ h$. For every $0 < r < \rho$, the submean inequality and $b'_j(0) = 1$ give

$$\int_{\mathbb{D}_r} |b'_j|^2 dA \geq \pi r^2, \quad \int_{\mathbb{D}_r} |b'|^2 dA = \lim_j \int_{\mathbb{D}_r} |b'_j|^2 dA \geq \pi r^2. \quad (6)$$

The equality follows from smooth convergence on the smaller disk. This is area in the fixed scalar base coordinate. It is carried by the retained principal map and does not locate a bubble selected elsewhere.

The following lemma supplies the weighted derivative bound for the principal map at a finite endpoint.

Lemma 2.7. *Let (X, ω) be a compact Hermitian manifold, let $0 < R < \infty$, and let $h : \mathbb{D}_R \rightarrow X$ be holomorphic with $\text{Area}_R(h) < \infty$. Then $B_R(h) < \infty$.*

The proof in Section 5.4 uses point selection: an unbounded weighted derivative for this fixed map would place a positive amount of area in arbitrarily thin boundary collars.

2.3. Continuation and the sharp area bound

Proof of Theorem 1.1. Keep the threshold A_* and the fixed projective embedding chosen at the beginning of this section. Choose $F \Subset U \Subset N$ with smooth boundary, keeping $\overline{U}$ inside the submersion neighborhood. The full-fiber hypothesis gives $y_0 \notin q(\partial U)$. We can therefore choose a coordinate ball Y' about y_0 such that $\overline{Y'} \cap q(\partial U) = \emptyset$ and $Y' \subset q(U)$. After replacing N by $U \cap q^{-1}(Y')$ and Y by Y' , the map q is proper: the inverse image of a compact subset of Y' is closed in $\overline{U}$ and avoids ∂U . In these restricted neighborhoods, take the coordinates $\psi : N \rightarrow \mathbb{C}^{d_Y}$ and a nonzero covector $\eta \in (\mathbb{C}^{d_Y})^*$, and use the classes (4).

Continuation under a strict cap.

Fix $A > A_*$ and abbreviate $a(R) = a_{F, \lambda^F}(R)$. Set

$$\mathcal{S}_A := \{R > 0 : a(R) < A\}, \quad R_A := \sup \mathcal{S}_A.$$

Small coordinate disks give $\mathcal{S}_A \neq \emptyset$, and restriction supplies every smaller positive radius. Each $R \in \mathcal{S}_A$ also admits a slightly larger radius: choose $f \in \mathcal{A}_R^{F, \lambda^F}$ with $\text{Area}_R(f) < A$, and extend the source slightly within the holomorphic neighborhood of $\overline{\mathbb{D}}_R$. Continuity of its area preserves the strict cap. Thus

$$\mathcal{S}_A = (0, R_A). \quad (7)$$

Consider first $R_A < \infty$. Choose $r_j \uparrow R_A$ and write

$$L := a(R_A^-) = \sup_{0 < s < R_A} a(s) \leq A.$$

Choose $0 < \varepsilon_j < \min\{j^{-1}, (A - a(r_j))/2\}$ and $f_j \in \mathcal{A}_{r_j}^{F, \lambda^F}$ such that

$$a(r_j) \leq \text{Area}_{r_j}(f_j) < a(r_j) + \varepsilon_j < A.$$

Then $\text{Area}_{r_j}(f_j) \rightarrow L$. Apply Lemma 2.6. Its first outcome already gives a sphere meeting but not contained in F , with area at most $L \leq A$. We show that the second outcome permits continuation beyond R_A .

The finite endpoint.

In the second outcome, let $h : \mathbb{D}_{R_A} \rightarrow X$ be the normalized principal map. For every $0 < s < R_A$, its restriction extends holomorphically past $\overline{\mathbb{D}}_s$ and belongs to $\mathcal{A}_s^{F, \lambda^F}$. Hence

$$a(s) \leq \text{Area}_s(h), \quad \text{Area}_{R_A}(h) \leq L.$$

Letting $s \uparrow R_A$ gives $\text{Area}_{R_A}(h) = L$. Together with Lemma 2.7, this yields precisely the endpoint data needed below:

$$\text{Area}_{R_A}(h) = L, \quad B_{R_A}(h) < \infty. \quad (8)$$

If $L > A_*$, set $\delta = (L - A_*)/2$. For $s < R_A$ sufficiently close to R_A , we have

$$A_* + \delta \leq \text{Area}_s(h) \leq A, \quad B_s(h) \leq B_{R_A}(h).$$

Proposition 2.1, with these common caps, gives a fixed $\sigma > 0$ and competitors $g_s \in \mathcal{A}_s^{F, \lambda^F}$ satisfying

$$a(s) \leq \text{Area}_s(g_s) \leq \text{Area}_s(h) - \sigma \leq L - \sigma.$$

Since $a(s) \rightarrow L$, this is impossible. Therefore

$$L \leq A_* < A. \quad (9)$$

Apply Lemma 2.2 to h . The strict margin in (9) and continuity of $\text{Area}_{R_A}(h_t)$ give a small $t > 0$ with $\text{Area}_{R_A}(h_t) < A$. The normalized map

$$G_t(z) = h_t(z/(1+t))$$

is holomorphic on $\mathbb{D}_{(1+t)R_A}$ and has this same total area. Choose $R_A < R' < (1+t)R_A$ and restrict G_t to $\mathbb{D}_{R'}$. The resulting competitor belongs to $\mathcal{A}_{R'}^{F, \lambda^F}$ and has area less than A , contrary to the definition of R_A . Thus at a finite endpoint the alternative supplies the required sphere.

Expanding source disks.

If $R_A = \infty$, choose $R_j \rightarrow \infty$ and $f_j \in \mathcal{A}_{R_j}^{F, \lambda^F}$ with $\text{Area}_{R_j}(f_j) < A$. Pass to a subsequence whose areas converge to $E \leq A$, and apply Lemma 2.6. The first outcome again supplies the required sphere. In the second, the principal map $u : \mathbb{C} \rightarrow X$ satisfies

$$\text{Area}(u) \leq E, \quad u(0) \in F, \quad (\eta \circ \psi \circ u)_z(0) = \lambda_{u(0)}^F(u_z(0)) = 1,$$

where the base composition is used only near zero. Theorem 2.4 extends u across infinity. Its nonzero base derivative shows that the resulting sphere meets F without being contained in F . We have therefore obtained such a sphere of area at most A for every $A > A_*$.

The sharp area bound.

To reach the exact bound A_* , we take a second limit, now retaining incidence through two marked values rather than requiring the scalar derivative normalization to survive. For each j , choose a sphere $u_j : \mathbb{P}^1 \rightarrow X$ meeting F without being contained in F , with

$$\int_{\mathbb{P}^1} u_j^* \omega \leq A_* + \frac{1}{j}.$$

Choose $F \subseteq W_F \subseteq N$ with smooth boundary. A sphere meeting F without being contained in F must leave W_F , and hence meet ∂W_F . Indeed, if $u_j(\mathbb{P}^1) \subset W_F$, the holomorphic map $\psi \circ u_j : \mathbb{P}^1 \rightarrow \mathbb{C}^{d_Y}$ is constant. Since it vanishes at a point mapped into F , it vanishes everywhere, forcing $u_j(\mathbb{P}^1) \subset F$, a contradiction.

To put the spheres in the form needed for two-marked compactness, choose distinct points $p_j, q_j \in \mathbb{P}^1$ mapped into F and ∂W_F , respectively, and reparametrize so that these source marks are 0 and 1. Let $f_j : \mathbb{C} \rightarrow X$ be the restriction of the reparametrized sphere to the affine chart. Taking $R_j = j + 1$ then gives

$$f_j(0) \in F, \quad f_j(1) \in \partial W_F, \quad \text{Area}_{R_j}(f_j) \leq A_* + \frac{1}{j}.$$

Applying Lemma 2.5 to these fixed marks with $K_0 = F$ and $K_1 = \partial W_F$ gives a chain $v_1, \dots, v_\ell$ such that

$$\sum_{\alpha=1}^{\ell} \int_{\mathbb{P}^1} v_\alpha^* \omega \leq A_*.$$

Follow the chain from F toward ∂W_F and take the first component not contained in F :

$$k := \min\{\alpha : v_\alpha(\mathbb{P}^1) \not\subset F\}.$$

Such a component exists because the last one meets ∂W_F . It still meets F : if it is the first component, this follows from the marked value; otherwise, it shares a nodal value with its predecessor, whose image lies in F . Hence

$$v_k(\mathbb{P}^1) \cap F \neq \emptyset, \quad v_k(\mathbb{P}^1) \not\subset F, \quad 0 < \text{Area}(v_k) \leq \sum_{\alpha=1}^{\ell} \text{Area}(v_\alpha) \leq A_*.$$

Thus v_k meets F without being contained in F and has area at most A_* , as required. The two marked values are retained by the chain; the selected component need not contain both of them or retain the original derivative normalization. $\square$

Proof of Theorem 1.2. Given $x \in X$, let N be a coordinate neighborhood of x and let $q : N \rightarrow Y \subset \mathbb{C}^n$ be its coordinate map. The full fiber over $q(x)$ is $F = \{x\}$, and q is a submersion. Theorem 1.1 therefore gives a nonconstant sphere through x with area at most A_* . $\square$

Remark 2.8. On $\mathbb{P}^n$, use the Fubini–Study form

$$\omega_{\text{FS}} = \frac{i}{2} \partial \bar{\partial} \log(1 + |w^1|^2 + \dots + |w^n|^2).$$

In the normalization (1), $\text{Ric}(\omega_{\text{FS}}) = 2(n+1)\omega_{\text{FS}}$ and $\int_{\mathbb{P}^1} \omega_{\text{FS}} = \pi$ on a projective line. Every nonconstant sphere in $\mathbb{P}^n$ has positive integer degree d and area $d\pi$. Thus for $\kappa = 2(n+1)$ the bound $2\pi(n+1)/\kappa = \pi$ is attained by a line and cannot be decreased.

Remark 2.9. The finite-radius bound in (8) does not assert a common weighted derivative bound on expanding normalized disks. Indeed, if $M_{\lambda,F} = \max_{x \in F} \|\lambda_x^F\|_{\omega,*}$, then normalization gives

$$B_R(f) \geq R|f_z(0)|_\omega \geq R/M_{\lambda,F} \quad (f \in \mathcal{A}_R^{F,\lambda^F}).$$

The expanding-source argument instead retains the local base derivative on the principal map.

Remark 2.10. For fixed $A > A_*$, one may assume that every sphere of area at most A meeting F is contained in F . This excludes the first outcome of Lemma 2.6. The same descent and enlargement then force $R_A = \infty$, and removal of the normalized entire principal map contradicts that assumption. The passage $A \downarrow A_*$ is unchanged.

The direct proof allows each compactness step to produce the sphere and makes no initial nonexistence assumption. It still uses infima, subsequences, and incompatible inequalities at a putative finite endpoint; it does not provide a finite algorithm for producing the sphere.

3. RICCI TRACES AND SCALAR-CONSTRAINED VARIATIONS

This section explains why positive Ricci curvature can force area decrease above A_* . The calculation has two ingredients: a boundary trace identity and a choice of holomorphic multipliers that fix the center and one scalar derivative. Their Dirichlet cost $\pi(n+1)$, combined with Proposition 3.4, gives the threshold $A_* = 2\pi(n+1)/\kappa$. Section 4 realizes the resulting variation fields by holomorphic families with uniform estimates. We retain the normalizations (1)–(2).

For $z = x + iy$, the complex derivative and the Euclidean Laplacian are

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \Delta = \partial_x^2 + \partial_y^2 = 4\partial_z\partial_{\bar{z}}.$$

On f^*T_X , ∇_z denotes the Chern covariant derivative induced by ω in the source direction. On $\partial\mathbb{D}_R$, ν is the outward Euclidean unit normal, ∂_ν the corresponding derivative, and ds Euclidean arclength. Let A^* denote the conjugate transpose of A . We use the Hilbert–Schmidt norm and the operator norm

$$\|A\|_{\text{HS}}^2 := \text{tr}(AA^*) = \sum_{j,k} |A_{jk}|^2, \quad \|A\|_{\text{op}} := \sup_{|v|=1} |Av|. \quad (10)$$

The symbol Id_n denotes the identity matrix. Distances and diameters use the specified metric.

For a Hermitian holomorphic line bundle (L, h) , let $\Theta(L)$ denote the curvature of its Chern connection. In Demailly’s convention, the real curvature form is $i\Theta(L) = -i\partial\bar{\partial} \log |s|_h^2$, where s is a nonvanishing local holomorphic frame, and its cohomology class is $2\pi c_1(L)$ [14, Ch. V, (12.8), (13.3)]. We write $c_1(X) = c_1(K_X^{-1})$.

Yau’s theorem supplies the Ricci lower bound that allows the relative sphere theorem to be applied to Fano manifolds.

Proposition 3.1. *Every complex Fano manifold admits a Kähler metric ω and a constant $\kappa > 0$ such that*

$$\text{Ric}(\omega) \geq \kappa\omega. \quad (11)$$

Proof. Ampleness of K_X^{-1} gives a positive closed $(1,1)$ -form ρ representing $2\pi c_1(X)$. In a fixed Kähler class, Yau’s prescribed-Ricci theorem [61] gives a metric ω with $\text{Ric}(\omega) = \rho$. The least eigenvalue of ρ relative to ω is positive and continuous on the compact manifold. Its positive minimum therefore supplies κ with $\text{Ric}(\omega) \geq \kappa\omega$. $\square$

Remark 3.2. A connected compact Hermitian manifold with a metric satisfying (11) is Fano. The induced Hermitian metric on K_X^{-1} has positive Chern curvature $\text{Ric}(\omega)$, so Kodaira’s embedding theorem makes K_X^{-1} ample and X projective [41, pp. 314–316].

In Euclidean space, the paired second area variation is Green's formula for the squared norm of the variation field. The *Kähler condition* $d\omega = 0$ makes the same boundary identity exact for a curved target, by the classical coefficient symmetries of closed $(1, 1)$ -forms [14, Ch. VI, §4]; the geometry is retained in the metric norm.

Lemma 3.3. *Assume that ω is Kähler, and fix $R > 0$. Let $\Delta_t \subset \mathbb{C}$ be a parameter disk centered at zero, $U \subset \mathbb{C}$ an open neighborhood of $\overline{\mathbb{D}}_R$, and $\mathcal{F} : U \times \Delta_t \rightarrow X$ a holomorphic map. Write $f_t(z) := \mathcal{F}(z, t)$ and $\xi := \partial_t \mathcal{F}(\cdot, 0)$. Denote the second derivatives along the two real parameter axes by*

$$H_{\mathbb{R}} := \left. \frac{d^2}{ds^2} \right|_{s=0} \text{Area}_R(f_s), \quad H_{i\mathbb{R}} := \left. \frac{d^2}{ds^2} \right|_{s=0} \text{Area}_R(f_{is}).$$

Then

$$H_{\mathbb{R}} + H_{i\mathbb{R}} = \int_{\partial \mathbb{D}_R} \partial_\nu |\xi|_\omega^2 ds. \quad (12)$$

In particular, two holomorphic families with the same central map f_0 and the same tangent field ξ have the same paired second variation $H_{\mathbb{R}} + H_{i\mathbb{R}}$.

Proof. Set $\tilde{\omega} := \mathcal{F}^* \omega$, and write its Hermitian coefficient matrix in the source coordinates $\zeta^1 = z$, $\zeta^2 = t$ as

$$\tilde{\omega} = \frac{i}{2} \sum_{\alpha, \beta=1}^2 \tilde{\omega}_{\alpha\bar{\beta}} d\zeta^\alpha \wedge d\bar{\zeta}^\beta.$$

The Kähler condition gives $d\tilde{\omega} = \mathcal{F}^*(d\omega) = 0$. Closedness relates derivatives in the parameter direction to derivatives in the source direction. Comparing the $(2, 1)$ and $(1, 2)$ coefficients yields

$$\partial_t \tilde{\omega}_{z\bar{z}} = \partial_z \tilde{\omega}_{t\bar{z}}, \quad \partial_{\bar{t}} \tilde{\omega}_{t\bar{z}} = \partial_{\bar{z}} \tilde{\omega}_{t\bar{t}}.$$

Differentiating these identities and commuting mixed derivatives gives

$$\partial_t \partial_{\bar{t}} \tilde{\omega}_{z\bar{z}} = \partial_z \partial_{\bar{t}} \tilde{\omega}_{t\bar{z}} = \partial_z \partial_{\bar{z}} \tilde{\omega}_{t\bar{t}}.$$

At $t = 0$, the coefficient $\tilde{\omega}_{t\bar{t}}$ is $|\xi|_\omega^2$. Consequently

$$\partial_t \partial_{\bar{t}} \text{Area}_R(f_t)|_{t=0} = \int_{\mathbb{D}_R} \partial_z \partial_{\bar{z}} |\xi|_\omega^2 dA = \frac{1}{4} \int_{\partial \mathbb{D}_R} \partial_\nu |\xi|_\omega^2 ds.$$

The sum of the second derivatives along the real and imaginary parameter axes is the parameter Laplacian $4\partial_t \partial_{\bar{t}}$. Hence

$$H_{\mathbb{R}} + H_{i\mathbb{R}} = 4 \partial_t \partial_{\bar{t}} \text{Area}_R(f_t)|_{t=0},$$

which yields (12). $\square$

The classical Riemannian second-variation formula contains interior curvature and second-fundamental-form terms; on Kähler submanifolds, holomorphic normal fields are Jacobi fields [53, §§3.2, 3.5]. Here holomorphicity and $d\omega = 0$ give the explicit paired boundary formula (12), allowing moving boundary values and branch points without a boundary stationarity assumption.

Fix $R > 0$ and a map f holomorphic on a neighborhood of $\overline{\mathbb{D}}_R$. To select variation fields before realizing them by holomorphic families, define the boundary quadratic form on holomorphic sections ξ of f^*T_X that are smooth on the closed disk:

$$\mathcal{H}_{R,f}(\xi) := \int_{\partial \mathbb{D}_R} \partial_\nu |\xi|_\omega^2 ds. \quad (13)$$

When ξ is the tangent field of a complex holomorphic family, Lemma 3.3 identifies $\mathcal{H}_{R,f}(\xi)$ with the sum of the second area derivatives along the real and imaginary parameter axes.

We use the Wiener–Masani boundary normalization in the form employed by Berndtsson and Rosay [3, Theorem 2.1 and p. 888]; Lemma 4.1 recalls the construction of the resulting boundary-unitary holomorphic frame. The following computation combines this normalization with Green's

formula and the classical determinant-curvature identities [14, Ch. V, (13.3); Ch. VIII, (6.6)–(6.7)]. We record the matrix form that separates the Ricci integral from the multiplier’s Dirichlet cost and will give the descent threshold below.

Proposition 3.4. *Let $e = (e_1, \dots, e_n)$ be a holomorphic frame of f^*T_X , smooth on the closed disk and unitary on $\partial\mathbb{D}_R$. For a positive integer N , let Φ be a holomorphic $\text{Mat}_{n \times N}(\mathbb{C})$ -valued map on a neighborhood of $\mathbb{D}_R$, satisfying the boundary coisometry condition*

$$\Phi(\zeta)\Phi(\zeta)^* = \text{Id}_n \quad (|\zeta| = R).$$

The columns Φ_a give sections $(e\Phi)_a = \sum_{j=1}^n e_j \Phi_{ja}$ satisfying the trace identity

$$\sum_{a=1}^N \mathcal{H}_{R,f}((e\Phi)_a) = 4 \int_{\mathbb{D}_R} \|\Phi'\|_{\text{HS}}^2 dA - 2 \int_{\mathbb{D}_R} f^* \text{Ric}(\omega). \quad (14)$$

Here Φ' is the entrywise derivative and $\|\cdot\|_{\text{HS}}$ is the norm in (10).

Proof. The Gram matrix $G := (\langle e_j, e_k \rangle_\omega)_{j,k}$ represents squared norms by $|e\bar{c}|_\omega^2 = c^\top G \bar{c}$. Thus

$$\sum_{a=1}^N |(e\Phi)_a|_\omega^2 = \sum_{a=1}^N \Phi_a^\top G \bar{\Phi}_a = \text{tr}(\Phi^\top G \bar{\Phi}) = \text{tr}(G \bar{\Phi} \Phi^\top).$$

The first trace sums the diagonal entries of $\Phi^\top G \bar{\Phi}$; the last equality uses cyclicity of the trace. Since finite summation commutes with the normal derivative and integration, (13) gives the boundary integral of $\partial_\nu \text{tr}(G \bar{\Phi} \Phi^\top)$. On the boundary, $G = \bar{\Phi} \Phi^\top = \text{Id}_n$, but their normal derivatives need not vanish. The product rule and Jacobi’s classical determinant formula (see, for instance, [4]) give, on $\partial\mathbb{D}_R$,

$$\begin{aligned} \partial_\nu \text{tr}(G \bar{\Phi} \Phi^\top) &= \text{tr}(\partial_\nu G) + \text{tr}(\partial_\nu(\bar{\Phi} \Phi^\top)), \\ \partial_\nu \log \det G &= \text{tr}(G^{-1} \partial_\nu G) = \text{tr}(\partial_\nu G). \end{aligned}$$

The boundary normalization is essential in the second line: tr and $\log \det$ have the same first differential at Id_n . The divergence theorem and $\Delta \text{tr}(\bar{\Phi} \Phi^\top) = 4\|\Phi'\|_{\text{HS}}^2$ give the multiplier term. Since $\det G$ is the metric coefficient on $f^*K_X^{-1}$, (1) gives

$$f^* \text{Ric}(\omega) = -\frac{1}{2} \Delta \log \det G dA.$$

Green’s formula therefore turns the determinant term $\partial_\nu \log \det G$ into $-2 \int f^* \text{Ric}(\omega)$. □

We first construct variations that fix the value of a disk at 0 and one nonzero scalar component of its derivative. Fix $x_0 \in X$ and a nonzero complex-linear covector

$$\lambda \in T_{x_0}^* X \setminus \{0\}. \quad (15)$$

The scalar condition $\lambda(f_z(0)) = 1$ fixes one component of the initial derivative and gives $|f_z(0)|_\omega \geq \|\lambda\|_{\omega,*}^{-1}$, where the norm on the covector is dual to ω at x_0 . The multiplier below imposes the linearized value and scalar-derivative constraints. Theorem 4.9 realizes these constraints by holomorphic families that preserve $f(0) = x_0$ and $\lambda(f_z(0)) = 1$ exactly, while leaving the other derivative components free. In Section 2, the center varies in a compact fiber and the covector is pulled back from the base. In the principal branch of Lemma 2.6, this base normalization passes to the limit. For a disk satisfying $f(0) = x_0$ and $\lambda(f_z(0)) = 1$, the frame value $e(0) : \mathbb{C}^n \rightarrow T_{x_0} X$ identifies the scalar constraint with the nonzero covector $L_f := \lambda \circ e(0)$. Choose $U_f \in U(n)$ so that its last $n-1$ columns form an orthonormal basis of $\ker L_f$. All columns must vanish at zero to fix the center. In the last $n-1$ directions, which lie in $\ker L_f$, a simple zero also preserves the scalar derivative to first order. In the remaining direction we impose a double zero so that its derivative at zero vanishes. This leads to the multiplier

$$\Phi_f(z) := U_f \text{diag} \left((z/R)^2, z/R, \dots, z/R \right). \quad (16)$$

The columns of Φ_f vanish at zero, so $(e\Phi_f)'(0) = e(0)\Phi_f'(0)$. The first column of Φ_f has zero derivative at zero; the derivatives of its remaining columns belong to $\ker L_f \subset \mathbb{C}^n$. Thus $\Phi_f(0) = 0$ and $L_f\Phi_f'(0) = 0$ impose the infinitesimal constraints. On $|z| = R$, the diagonal entries have modulus one, so $\Phi_f\Phi_f^* = \text{Id}_n$. For each integer $d \geq 1$, scaling $z = R\zeta$ and integrating in polar coordinates gives $\int_{\mathbb{D}_R} |(z/R)^d|'^2 dA = \pi d$, independently of R . Since U_f is unitary, the quadratic column contributes 2π and each of the $n - 1$ linear columns contributes π . Hence

$$\int_{\mathbb{D}_R} \|\Phi_f'\|_{\text{HS}}^2 dA = 2\pi + (n - 1)\pi = \pi(n + 1). \quad (17)$$

If $\text{Ric}(\omega) \geq \kappa\omega$, then

$$\sum_{a=1}^n \mathcal{H}_{R,f}((e\Phi_f)_a) \leq 4\pi(n + 1) - 2\kappa \text{Area}_R(f). \quad (18)$$

The cost in (17) is optimal within the prescribed multiplier class. After a unitary change of rows, Parseval's identity gives unit total squared coefficient norm in each row, while the vanishing conditions give its lowest possible degree. Indeed, fix $R > 0$, $N \geq 1$, and $0 \neq L \in (\mathbb{C}^n)^*$, and let Φ be holomorphic near $\mathbb{D}_R$ with values in $\text{Mat}_{n \times N}(\mathbb{C})$, satisfying

$$\Phi\Phi^* = \text{Id}_n \quad \text{on } \partial\mathbb{D}_R, \quad \Phi(0) = 0, \quad L\Phi'(0) = 0.$$

The boundary identity forces $N \geq n$. Choose $Q \in U(n)$ with first row $L/\|L\|$, where $\|L\|$ is the Euclidean dual norm. Multiplication by Q preserves that identity and the Hilbert–Schmidt Dirichlet integral. The rows r_i of $Q\Phi$ have norm one on $\partial\mathbb{D}_R$ and vanish to orders at least $m_1 = 2$ and $m_i = 1$ for $i \geq 2$. Writing $r_i(z) = \sum_{k \geq m_i} a_{ik}(z/R)^k$, with $a_{ik} \in \mathbb{C}^N$, Parseval's identity and radial integration give

$$\sum_{k \geq m_i} \|a_{ik}\|^2 = 1, \quad \int_{\mathbb{D}_R} |r_i'|^2 dA = \pi \sum_{k \geq m_i} k \|a_{ik}\|^2 \geq \pi m_i.$$

Summing proves the lower bound $\pi(n + 1)$. It is attained by $U \text{diag}((z/R)^2, z/R, \dots, z/R)$ when the last $n - 1$ columns of $U \in U(n)$ span $\ker L$, appending $N - n$ zero columns if $N > n$. This multiplier optimality leaves the holomorphic realization and its remainder estimates to the next section.

4. UNIFORM HOLOMORPHIC REALIZATION AND DISK OPERATIONS

We realize the trace calculation by holomorphic families and then prove the two disk operations: uniform descent and scalar enlargement. Holomorphic mapping spaces and Stein neighborhoods of graphs provide a qualitative deformation theory [21]. Here the parameter radius and cubic area remainder must remain uniform as the disk and its source radius vary. We obtain this control by lifting to a fixed Stein submanifold of affine space, perturbing there, and projecting back to X .

Assume in this section that X is projective. Keep the projective embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$ fixed as in Section 2; it will be used to represent $\iota \circ f$ by homogeneous coordinates and to construct the lift space $\mathcal{Y}$. The embedding ι is kept fixed as f and R vary, so the constants below may depend on ι .

We use the standard L^p and C^k norms on source regions, with Euclidean area measure. Vector and operator norms are induced by the fixed Hermitian and Euclidean metrics; matrix-valued L^2 estimates use the Hilbert–Schmidt norm.

4.1. Frame and lifting estimates

Bounds on area and weighted derivative do not control derivatives on the source boundary. The following example explains why uniform realization requires direct estimates. For $0 < r < 1$ and $\varepsilon > 0$, the functions $h_r : \mathbb{D} \rightarrow \mathbb{C}$ given by

$$h_r(z) := \varepsilon \frac{1 - r^2}{1 - rz}$$

are holomorphic near $\overline{\mathbb{D}}$ and satisfy

$$\int_{\mathbb{D}} |h'_r|^2 dA = \pi \varepsilon^2 r^2, \quad B_1(h_r) \leq \frac{\varepsilon}{2}, \quad |h'_r(1)| = \varepsilon \frac{r(1+r)}{1-r} \longrightarrow \infty.$$

Thus bounded area and bounded B_1 do not imply relative compactness in $C^1(\overline{\mathbb{D}})$ (with the Euclidean metric on $\mathbb{C}$); the uniform estimates below must be proved directly.

Boundary spectral factorization and a Green potential bound control a normalized frame. Homogeneous coordinates and Nicolau's corona theorem then give a uniformly bounded affine lift with bounded Dirichlet integral.

Fix $A_0, B_0 \geq 0$. In the auxiliary results below, f is holomorphic near $\overline{\mathbb{D}}_R$ and satisfies $\text{Area}_R(f) \leq A_0$ and $B_R(f) \leq B_0$. Norms on fixed finite-rank bundles are chosen once and for all.

The spectral factorization theorem is due to Wiener and Masani [56, Theorem 7.13]. We use the smooth boundary version stated by Berndtsson and Rosay [3, Theorem 2.1]. It makes the frame in Theorem 4.7 unitary on the boundary, as required by the Ricci-trace identity.

Lemma 4.1. *Let $H : \partial\mathbb{D} \rightarrow \text{Mat}_{n \times n}(\mathbb{C})$ be a smooth positive-definite Hermitian matrix function. There is a map $A : \mathbb{D} \rightarrow \text{GL}_n(\mathbb{C})$, holomorphic in $\mathbb{D}$ and smooth on $\overline{\mathbb{D}}$ with invertible boundary values, such that*

$$A(\zeta)^* H(\zeta) A(\zeta) = \text{Id}_n \quad (|\zeta| = 1).$$

Proof. The Wiener–Masani theorem in the smooth boundary form of [3, Theorem 2.1] gives $H = h^* h$, where h and h^{-1} are holomorphic on $\mathbb{D}$ and smooth on $\overline{\mathbb{D}}$. Thus $A := h^{-1}$ has the required boundary normalization. $\square$

For $z \in \mathbb{D}$, let δ_z be the unit point mass at z . The Dirichlet Green function of the unit disk, with the sign and normalization used here, is [14, Ch. I, §4.B]

$$g(z, w) = \log \left| \frac{1 - \bar{z}w}{z - w} \right|, \quad -\Delta_w g = 2\pi \delta_z.$$

On $\mathbb{D}_R$ we use $g_R(z, w) := g(z/R, w/R)$.

The Green potential controls both the frame and the homogeneous lift. The following elementary estimate separates a neighborhood of the logarithmic singularity, controlled by B_0 , from its complement, controlled by the total area.

Lemma 4.2. *Let $\varepsilon_f^\omega(w) := |f_z(w)|_\omega^2$ be the area density of f , so that $f^* \omega = \varepsilon_f^\omega dA$ and $\text{Area}_R(f) = \int_{\mathbb{D}_R} \varepsilon_f^\omega dA$. If $\text{Area}_R(f) \leq A_0$ and $B_R(f) \leq B_0$, then*

$$\sup_{z \in \mathbb{D}_R} \int_{\mathbb{D}_R} g_R(z, w) \varepsilon_f^\omega(w) dA(w) \leq C_g(A_0, B_0) := C_G(B_0^2 + A_0), \quad (19)$$

where C_G is a universal constant, independent of R and f .

Proof. First reduce to the unit disk. Under the dilation $h(\zeta) := f(R\zeta)$, the identities

$$\text{Area}_1(h) = \text{Area}_R(f), \quad B_1(h) = B_R(f), \quad \varepsilon_h^\omega(\eta) = R^2 \varepsilon_f^\omega(R\eta)$$

show that the hypotheses are unchanged. Since $g_R(R\zeta, R\eta) = g(\zeta, \eta)$, change of variables also preserves the Green-potential integral. We may therefore assume $R = 1$.

Fix $z \in \mathbb{D}$ and set $\delta := 1 - |z|$, its distance to the boundary. Split the integral into the disk $|w - z| < \delta/2$ and its complement in $\mathbb{D}$. On the first region,

$$1 - |w| \geq 1 - |z| - |w - z| > \frac{\delta}{2}, \quad |f_z(w)|_\omega \leq \frac{B_0}{1 - |w|}.$$

Thus $\varepsilon_f^\omega(w) \leq 4B_0^2/\delta^2$. To bound the Green kernel, write

$$1 - \bar{z}w = (1 - |z|^2) + \bar{z}(z - w).$$

Since $1 - |z|^2 = (1 + |z|)\delta \leq 2\delta$, the triangle inequality gives

$$|1 - \bar{z}w| \leq 2\delta + |z - w| \leq \frac{5\delta}{2}, \quad g(z, w) \leq \log \frac{5\delta}{2|z - w|}.$$

Integration in polar coordinates bounds this contribution by

$$\frac{4B_0^2}{\delta^2} 2\pi \int_0^{\delta/2} r \log \frac{5\delta}{2r} dr = \pi B_0^2 \left(\log 5 + \frac{1}{2} \right).$$

On the complementary region $|z - w| \geq \delta/2$, the same triangle inequality instead yields

$$\frac{|1 - \bar{z}w|}{|z - w|} \leq 1 + \frac{2\delta}{|z - w|} \leq 5.$$

Thus $g(z, w) \leq \log 5$ there, and the remaining contribution is at most

$$(\log 5) \int_{\mathbb{D}} \varepsilon_f^\omega dA \leq A_0 \log 5.$$

Adding the two bounds proves (19); one may take the universal constant $C_G := \pi(\log 5 + 1/2)$, independently of z , R , and f . $\square$

We use the following Chern connection and curvature convention [14, Ch. V, (12.2)–(12.4)]. In the coordinates of (1), a section with coefficient column c has squared norm $c^\top h \bar{c}$. Writing $\Theta(T_X) = \nabla^2$, the connection and curvature matrices are

$$\nabla' c = \partial c + \bar{h}^{-1}(\partial \bar{h})c, \quad \nabla'' c = \bar{\partial} c, \quad \Theta(T_X) = \bar{\partial}(\bar{h}^{-1} \partial \bar{h}).$$

The coefficient of $dw^\alpha \wedge d\bar{w}^\beta$ in $\Theta(T_X)$ is $-\partial_{\bar{w}^\beta}(\bar{h}^{-1} \partial_{w^\alpha} \bar{h})$; the minus sign comes from reversing the order of the differentials. For $v = \sum_\alpha v^\alpha \partial_{w^\alpha}$ and a tangent vector u with coefficient column c , our curvature tensor is

$$R^\omega(v, \bar{v}, u, \bar{u}) := \sum_{\alpha, \beta} v^\alpha \bar{v}^\beta c^\top [-\partial_{w^\alpha} \partial_{\bar{w}^\beta} h + (\partial_{w^\alpha} h) h^{-1} (\partial_{\bar{w}^\beta} h)] \bar{c}. \quad (20)$$

Thus positive holomorphic bisectional curvature means that (20) is positive for every pair of nonzero tangent vectors v, u at the same point. For a holomorphic disk f , the curvature endomorphism $\mathcal{R}_f^\omega$ is characterized by

$$\langle \mathcal{R}_f^\omega v, v \rangle_\omega = R^\omega(f_z, \bar{f}_z, v, \bar{v}). \quad (21)$$

The Bochner formula and the curvature formula for the determinant bundle [14, Ch. VIII, proof of Lemma 7.2; Ch. V, (4.6)] give, for a holomorphic section s of f^*T_X ,

$$\partial_z \partial_{\bar{z}} |s|_\omega^2 = |\nabla_z s|_\omega^2 - \langle \mathcal{R}_f^\omega s, s \rangle_\omega, \quad f^* \text{Ric}(\omega) = 2 \text{tr}(\mathcal{R}_f^\omega) dA. \quad (22)$$

Choose C_ω so that

$$\|\mathcal{R}_f^\omega(z)\|_{\text{op}} \leq C_\omega \varepsilon_f^\omega(z).$$

Compactness of X gives such a constant.

For a frame $e = (e_1, \dots, e_n)$, write $G = (\langle e_j, e_k \rangle_\omega)$. Matrix inequalities refer to Hermitian quadratic forms. The Green bound and Bochner identity control G throughout the disk and bound the frame's covariant Dirichlet integral. Both estimates enter the lifted-field bounds in Theorem 4.7. The two-sided metric control follows the boundary normalization and curvature-potential method of Berndtsson and Rosay [3, Lemma 2.2 and Proposition 2.3]. We give the calculation with the caps A_0, B_0 and record also the covariant Dirichlet bound.

Proposition 4.3. *There are constants $\Gamma \geq 1$ and $D_1 < \infty$, depending only on (X, ω, A_0, B_0) , such that every holomorphic frame e of f^*T_X that is smooth on the closed disk and unitary on its boundary has Gram matrix G satisfying*

$$\Gamma^{-1} \text{Id}_n \leq G \leq \Gamma \text{Id}_n. \quad (23)$$

Moreover

$$\sum_{a=1}^n \int_{\mathbb{D}_R} |\nabla_z e_a|^2 dA \leq D_1. \quad (24)$$

In particular, these constants are independent of R , f , and the boundary-unitary frame.

Proof. Set

$$u_G := \operatorname{tr} G = \sum_{a=1}^n |e_a|_\omega^2 > 0, \quad q := \sum_{a=1}^n |\nabla_z e_a|_\omega^2.$$

The curvature bound and (22) give

$$\begin{aligned} \partial_z \partial_{\bar{z}} u_G &= q - \sum_a \langle \mathcal{R}_f^\omega e_a, e_a \rangle_\omega \\ &\geq q - C_\omega \varepsilon_f^\omega \sum_a |e_a|_\omega^2 = q - C_\omega \varepsilon_f^\omega u_G. \end{aligned}$$

Because e_a is holomorphic and the connection preserves the metric, $\partial_z u_G = \sum_a \langle \nabla_z e_a, e_a \rangle_\omega$. Cauchy–Schwarz, applied to the direct sum of the n fibers, gives

$$|\partial_z u_G|^2 \leq \left(\sum_a |e_a|_\omega^2 \right) \left(\sum_a |\nabla_z e_a|_\omega^2 \right) = u_G q.$$

Apply the classical logarithmic norm calculation to the section $(e_1, \dots, e_n)$ of the direct-sum bundle [14, Ch. VIII, proof of Lemma 7.2]. Since $u_G > 0$, it gives

$$\begin{aligned} \partial_z \partial_{\bar{z}} \log u_G &= \frac{\partial_z \partial_{\bar{z}} u_G}{u_G} - \frac{|\partial_z u_G|^2}{u_G^2} \\ &\geq \frac{q}{u_G} - C_\omega \varepsilon_f^\omega - \frac{q}{u_G} = -C_\omega \varepsilon_f^\omega. \end{aligned}$$

Hence

$$\Delta \log u_G \geq -4C_\omega \varepsilon_f^\omega, \quad u_G|_{\partial \mathbb{D}_R} = n.$$

Green representation, with $-\Delta_w g_R(z, w) = 2\pi \delta_z$, reads

$$\log \frac{u_G(z)}{n} = -\frac{1}{2\pi} \int_{\mathbb{D}_R} g_R(z, w) \Delta_w \log u_G(w) dA(w).$$

Since $g_R \geq 0$, Lemma 4.2 yields

$$\log \frac{u_G(z)}{n} \leq \frac{2C_\omega}{\pi} \int_{\mathbb{D}_R} g_R(z, w) \varepsilon_f^\omega(w) dA(w) \leq \frac{2C_\omega C_g(A_0, B_0)}{\pi}.$$

The algebraic dual holomorphic frame $e^a(e_b) = \delta_b^a$ has Gram matrix $G^{-\top} := (G^{-1})^\top$, where $\top$ denotes ordinary transpose. The curvature of the dual bundle is the negative transpose of the original curvature [14, Ch. V, (4.3')], so its operator norm is bounded by $C_\omega \varepsilon_f^\omega$. Applying the logarithmic trace estimate to the dual frame gives

$$\operatorname{tr} G \leq \Gamma, \quad \operatorname{tr}(G^{-1}) \leq \Gamma, \quad \Gamma := n \exp\left(\frac{2C_\omega C_g(A_0, B_0)}{\pi}\right).$$

For every eigenvalue $\mu > 0$ of G , $\mu \leq \operatorname{tr} G \leq \Gamma$ and $\mu^{-1} \leq \operatorname{tr}(G^{-1}) \leq \Gamma$. These two inequalities give (23).

It remains to bound the frame's covariant Dirichlet integral. We first sum the Bochner identity; after integration, the boundary determinant calculation from Proposition 3.4 will identify its left-hand side:

$$\partial_z \partial_{\bar{z}} \operatorname{tr} G = \sum_a |\nabla_z e_a|^2 - \operatorname{tr}(\mathcal{R}_f^\omega S), \quad S := \sum_a e_a \otimes e_a^b.$$

Here $e_a^\flat(v) = \langle v, e_a \rangle_\omega$ denotes metric duality. Since $G = \text{Id}_n$ on $\partial\mathbb{D}_R$, Jacobi's formula gives

$$\partial_\nu \log \det G = \text{tr}(G^{-1} \partial_\nu G) = \partial_\nu \text{tr} G \quad \text{on } \partial\mathbb{D}_R.$$

Consequently Green's formula and (1) give

$$\int_{\mathbb{D}_R} \partial_z \partial_{\bar{z}} \text{tr} G dA = \frac{1}{4} \int_{\partial\mathbb{D}_R} \partial_\nu \log \det G ds = -\frac{1}{2} \int_{\mathbb{D}_R} f^* \text{Ric}(\omega).$$

Integrating the Bochner identity and solving for the nonnegative derivative term therefore gives

$$\sum_a \int_{\mathbb{D}_R} |\nabla_z e_a|^2 dA = \int_{\mathbb{D}_R} \text{tr}(\mathcal{R}_f^\omega \mathbf{S}) dA - \frac{1}{2} \int_{\mathbb{D}_R} f^* \text{Ric}(\omega).$$

Now $\text{tr} \mathbf{S} \leq n\Gamma$, while the absolute Ricci density is at most $2nC_\omega \varepsilon_f^\omega$, since $f^* \text{Ric} = 2 \text{tr}(\mathcal{R}_f^\omega) dA$. Taking absolute values and using $\text{Area}_R(f) \leq A_0$ gives

$$\sum_a \int_{\mathbb{D}_R} |\nabla_z e_a|^2 dA \leq nC_\omega(\Gamma + 1)A_0.$$

Thus $D_1 := nC_\omega(\Gamma + 1)A_0$ is an admissible choice in (24). $\square$

Recall the projective embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$ chosen at the beginning of this section. Let ω_{FS} be the Fubini–Study form on its target, with normalization

$$\omega_{\text{FS}} = \frac{i}{2} \partial \bar{\partial} \log \sum_{j=1}^m |Z_j|^2.$$

Choose $c_\iota > 0$ such that $\iota^* \omega_{\text{FS}} \leq c_\iota \omega$; such a constant exists by compactness of X . The displayed Fubini–Study formula is read in homogeneous coordinates $Z = (Z_1, \dots, Z_m) \neq 0$; it is unchanged as a form on projective space when the homogeneous representative is multiplied by a nonvanishing holomorphic function.

In rank one, boundary normalization subtracts a harmonic function from the logarithm of the metric, as in Berndtsson and Rosay [3, p. 886]. Together with the classical Fubini–Study potential [14, Ch. VI, (4.4)], this gives homogeneous coordinates with the bounds needed for the corona theorem. We record the area identity and the uniform lower bound.

Lemma 4.4. *Every disk satisfying $\text{Area}_R(f) \leq A_0$ and $B_R(f) \leq B_0$ has a homogeneous lift $\mathbf{s}$ from an open neighborhood of $\overline{\mathbb{D}}_R$ to $\mathbb{C}^m \setminus \{0\}$, holomorphic on that neighborhood, with $[\mathbf{s}] = \iota \circ f$ and $|\mathbf{s}| = 1$ on the boundary. It satisfies*

$$|\mathbf{s}| \leq 1, \quad \int_{\mathbb{D}_R} |\mathbf{s}'|^2 dA = \int_{\mathbb{D}_R} f^* \iota^* \omega_{\text{FS}} \leq c_\iota A_0, \quad (25)$$

and

$$|\mathbf{s}(z)| \geq \delta_{\mathbf{s}} > 0, \quad (26)$$

where $\delta_{\mathbf{s}} = \delta_{\mathbf{s}}(X, \omega, \iota, A_0, B_0)$ is independent of R and f .

Proof. The tautological line bundle on $\mathbb{P}^{m-1}$ has fiber $\mathbb{C}Z$ over $[Z]$. Trivialize its pullback by $\iota \circ f$ on a slightly larger disk, using the Oka–Grauert principle [28, Satz 6], and choose a nonvanishing section, represented by a lift v in $\mathbb{C}^m \setminus \{0\}$. The positive real-analytic boundary function $|v|$ has a logarithm whose harmonic extension is the real part of a holomorphic function a on a slightly larger disk. Then $\mathbf{s} := e^{-a}v$ has boundary norm one, and the maximum principle gives $|\mathbf{s}| \leq 1$.

The area density of $\iota \circ f$ for the Fubini–Study metric is $\varepsilon_f^{\text{FS}} := |(\iota \circ f)'|_{\text{FS}}^2$, so that $f^* \iota^* \omega_{\text{FS}} = \varepsilon_f^{\text{FS}} dA$. Set $u := \log |\mathbf{s}|^2$. The boundary normalization $|\mathbf{s}| = 1$ gives $u = 0$ and $\partial_\nu u = \partial_\nu |\mathbf{s}|^2$ there. Since $\mathbf{s}$ has no common zero, the Fubini–Study formula gives $\Delta u = 4\varepsilon_f^{\text{FS}}$, while holomorphicity gives $\Delta |\mathbf{s}|^2 = 4|\mathbf{s}'|^2$. Green's formula therefore yields

$$\int_{\mathbb{D}_R} |\mathbf{s}'|^2 dA = \frac{1}{4} \int_{\partial\mathbb{D}_R} \partial_\nu |\mathbf{s}|^2 ds = \frac{1}{4} \int_{\partial\mathbb{D}_R} \partial_\nu u ds = \int_{\mathbb{D}_R} \varepsilon_f^{\text{FS}} dA,$$

which proves (25). Green representation for u with zero boundary values [14, Ch. I, (4.5)] gives

$$-u(z) = \frac{2}{\pi} \int g_R(z, w) \varepsilon_f^{\text{FS}}(w) dA(w) \leq \frac{2c_\iota C_g(A_0, B_0)}{\pi}.$$

Thus (26) holds with $\delta_{\mathbf{s}} = \exp(-c_\iota C_g(A_0, B_0)/\pi)$. $\square$

Write $H^\infty(\mathbb{D})$ for the bounded holomorphic functions with the supremum norm, and let

$$\mathcal{D}(\mathbb{D}) := \left\{ h : \mathbb{D} \rightarrow \mathbb{C} \text{ holomorphic} : \int_{\mathbb{D}} |h'|^2 dA < \infty \right\}.$$

The Dirichlet algebra is

$$\mathfrak{A} := H^\infty(\mathbb{D}) \cap \mathcal{D}(\mathbb{D}), \quad \|h\|_{\mathfrak{A}} := \|h\|_\infty + \left(\int_{\mathbb{D}} |h'|^2 dA \right)^{1/2}.$$

To construct the affine lift, we need a holomorphic dual vector χ with $\chi \cdot \mathbf{s} = 1$ and bounds on both its supremum norm and Dirichlet integral. Nicolau's theorem and the bound at the beginning of its proof [49, pp. 135–136] give the following statement.

Theorem 4.5. *For every $m \geq 1$ and $A, \delta > 0$ there is $C_N(m, A, \delta) < \infty$ such that, whenever $u_1, \dots, u_m \in \mathfrak{A}$ satisfy*

$$\|u_j\|_{\mathfrak{A}} \leq A, \quad \max_j |u_j(z)| \geq \delta \quad (z \in \mathbb{D}),$$

there are $v_1, \dots, v_m \in \mathfrak{A}$ with

$$\sum_j u_j v_j = 1, \quad \|v_j\|_{\mathfrak{A}} \leq C_N(m, A, \delta).$$

The same statement holds on every disk after dilation.

Proof. The case $\delta > A$ is vacuous; assume $\delta \leq A$. For $p = 2$, Nicolau uses the norm

$$\|h\|_{\text{Nic}} := \|h\|_\infty + \left(\frac{1}{\pi} \int_{\mathbb{D}} |h'|^2 dA \right)^{1/2}, \quad \|h\|_{\text{Nic}} \leq \|h\|_{\mathfrak{A}} \leq \sqrt{\pi} \|h\|_{\text{Nic}}.$$

His corona condition is the same maximum lower bound as above [49, p. 135, (1)]. The generators u_j/A therefore have Nicolau norm at most 1 and lower bound δ/A . The estimate at the beginning of his proof [49, p. 136] gives solutions b_j with $\sum_j (u_j/A) b_j = 1$ and $\|b_j\|_{\text{Nic}} \leq K_m(\delta/A)$, where K_m depends only on the indicated data. Setting $v_j := b_j/A$ gives $\sum_j u_j v_j = 1$ and

$$\|v_j\|_{\mathfrak{A}} \leq \frac{\sqrt{\pi}}{A} K_m(\delta/A),$$

which is an admissible choice of $C_N(m, A, \delta)$. For a disk of radius R ,

$$\|h(R \cdot)\|_{\infty, \mathbb{D}} = \|h\|_{\infty, \mathbb{D}_R}, \quad \int_{\mathbb{D}} |(h(R\zeta))'|^2 dA(\zeta) = \int_{\mathbb{D}_R} |h'(z)|^2 dA(z),$$

so the same bound applies after dilation. $\square$

For a normalized lift $\mathbf{s}$ from Lemma 4.4, its coordinates have algebra norm at most $C_{\mathbf{s}} := 1 + \sqrt{c_\iota A_0}$, and at least one has modulus at least $\delta_{\mathbf{s}}/\sqrt{m}$ at every point, because $|\mathbf{s}|^2 \leq m \max_j |\mathbf{s}_j|^2$. Theorem 4.5 gives a holomorphic map $\chi : \mathbb{D}_R \rightarrow (\mathbb{C}^m)^*$, whose components belong to the algebra on $\mathbb{D}_R$ obtained from $\mathfrak{A}$ by dilation, satisfying

$$\chi \cdot \mathbf{s} = 1, \quad \|\chi\|_\infty + \|\chi'\|_2 \leq C_\chi \tag{27}$$

for a constant $C_\chi = C_\chi(X, \omega, \iota, A_0, B_0)$. The dot denotes the bilinear pairing $\chi \cdot \mathbf{s} = \sum_{j=1}^m \chi_j \mathbf{s}_j$, without complex conjugation. The constant absorbs the fixed dimension factors in passing from scalar to vector norms.

To extend the lift past $\overline{\mathbb{D}}_R$, dilate χ radially while keeping $\mathbf{s}$ fixed, then restore their pairing by division. For $0 < \rho < 1$ sufficiently close to 1, set

$$\chi_\rho(z) := \chi(\rho z), \quad \gamma_\rho := \chi_\rho \cdot \mathbf{s}, \quad \widehat{\chi} := \frac{\chi_\rho}{\gamma_\rho}.$$

Since $\chi(\rho z) \cdot \mathbf{s}(\rho z) = 1$, we have

$$\gamma_\rho(z) - 1 = \chi(\rho z) \cdot (\mathbf{s}(z) - \mathbf{s}(\rho z)).$$

The fixed lift $\mathbf{s}$ is uniformly continuous on $\overline{\mathbb{D}}_R$, and χ is bounded. Thus $\gamma_\rho \rightarrow 1$ uniformly there as $\rho \uparrow 1$, and we may require $|\gamma_\rho| \geq 1/2$. The quotient is therefore holomorphic near $\overline{\mathbb{D}}_R$ and satisfies $\widehat{\chi} \cdot \mathbf{s} = 1$ and $\|\widehat{\chi}\|_\infty \leq 2C_\chi$. Dilation gives $\|\chi'_\rho\|_2 \leq C_\chi$; the product and quotient rules give

$$\begin{aligned} \|\gamma'_\rho\|_2 &\leq C_\chi(1 + \sqrt{c_i A_0}), \\ \|\widehat{\chi}'\|_2 &\leq 2C_\chi + 4C_\chi^2(1 + \sqrt{c_i A_0}) =: D_\chi. \end{aligned} \tag{28}$$

All norms here are on $\mathbb{D}_R$. The bound depends only on the fixed target, embedding, and A_0, B_0 ; the base lift $\mathbf{s}$ is unchanged.

4.2. Uniform lifts and holomorphic variations

The affine model below is a homogeneous version of the classical incidence-complement construction known as the Jouanolou trick; see [23, §2.6, p. 771]. The pairs $(\mathbf{s}, \chi)$ lie in this fixed model: the equation $\chi \cdot \mathbf{s} = 1$ excludes the cone vertex. A holomorphic right inverse lifts tangent fields to this model. The model depends only on the embedding; the area and weighted-derivative caps determine a compact subset and a retraction near it.

Lemma 4.6. *For the projective embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$ chosen above, there are a closed Stein submanifold $\mathcal{Y} \subset \mathbb{C}^{2m}$, a holomorphic submersion $\pi_X : \mathcal{Y} \rightarrow X$, and a holomorphic bundle map $\mathcal{L} : \pi_X^* T_X \rightarrow T_{\mathcal{Y}}$ such that $d\pi_X \circ \mathcal{L} = \text{Id}$. These objects depend only on the fixed embedding.*

Proof. Consider the affine cone of the embedding,

$$\widehat{X} := \{0\} \cup \{\mathbf{s} \in \mathbb{C}^m \setminus \{0\} : [\mathbf{s}] \in \iota(X)\},$$

and set

$$\mathcal{Y} := \{(\mathbf{s}, \chi) \in \widehat{X} \times (\mathbb{C}^m)^* : \chi \cdot \mathbf{s} = 1\}, \quad \pi_X(\mathbf{s}, \chi) := \iota^{-1}[\mathbf{s}]. \tag{29}$$

The defining equation excludes the cone vertex. On a chart $\mathbf{s}_j \neq 0$, local coordinates consist of coordinates on X , a nonzero scale factor, and $m - 1$ unconstrained dual coordinates. Hence $\mathcal{Y}$ is a smooth closed affine algebraic variety and therefore Stein, while π_X is a holomorphic submersion.

For the holomorphic splitting, set $V := \mathbb{C}^m$ and let $\ell := \iota^* \mathcal{O}_{\mathbb{P}^{m-1}}(-1)$ be the tautological line bundle on X ; its fiber $\ell_x \subset V$ is the line representing $\iota(x)$. Define

$$\mathcal{U} := \{(x, \chi) \in X \times V^* : \chi|_{\ell_x} \neq 0\}.$$

The map

$$\Theta : \mathcal{Y} \longrightarrow \mathcal{U}, \quad (\mathbf{s}, \chi) \longmapsto (\pi_X(\mathbf{s}, \chi), \chi),$$

is biholomorphic. Indeed, if s_0 is a local nonvanishing holomorphic section of ℓ , then

$$\sigma(x, \chi) := \frac{s_0(x)}{\chi(s_0(x))}$$

is independent of the choice of s_0 , is holomorphic, lies in ℓ_x , and satisfies $\chi(\sigma(x, \chi)) = 1$. Thus

$$\Theta^{-1}(x, \chi) = (\sigma(x, \chi), \chi).$$

Since $\mathcal{U}$ is open in $X \times V^*$, its tangent space at (x, χ) is $T_x X \oplus V^*$. Transporting the vectors $(v, 0)$ by $d\Theta^{-1}$ gives the required holomorphic right inverse

$$\mathcal{L} : \pi_X^* T_X \rightarrow T_{\mathcal{Y}}, \quad \mathcal{L}_y(v) := d(\Theta^{-1})_{\Theta(y)}(v, 0), \quad d\pi_X \circ \mathcal{L} = \text{Id}. \tag{30}$$

□

We assemble three ingredients for the holomorphic variations: a controlled boundary-unitary frame, a lift of the disk, and lifts of the frame vectors. Under fixed area and weighted-derivative caps, the lifted disks lie in a common compact subset of the fixed affine model, and the estimates for all three ingredients are uniform.

Theorem 4.7. *Fix the affine model $\mathcal{Y}$, the submersion π_X , and the right inverse $\mathcal{L}$ supplied by Lemma 4.6. Given $A_0, B_0 \geq 0$, one can choose a compact set $K_{A_0, B_0} \Subset \mathcal{Y}$, an open neighborhood Ω of K_{A_0, B_0} in $\mathbb{C}^{2m}$, a holomorphic map $\tilde{\pi}_X : \Omega \rightarrow X$ with $\tilde{\pi}_X|_{\mathcal{Y} \cap \Omega} = \pi_X$, and constants*

$$\Gamma, D_1, D_{\tilde{f}}, C_{\theta, \infty}, D_\theta < \infty.$$

These cap-dependent choices depend only on $(X, \omega, \iota, A_0, B_0)$ and are independent of the source radius and the individual disk; the affine model and its splitting depend only on ι . For every $R > 0$ and every map f holomorphic on a neighborhood of $\overline{\mathbb{D}}_R$ with $\text{Area}_R(f) \leq A_0$ and $B_R(f) \leq B_0$, one can choose a frame e , a lifted disk $\tilde{f}$, and a matrix θ compatibly so that the following properties hold:

- (i) *e is a holomorphic frame of f^*T_X , smooth on the closed disk and unitary on its boundary, whose Gram matrix $G(z) := (\langle e_j(z), e_k(z) \rangle_\omega)_{j,k=1}^n$ satisfies*

$$\Gamma^{-1} \text{Id}_n \leq G \leq \Gamma \text{Id}_n, \quad \sum_{a=1}^n \int_{\mathbb{D}_R} |\nabla_z e_a|^2 dA \leq D_1;$$

- (ii) *$\tilde{f}$ is a map from a neighborhood of $\overline{\mathbb{D}}_R$ to $\mathcal{Y}$, holomorphic on that neighborhood, such that*

$$\pi_X \circ \tilde{f} = f, \quad \tilde{f}(\overline{\mathbb{D}}_R) \subset K_{A_0, B_0}, \quad \int_{\mathbb{D}_R} |\tilde{f}'|^2 dA \leq D_{\tilde{f}},$$

- (iii) *the frame vectors have holomorphic lifts $\theta_1, \dots, \theta_n : \mathbb{D}_R \rightarrow \mathbb{C}^{2m}$ along $\tilde{f}$. Their column matrix and the lifting identities are*

$$\begin{aligned} \theta &:= (\theta_1, \dots, \theta_n) : \mathbb{D}_R \rightarrow \text{Mat}_{2m \times n}(\mathbb{C}), \\ d(\tilde{\pi}_X)_{\tilde{f}(z)} \theta_a(z) &= e_a(z) \quad (1 \leq a \leq n). \end{aligned} \tag{31}$$

The matrix θ is smooth on the closed disk and satisfies

$$\sup_{z \in \mathbb{D}_R} \|\theta(z)\|_{\text{op}} \leq C_{\theta, \infty}, \quad \int_{\mathbb{D}_R} \|\theta'\|_{\text{HS}}^2 dA \leq D_\theta.$$

Proof. Fix a disk f satisfying the caps A_0, B_0 .

- (i) Choose $R' > R$ so that $\overline{\mathbb{D}}_{R'}$ lies in the given source neighborhood, and choose a holomorphic frame $\tilde{e}$ of f^*T_X on $\mathbb{D}_{R'}$; such a frame exists by the Oka–Grauert principle [28, Satz 6, p. 270], since the source is contractible and Stein. Restrict its Gram matrix to $\partial\mathbb{D}_R$ and apply Lemma 4.1 to $H(\zeta) = \overline{G(R\zeta)}$. For the resulting factor A , set $e(z) := \tilde{e}(z)A(z/R)$. Conjugating the factorization identity gives its boundary Gram matrix $A(z/R)^\top \tilde{G}(z) A(z/R) = \text{Id}_n$. Thus e is holomorphic, smooth on the closed disk, and unitary on its boundary. Proposition 4.3 gives (23) and (24), proving (i).

- (ii) By Lemma 4.4, f has a homogeneous lift $\mathbf{s}$ with

$$|\mathbf{s}| \leq 1, \quad |\mathbf{s}| \geq \delta_{\mathbf{s}}, \quad \int_{\mathbb{D}_R} |\mathbf{s}'|^2 dA \leq c_\iota A_0.$$

Its coordinates satisfy the hypotheses of Theorem 4.5 with algebra-norm bound $C_{\mathbf{s}}$ and corona lower bound $\delta_{\mathbf{s}}/\sqrt{m}$. Thus (27) supplies a vector χ satisfying $\chi \cdot \mathbf{s} = 1$, and the dilation construction leading to (28) gives a vector $\hat{\chi}$, holomorphic near the closed disk, with

$$\hat{\chi} \cdot \mathbf{s} = 1, \quad \|\hat{\chi}\|_{L^\infty(\mathbb{D}_R)} \leq 2C_\chi, \quad \|\hat{\chi}'\|_{L^2(\mathbb{D}_R)} \leq D_\chi.$$

The identity $\hat{\chi} \cdot \mathbf{s} = 1$ shows that $\tilde{f} := (\mathbf{s}, \hat{\chi})$ takes values in the fixed space $\mathcal{Y}$ of (29). It satisfies

$$\tilde{f}(\overline{\mathbb{D}}_R) \subset K_{A_0, B_0} := \mathcal{Y} \cap \{|\mathbf{s}| \leq 1, |\chi| \leq 2C_\chi\}, \quad \int_{\overline{\mathbb{D}}_R} |\tilde{f}'|^2 dA \leq c_\iota A_0 + D_\chi^2 =: D_{\tilde{f}}. \quad (32)$$

The set K_{A_0, B_0} is compact: it is closed and bounded, and $\chi \cdot \mathbf{s} = 1$ prevents $\mathbf{s}$ from approaching the cone vertex.

Since $\mathcal{Y}$ is a closed Stein submanifold of $\mathbb{C}^{2m}$, the Docquier–Grauert tubular neighborhood theorem [16, Satz 3], in the holomorphic deformation-retraction formulation [22, Theorem 3.1], gives a holomorphic retraction $\mathfrak{r} : \Omega \rightarrow \mathcal{Y} \cap \Omega$ from a neighborhood Ω of K_{A_0, B_0} , equal to the identity on $\mathcal{Y} \cap \Omega$. Its derivatives are bounded on smaller compact neighborhoods of K_{A_0, B_0} . Set $\tilde{\pi}_X := \pi_X \circ \mathfrak{r}$. Since $\pi_X \circ \tilde{f} = f$ and $\tilde{\pi}_X|_{\mathcal{Y} \cap \Omega} = \pi_X$, (32) proves (ii). The choices of K_{A_0, B_0} , Ω , and $\tilde{\pi}_X$ depend only on the fixed target data and the two caps; they are fixed before any individual lifted disk $\tilde{f}$ is chosen. The space $\mathcal{Y}$, the projection π_X , and the splitting $\mathcal{L}$ were already fixed from the embedding alone. (iii) Regard the splitting $\mathcal{L}$ from (30) as a bundle map from $\pi_X^* T_X$ to the ambient trivial bundle $\mathcal{Y} \times \mathbb{C}^{2m}$. In $\nabla \mathcal{L}$, use the pullback Chern connection on $\pi_X^* T_X$ and the flat connection on this ambient bundle. Let

$$L_0 := \sup_{K_{A_0, B_0}} \|\mathcal{L}\|, \quad L_1 := \sup_{K_{A_0, B_0}} \|\nabla \mathcal{L}\|.$$

Set

$$\theta_a(z) := \mathcal{L}_{\tilde{f}(z)} e_a(z).$$

Then $d(\tilde{\pi}_X)_\# \theta_a = e_a$. With the preceding connection convention, ambient differentiation gives

$$\theta'_a = \mathcal{L}_{\tilde{f}} \nabla_z e_a + (\nabla_{\tilde{f}} \mathcal{L}) e_a. \quad (33)$$

The frame and lifted-disk bounds, with $|a + b|^2 \leq 2|a|^2 + 2|b|^2$, give

$$\|\theta\|_{L^\infty, \text{op}} \leq L_0 \sqrt{\Gamma} =: C_{\theta, \infty}, \quad (34)$$

$$\int_{\overline{\mathbb{D}}_R} \|\theta'\|_{\text{HS}}^2 dA \leq 2L_0^2 D_1 + 2n\Gamma L_1^2 D_{\tilde{f}} =: D_\theta. \quad (35)$$

This proves (iii). $\square$

We now add the lifted variation fields and project to X . Radial regularization extends the fields past the closed disk while keeping the lifted map fixed. The multiplier retains its vanishing orders, and the paired trace converges to the value computed in Section 3. The parameter range and cubic remainder remain uniform throughout this approximation.

The embedding fixes $\mathcal{Y}, \pi_X, \mathcal{L}$, independently of the disks and caps. The caps A_0, B_0 then fix $K_{A_0, B_0}, \Omega, \tilde{\pi}_X$; together with D_Φ, N_Φ , these data determine t_0, C_T . Each disk has its own compatible frame, lift, and lifted fields satisfying Theorem 4.7. The approximation parameter ρ may depend on this triple, on Φ , and on the prescribed trace error ε , whereas t_0, C_T are independent of R, f, Φ, ρ , and ε . The extension neighborhood may depend on the family, Φ , and ρ ; its width need not be uniform.

Theorem 4.8. *Fix (X, ω) , the projective embedding $\iota : X \hookrightarrow \mathbb{P}^{m-1}$, and the bounds $A_0, B_0 \geq 0$ as in Theorem 4.7. Let $D_\Phi \geq 0$, and let $N_\Phi \geq 1$ be an integer. There are constants $t_0 > 0$ and $C_T < \infty$, depending only on $(X, \omega, \iota, A_0, B_0, D_\Phi, N_\Phi)$, with the following property.*

Let $R > 0$, and let $f : \mathbb{D}_R \rightarrow X$ extend holomorphically to a neighborhood of $\overline{\mathbb{D}}_R$ and satisfy $\text{Area}_R(f) \leq A_0$ and $B_R(f) \leq B_0$. Use the fixed K_{A_0, B_0}, Ω , and $\tilde{\pi}_X$ from Theorem 4.7, and choose any compatible disk-dependent triple $e, \tilde{f}, \theta$ supplied by that theorem. Suppose that $\Phi = (\Phi_1, \dots, \Phi_{N_\Phi}) : \mathbb{D}_R \rightarrow \text{Mat}_{n \times N_\Phi}(\mathbb{C})$ extends holomorphically to an open neighborhood of $\overline{\mathbb{D}}_R$ and satisfies

$$\sup_{z \in \overline{\mathbb{D}}_R} \|\Phi(z)\|_{\text{op}} \leq 1, \quad \int_{\overline{\mathbb{D}}_R} \|\Phi'\|_{\text{HS}}^2 dA \leq D_\Phi,$$

where the column Φ_a determines the section $e\Phi_a = \sum_{j=1}^n e_j \Phi_{ja}$ of f^*T_X . The constants t_0 and C_T are independent of R, f, Φ and of this compatible triple.

For each $0 < \rho < 1$, write

$$\Xi_\rho(z) := \theta(\rho z)\Phi(z), \quad (\Xi_\rho)_a(z) := \sum_{j=1}^n \theta_j(\rho z)\Phi_{ja}(z).$$

(i) Holomorphic realization. For every $0 < \rho < 1$, each column defines the family

$$f_{a,t}^\rho(z) := \tilde{\pi}_X(\tilde{f}(z) + t(\Xi_\rho)_a(z)), \quad t \in \mathbb{C}, \quad |t| \leq t_0, \quad 1 \leq a \leq N_\Phi.$$

All these families are jointly holomorphic on one open neighborhood of $\overline{\mathbb{D}}_R \times \{t \in \mathbb{C} : |t| \leq t_0\}$; for every $0 < \rho < 1$ that neighborhood may depend on the selected disk-dependent triple, Φ , and ρ , but not on the column a .

(ii) Uniform cubic remainder. For every $0 < \rho < 1$, $1 \leq a \leq N_\Phi$, and $\tau \in \{1, i\}$, write $t = \tau s$ with $s \in \mathbb{R}$ and set

$$\ell_{a,\tau}^\rho := \left. \frac{d}{ds} \right|_{s=0} \text{Area}_R(f_{a,\tau s}^\rho), \quad H_{a,\tau}^\rho := \left. \frac{d^2}{ds^2} \right|_{s=0} \text{Area}_R(f_{a,\tau s}^\rho).$$

For $|s| \leq t_0$,

$$\text{Area}_R(f_{a,\tau s}^\rho) \leq \text{Area}_R(f) + \ell_{a,\tau}^\rho s + \frac{1}{2} H_{a,\tau}^\rho s^2 + C_T |s|^3 \quad (36)$$

with the same constant C_T for both choices of τ .

(iii) Trace approximation. If Φ is boundary coisometric, meaning $\Phi\Phi^* = \text{Id}_n$ on $\partial\mathbb{D}_R$, then for every $\varepsilon > 0$ there is $0 < \rho_\varepsilon < 1$, depending only on the individual data displayed above and on ε , such that

$$\left| \sum_{a=1}^{N_\Phi} (H_{a,1}^{\rho_\varepsilon} + H_{a,i}^{\rho_\varepsilon}) - \left(4 \int_{\mathbb{D}_R} \|\Phi'\|_{\text{HS}}^2 dA - 2 \int_{\mathbb{D}_R} f^* \text{Ric}(\omega) \right) \right| \leq \varepsilon. \quad (37)$$

Only this conclusion requires choosing ρ_ε close to 1; t_0 and C_T remain independent of ε and ρ_ε .

Proof. Use the fixed cap-dependent ambient data and the selected compatible triple from Theorem 4.7. Set

$$\delta_0 := \min\{1, \text{dist}(K_{A_0, B_0}, \mathbb{C}^{2m} \setminus \Omega)\} > 0,$$

and choose a compact $K' \Subset \Omega$ whose interior contains the closed Euclidean $3\delta_0/4$ -thickening of K_{A_0, B_0} . In the standard coordinate frame of $\mathbb{C}^{2m}$, let g_0 be the positive-semidefinite Hermitian coefficient matrix of $\tilde{\pi}_X^* \omega$ on K' , so

$$v^\top g_0(w) \bar{v} = |d(\tilde{\pi}_X)_w(v)|_\omega^2$$

for a coefficient column v . Only smooth coefficient bounds are used; the pullback form may be degenerate along the fibers. Take $C_{g,3}$ to bound the coefficients and their derivatives through order three on K' .

Take a multiplier Φ satisfying the stated bounds and, for $0 < \rho < 1$, set $\Xi_\rho := \theta(\rho \cdot)\Phi$. Dilation gives $\int_{\mathbb{D}_R} |\rho\theta'(\rho z)|^2 dA(z) = \int_{\mathbb{D}_{\rho R}} |\theta'(w)|^2 dA(w) \leq D_\theta$. Use this identity, $\|\Phi\|_\infty \leq 1$, and $(\theta(\rho z)\Phi)' = \rho\theta'(\rho z)\Phi + \theta(\rho z)\Phi'$ to obtain

$$\sup_{z \in \mathbb{D}_R} \|\Xi_\rho(z)\|_{\text{op}} \leq C_{\theta, \infty}, \quad \int_{\mathbb{D}_R} \|\Xi'_\rho\|_{\text{HS}}^2 dA \leq 2D_\theta + 2C_{\theta, \infty}^2 D_\Phi =: D_\Xi. \quad (38)$$

Choose

$$t_0 := \min \left\{ 1, \frac{\delta_0}{4 \max\{1, C_{\theta, \infty}\}} \right\}.$$

For $|t| \leq 2t_0$, the displacement of each column perturbation is at most $2t_0 C_{\theta, \infty} \leq \delta_0/2$. Thus $\tilde{f} + t(\Xi_\rho)_a$ stays in K' , uniformly in the disk and the column.

(i) Fix a column and write $\Xi := (\Xi_\rho)_a$. The function $\theta(\rho z)$ is holomorphic for $|z| < R/\rho$; since $\tilde{f}$ and Φ extend past the closed disk, so does $\tilde{f} + t\Xi$. On the closed disk and for $|t| \leq t_0$ its image lies in the closed $\delta_0/2$ -thickening of K_{A_0, B_0} , which is compactly contained in $\text{int } K' \Subset \Omega$. Compactness gives a neighborhood of $\mathbb{D}_R \times \{|t| \leq t_0\}$ on which the image remains in Ω , so the projected family is jointly holomorphic there. Intersect these neighborhoods over the finitely many columns.

(ii) In ambient coordinates write $w(t) := \tilde{f} + t\Xi$ and $v(t) := \tilde{f}' + t\Xi'$. The area density is $v(t)^\top g_0(w(t)) \overline{v(t)}$. Along either real parameter axis $t = \tau s$, with $\tau \in \{1, i\}$, the functions $w(\tau s)$ and $v(\tau s)$ are affine in s , with first derivatives $\tau\Xi$ and $\tau\Xi'$. For fixed z , put $G(s) := g_0(w(\tau s))$, $V(s) := v(\tau s)$, and $Q := \tau\Xi'$. Ordinary differentiation in the real variable s gives

$$\frac{d^3}{ds^3}(V^\top G \bar{V}) = V^\top G''' \bar{V} + 3Q^\top G'' \bar{V} + 3V^\top G'' \bar{Q} + 6Q^\top G' \bar{Q}.$$

The chain rule bounds $|G^{(k)}(s)|$ by $C_{g,3} |\Xi|^k$ for $1 \leq k \leq 3$, up to fixed dimension factors. The bounds $\|\Xi\|_\infty \leq C_{\theta, \infty}$ and $|t| \leq t_0$ consequently give the integrable estimate

$$\left| \frac{d^3}{ds^3}(v(\tau s)^\top g_0(w(\tau s)) \overline{v(\tau s)}) \right| \leq C(C_{g,3}, C_{\theta, \infty}, t_0)(|\tilde{f}'|^2 + |\Xi'|^2).$$

The right-hand side has a uniformly bounded integral by (32) and (38). Thus the third derivative of the area is uniformly bounded along both real parameter axes. Taylor's theorem gives (36) with one remainder constant C_T .

(iii) The tangent fields of the actual families satisfy, for each fixed f ,

$$\xi_{\rho, a} := d(\tilde{\pi}_X)_{\tilde{f}}(\theta(\rho \cdot) \Phi_a) \longrightarrow e\Phi_a \quad \text{in } C^1(\mathbb{D}_R) \quad (\rho \uparrow 1),$$

because $\theta(\rho \cdot) \rightarrow \theta$ in the same norm. Here the Kähler condition enters through Lemma 3.3. If Φ is boundary coisometric, that lemma, continuity of (13), and Proposition 3.4 therefore give

$$\sum_a \mathcal{H}_{R, f}(\xi_{\rho, a}) \longrightarrow 4 \int_{\mathbb{D}_R} \|\Phi'\|_{\text{HS}}^2 dA - 2 \int_{\mathbb{D}_R} f^* \text{Ric}(\omega).$$

The summand is $H_{a,1}^\rho + H_{a,i}^\rho$ by Lemma 3.3. Choose ρ_ε close enough to one to obtain the prescribed error. Parts (i)–(ii) and (38) hold for every $\rho \in (0, 1)$ with the same t_0, C_T . Their dependence is only on the fixed target data and A_0, B_0, D_Φ, N_Φ . $\square$

4.3. Uniform descent and enlargement

When $\text{Ric}(\omega) \geq \kappa\omega$ and $\text{Area}_R(f) > A_*$, the trace is negative. Selecting one real direction and the sign of its parameter gives descent; the common parameter radius and Taylor remainder give a uniform step size under fixed area and weighted-derivative bounds.

Theorem 4.9. *Let (X, ω) be a compact Kähler n -fold with $\text{Ric}(\omega) \geq \kappa\omega$ for some $\kappa > 0$, and set $A_* = 2\pi(n+1)/\kappa$ as in (3). Fix the point $x_0 \in X$ and the nonzero complex-linear covector $\lambda \in T_{x_0}^* X$ from (15). For every $A_0, B_0, \delta > 0$ there is $\sigma > 0$ with the following property. For every $R > 0$ and every holomorphic map $f : U \rightarrow X$, where $U \subset \mathbb{C}$ is open and $\mathbb{D}_R \Subset U$, satisfying*

$$f(0) = x_0, \quad \lambda(f_z(0)) = 1, \quad \text{Area}_R(f) \leq A_0, \quad B_R(f) \leq B_0, \quad \text{Area}_R(f) \geq A_* + \delta,$$

there is a holomorphic map $g : U' \rightarrow X$, with U' an open neighborhood of $\mathbb{D}_R$, such that

$$g(0) = x_0, \quad \lambda(g_z(0)) = 1, \quad \text{Area}_R(g) \leq \text{Area}_R(f) - \sigma.$$

Proof. For the multiplier (16), one has $N_\Phi = n$, $D_\Phi = \pi(n+1)$, and, by (18),

$$4\pi(n+1) - 2\kappa \text{Area}_R(f) \leq -2\kappa\delta.$$

Apply Theorem 4.8 with trace error $\varepsilon = \kappa\delta$, choose the value ρ_ε supplied by part (iii), and abbreviate $f_{a,t} = f_{a,t}^{\rho_\varepsilon}$, $\ell_{a,\tau} = \ell_{a,\tau}^{\rho_\varepsilon}$, and $H_{a,\tau} = H_{a,\tau}^{\rho_\varepsilon}$. For every column a and every $|t| \leq t_0$, the identities $\Phi_a(0) = 0$ and $d(\tilde{\pi}_X)_{\tilde{f}(0)}\theta(0) = e(0)$ give the exact interpolation formula

$$\begin{aligned} f_{a,t}(0) &= f(0), \\ (f_{a,t})_z(0) &= f_z(0) + t e(0) \Phi'_a(0), \\ \lambda((f_{a,t})_z(0)) &= \lambda(f_z(0)) + t \lambda(e(0) \Phi'_a(0)) = 1. \end{aligned} \tag{39}$$

Indeed, the ambient perturbation at $z = 0$ equals $\tilde{f}(0)$ for every t , so the chain rule always evaluates $d\tilde{\pi}_X$ at the same point $\tilde{f}(0)$. Moreover, the product rule gives $(\Xi_{\rho_\varepsilon})'_a(0) = \theta(0) \Phi'_a(0)$, so radial regularization does not alter the derivative of the increment at the origin. Formula (39) fixes the displayed scalar component, not the entire first jet. For this choice, part (iii) and the preceding trace bound give $\sum_a (H_{a,1} + H_{a,i}) \leq -\kappa\delta$. One of the $2n$ real axes therefore has second area derivative $H_{a,\tau} \leq -\kappa\delta/(2n)$. Choose the sign of s so that $\ell_{a,\tau}s \leq 0$. Then (36) gives

$$\text{Area}_R(f_{a,\tau s}) - \text{Area}_R(f) \leq -\frac{\kappa\delta}{4n} s^2 + C_T |s|^3.$$

Set

$$t_* := \min \left\{ \frac{t_0}{2}, \frac{\kappa\delta}{8n \max\{C_T, 1\}} \right\}, \quad \sigma := \frac{\kappa\delta}{8n} t_*^2.$$

Taking $s \in \{t_*, -t_*\}$ with the chosen sign and $g = f_{a,\tau s}$ proves the assertion. The constants t_0, C_T , hence t_* and σ , are independent of R, f , and ρ . $\square$

The same trace cost and Taylor bounds apply when the covector is $\lambda_{f(0)}^F$. This gives the relative descent used in continuation, with constants independent of the center in F .

Proof of Proposition 2.1. For descent, use the multiplier (16) with $L_f = \lambda_{f(0)}^F \circ e(0)$. It has

$$\Phi_f(0) = 0, \quad L_f \Phi'_f(0) = 0, \quad \int_{\mathbb{D}_R} \|\Phi'_f\|_{\text{HS}}^2 dA = \pi(n+1).$$

By (39), each variation fixes $f(0)$ and the scalar $\lambda_{f(0)}^F(f_z(0)) = 1$. The choice of t_* and σ in the proof of Theorem 4.9 uses only the trace bound and the uniform Taylor estimate. In the present family, the parameter radius and cubic remainder in Theorem 4.8 depend only on A_0, B_0 and the fixed target data. Hence the same area drop works for all $f(0) \in F$. $\square$

For scalar enlargement, it suffices to lift one finite-area disk and perturb its lift in one constant ambient direction.

Proof of Lemma 2.2. By Lemma 2.7, $B_R(h) < \infty$. Choose $s_j \uparrow R$. Each restriction $h|_{\mathbb{D}_{s_j}}$ extends holomorphically past its closed disk and has the common bounds

$$\text{Area}_{s_j}(h) \leq A_0 := \text{Area}_R(h), \quad B_{s_j}(h) \leq B_0 := B_R(h).$$

Apply Theorem 4.7(ii) with these caps. It supplies one compact set $K = K_{A_0, B_0} \Subset \mathcal{Y}$, one ambient neighborhood $\Omega \subset \mathbb{C}^{2m}$ of K , and one holomorphic projection $\Pi = \tilde{\pi}_X : \Omega \rightarrow X$ satisfying $\Pi|_{\mathcal{Y} \cap \Omega} = \pi_X$. The corresponding lifts H_j obey

$$H_j(\overline{\mathbb{D}_{s_j}}) \subset K, \quad \Pi \circ H_j = h \quad \text{on } \mathbb{D}_{s_j}, \quad \int_{\mathbb{D}_{s_j}} |H'_j|^2 dA \leq D_{\tilde{f}},$$

where the last constant is independent of j .

The compact range makes the coordinate functions of H_j a normal family on every compact subdisk of $\mathbb{D}_R$. A diagonal subsequence therefore converges locally uniformly, together with its derivatives, to a holomorphic map $H : \mathbb{D}_R \rightarrow \mathbb{C}^{2m}$. Since K is closed, $H(\mathbb{D}_R) \subset K$, and continuity gives $\Pi \circ H = h$. For every $s < R$, convergence of derivatives on $\overline{\mathbb{D}}_s$ gives $\int_{\mathbb{D}_s} |H'|^2 dA \leq D_{\tilde{f}}$. Exhausting $\mathbb{D}_R$ yields

$$H(\mathbb{D}_R) \subset K, \quad \Pi \circ H = h, \quad \int_{\mathbb{D}_R} |H'|^2 dA \leq D_{\tilde{f}}. \quad (40)$$

No compatibility between the separately chosen lifts is required.

Set $x = h(0)$, $y = H(0)$, and take the constant ambient vector $v = H'(0)$. Since $\Pi \circ H = h$, this choice satisfies $\lambda_x^F(d\Pi_y v) = 1$. Choose $\epsilon > 0$ such that the closed ambient ϵ -neighborhood of K is compact and contained in Ω . Since $R < \infty$, there is $t_0 > 0$ with $t_0 R |v| < \epsilon$. For $0 \leq t < t_0$ define

$$h_t(z) := \Pi(H(z) + tzv), \quad z \in \mathbb{D}_R.$$

This is holomorphic on the whole open disk and fixes x . At zero its derivative satisfies the exact identity

$$(h_t)_z(0) = d\Pi_y((1+t)H'(0)) = (1+t)h_z(0), \quad \lambda_x^F((h_t)_z(0)) = 1+t. \quad (41)$$

The pullback Hermitian form $\Pi^*\omega$ has bounded coefficients on the chosen compact ambient neighborhood. After decreasing t_0 to at most 1, the area density of h_t is consequently bounded by

$$C(|H'(z)|^2 + |v|^2)$$

for a constant C independent of z and t . This function is integrable on $\mathbb{D}_R$, by (40) and $R < \infty$. The densities converge pointwise to the density of h as $t \downarrow 0$, so dominated convergence gives $\text{Area}_R(h_t) \rightarrow \text{Area}_R(h)$. Finally, source dilation gives

$$G_t(0) = h(0), \quad \lambda_{h(0)}^F((G_t)_z(0)) = 1, \quad \text{Area}_{(1+t)R}(G_t) = \text{Area}_R(h_t).$$

In fact, $(G_t)_z(0) = h_z(0)$ for this choice of v ; only its scalar component is needed in the continuation argument. Restriction to any radius strictly between R and $(1+t)R$ gives the closed-disk extension required for an admissible competitor. $\square$

5. COMPACTNESS, MARKED VALUES, AND FINITE-AREA DISKS

The main proof uses compactness to obtain a principal map in the original source coordinate and a rational chain retaining two marked values. We obtain both from one fixed-mark exhaustion statement, with a joint area bound for the principal map and the attached spheres. Point selection then gives the weighted derivative bound for an individual finite-area disk. Throughout this section, X is a compact complex manifold equipped with a smooth Hermitian metric ω .

5.1. Finite-area removal

Finite area fills the punctures of a principal limit, including infinity when its source is $\mathbb{C}$. We first prove removal so that the compactness statement can include the extended principal map.

With $g_\omega(v, w) = \omega(v, Jw)$, the energy convention in the compactness and removal results used below is

$$\|df\|_{L^2}^2 = \int (|f_x|_{g_\omega}^2 + |f_y|_{g_\omega}^2) dA = 2 \int f^* \omega = 2 \text{Area}(f), \quad (42)$$

because $f_y = Jf_x$. Thus their energy hypotheses are equivalent to our area hypotheses, with the fixed factor 2.

Proof of Theorem 2.4. Finite area gives $\int_{0 < |z| < r} f^* \omega \rightarrow 0$ as $r \downarrow 0$, so every sufficiently small concentric annulus has small energy by (42). Compactness of X supplies completeness of the target metric and uniform continuity of its complex structure. The removal theorem of Ivashkovich and Shevchishin [36, Corollary 3.6] therefore gives a continuous extension at the puncture. In a target chart about the limiting value, the coordinate functions extend holomorphically by the Riemann

removable singularity theorem. For an entire map, $\zeta = 1/z$ identifies infinity with a puncture and preserves area, so the same argument applies there. See also [45, Theorem 4.2.1]. $\square$

5.2. Marked compactness on an exhaustion

Spheres obtained by rescaling near energy concentration points go back to Sacks and Uhlenbeck [52, pp. 2–3 and Section 4]; Gromov’s compactness theory records how such components attach [29]. We use the theorem of Ivashkovich and Shevchishin [37]. For precise locators, we refer to its preprint formulation, whose title and numbering differ from the published version: Definition 1.6, Theorem 1.1, and the marked-point construction in Remark 2 following that theorem [36].

The *principal map* is the limit in the original source coordinate and may be constant. A *bubble* is a sphere obtained by rescaling; the finite-area extension of an entire principal map is also a sphere, but need not be a bubble.

A *ghost sphere* is a constant sphere component retaining marks or separating nodes. It has at least three special points, namely marks or nodes; the fixed-coordinate root disk has no such stability requirement. The dual graph records the attachments, with one vertex for each component and one edge for each node. Here it is a rooted tree.

The following proposition combines the unmarked principal limit needed for the area profile with the marked limit needed for incidence. Fixed distinct source marks suffice for both applications. Constant components inside concentration trees must still be retained, since they can carry a marked value or join nonconstant components.

Proposition 5.1. *Let $R_j > 0$ and let $f_j : U_j \rightarrow X$ be holomorphic, where $U_j \subset \mathbb{C}$ is open and $\mathbb{D}_{R_j} \Subset U_j$. Suppose $\sup_j \text{Area}_{R_j}(f_j) < \infty$, and set $A_\infty = \liminf_j \text{Area}_{R_j}(f_j)$. Choose $R_\infty \in (0, \infty]$ with $R_\infty \leq \liminf_j R_j$, interpreting $\mathbb{D}_\infty$ as $\mathbb{C}$. Fix finitely many pairwise distinct points $a_1, \dots, a_m \in \mathbb{D}_{R_\infty}$, allowing $m = 0$. After first passing to a subsequence realizing A_∞ and then to a further subsequence, the following data exist simultaneously.*

- (i) *A finite set $\Sigma_\infty \subset \mathbb{D}_{R_\infty}$ and a holomorphic principal map $f_\infty : \mathbb{D}_{R_\infty} \rightarrow X$ satisfy*

$$f_j \longrightarrow f_\infty \quad \text{smoothly on compact subsets of } \mathbb{D}_{R_\infty} \setminus \Sigma_\infty$$

in the original source coordinate.

- (ii) *Finitely many holomorphic sphere components $u_\alpha : \mathbb{P}^1 \rightarrow X$, possibly constant, form trees attached to the principal root at points of Σ_∞ . Each attached tree contains a nonconstant component; each constant sphere is stable. Let N_α be the set of nodes on the component, and let z_α be the root attachment point of the tree containing it. There are source reparametrizations $\varphi_{j,\alpha}$, defined near every compact subset of $\mathbb{P}^1 \setminus N_\alpha$ for all sufficiently large j , such that*

$$f_j \circ \varphi_{j,\alpha} \longrightarrow u_\alpha \quad \text{smoothly}, \quad \varphi_{j,\alpha} \longrightarrow z_\alpha \quad \text{locally uniformly}$$

on $\mathbb{P}^1 \setminus N_\alpha$.

- (iii) *Root, spheres, nodes and marks form one nodal map. Writing u_v for its component maps, p_{vw}, p_{wv} for the two points of a node, and p_ℓ on v_ℓ for the limiting ℓ th mark, one has*

$$u_v(p_{vw}) = u_w(p_{wv}), \quad f_j(a_\ell) \longrightarrow u_{v_\ell}(p_\ell).$$

A marked value therefore propagates along any path of constant components to its first nonconstant component.

- (iv) *The areas, counted with the multiplicities of the maps, satisfy*

$$\text{Area}_{R_\infty}(f_\infty) + \sum_{\alpha} \text{Area}(u_\alpha) \leq A_\infty. \tag{43}$$

Proof. Retain a subsequence whose total areas converge to A_∞ . The proof first selects the principal map on the whole exhaustion, then makes one marked extraction on a fixed disk, and finally counts their areas together.

The original-coordinate principal map.

A diagonal weak selection of the area measures on compact subdisks gives a Radon measure μ on $\mathbb{D}_{R_\infty}$. For every continuous compactly supported χ with $0 \leq \chi \leq 1$,

$$\int \chi d\mu = \lim_j \int \chi f_j^* \omega \leq A_\infty,$$

so $\mu(\mathbb{D}_{R_\infty}) \leq A_\infty$ by exhaustion. Fix $p > 2$ and an area threshold $\epsilon_0 > 0$ small enough for the local compactness result [36, Lemma 3.1 and Corollary 3.3], using (42), and set

$$\Sigma_\infty = \{z : \mu(\{z\}) \geq \epsilon_0\}.$$

The mass bound makes this set finite. For $z \notin \Sigma_\infty$, choose a source disk with μ -null boundary and μ -mass less than ϵ_0 . Weak convergence puts its area below that threshold for all large j . The cited result gives strong local $W^{1,p}$ convergence on a smaller disk; Sobolev embedding gives uniform convergence, and target charts and Cauchy estimates give smooth convergence. A diagonal choice on a countable cover gives f_∞ on the complement of Σ_∞ in the original coordinate. Its local definitions agree because they are limits of the same subsequence. Smooth convergence and exhaustion give $\text{Area}(f_\infty) \leq A_\infty$. Theorem 2.4 fills the finitely many punctures without changing area.

One marked extraction and its root.

Choose a fixed disk $\mathbb{D}_r$ containing all concentration points and source marks, and then choose $r < \rho < r' < R_\infty$. Apply marked compactness to the already selected subsequence on $\mathbb{D}_{r'}$, and retain the limit over $\mathbb{D}_\rho$, whose interior contains all the concentration points and source marks. The standard complex structure is fixed on this source disk, so the boundary-annulus nondegeneration condition is automatic [36, Definition 1.7 and Theorem 1.1]. The positive collar between $\mathbb{D}_\rho$ and $\partial\mathbb{D}_{r'}$ keeps this application strictly inside the available domains. The target is compact, and the energy bound follows from (42).

The fixed-domain construction uses the original coordinate away from concentration points and creates sphere components by rescaling shrinking regions at those points [36, proof of Theorem 1.1, Case 1, and the remark following its proof]. Its root coordinate is identified with the original disk: the holomorphic identification agrees with the identity on an outer open set and hence everywhere. Uniqueness of the already selected limit then identifies the root map with f_∞ off Σ_∞ , and holomorphic extension identifies it throughout $\mathbb{D}_\rho$. We extend this root by the global map f_∞ .

The marked construction in Remark 2 following the same theorem retains the marked evaluations, inserting stable constant components when a mark approaches a node in a rescaled coordinate. Nonconstant components can attach only in Σ_∞ , since convergence is smooth elsewhere. Nor can a tree of constant components attach to the root. If it had $g \geq 1$ vertices and m_z marks, its $g - 1$ internal edges and one edge to the root would contribute $2(g - 1) + 1$ nodal special points. Stability would therefore give

$$3g \leq 2(g - 1) + 1 + m_z, \quad m_z \geq g + 1 \geq 2.$$

All these marks would have the same original-coordinate limit, contrary to the fixed distinct marks. Thus all attached trees lie over Σ_∞ and contain a nonconstant sphere. The shrinking-region construction gives the reparametrizations and their collapse asserted in (ii), including for internal ghosts. The compactness theorem gives finiteness of the marked tree.

By Definition 1.6 of the cited preprint, all components belong to one continuous nodal limit. Its two branches therefore have equal values at every node. Marked convergence gives the second identity in (iii); repeatedly applying the node identity propagates a marked value through constant components. No derivative at a marked point is asserted to survive.

Joint global area accounting.

Choose a compact subset $Q \subset \mathbb{D}_{R_\infty} \setminus \Sigma_\infty$ and compact subsets $Q_\alpha \subset \mathbb{P}^1 \setminus N_\alpha$ on the sphere components. The fixed extraction represents the Q_α in mutually disjoint shrinking source regions

at separated scales. For all large j these regions also avoid Q , even when Q extends outside the retained disk $\mathbb{D}_\rho$, because Q stays a positive distance from Σ_∞ . Taking the sequence limit first gives

$$\int_Q f_\infty^* \omega + \sum_\alpha \int_{Q_\alpha} u_\alpha^* \omega = \lim_j \left(\int_Q f_j^* \omega + \sum_\alpha \int_{\varphi_{j,\alpha}(Q_\alpha)} f_j^* \omega \right) \leq A_\infty.$$

Now exhaust the global principal domain minus Σ_∞ and each sphere minus its nodes. The omitted points have zero area, and all omitted source areas are nonnegative. This proves (43). Constant components contribute zero; multiplicities are retained because we integrate pullbacks by the component maps. $\square$

Taking no marks gives exactly the principal compactness statement used in the continuation argument.

Proof of Proposition 2.3. Apply Proposition 5.1 with $m = 0$ and omit the nonnegative sphere areas in (43). Its principal map and convergence have the required properties. $\square$

5.3. Two marked values and incidence

Fixing the complete first jet at 0 does not ensure that a nonconstant bubble retains the prescribed tangent direction. In the following example, the marked value lies on the bubble, while that direction remains on the principal component. On the unit disk, with the product Fubini–Study metric, consider

$$\widehat{f}_j(z) := ([jz^2 : 1], [z : 1]) : \mathbb{D} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

The areas are uniformly bounded, and in the affine coordinates $([w : 1], [v : 1])$ the complete first jet is fixed:

$$\widehat{f}_j(0) = ([0 : 1], [0 : 1]), \quad (\widehat{f}_j)_z(0) = (0, 1).$$

Away from zero, the principal limit is

$$\widehat{f}_\infty(z) = ([1 : 0], [z : 1]).$$

Its extension has value $([1 : 0], [0 : 1])$ at zero, different from the marked value. Rescaling by $z = \zeta/\sqrt{j}$ instead gives

$$\widehat{f}_j(\zeta/\sqrt{j}) = ([\zeta^2 : 1], [\zeta/\sqrt{j} : 1]) \longrightarrow ([\zeta^2 : 1], [0 : 1]).$$

The prescribed second-factor direction remains on the residual principal component. Correspondingly, $B_1(\widehat{f}_j)/|(\widehat{f}_j)_z(0)|_\omega \rightarrow \infty$.

Marked and nodal values retain incidence. Lemma 2.5 gives a chain joining two compact target sets with controlled total area. It supplies the compactness alternative in Section 2 and the final passage to the sharp area bound.

Proof of Lemma 2.5. Pass first to a subsequence realizing the original lower limit $A_\infty := \liminf_j \text{Area}_{R_j}(f_j)$. Apply Proposition 5.1 with $R_\infty = \infty$ and the fixed marks $0, 1$. It supplies an entire principal map f_∞ and one finite marked tree, with the joint area bound (43). Theorem 2.4 fills infinity, producing the principal sphere. Replace the principal root of the marked limit by this extension and retain its attached sphere trees. Each mark lies either on the principal sphere or on one of these trees. Its marked value lies in K_0 or K_1 because these sets are closed. On the resulting finite tree, take the path between the marked components and omit constant components. Nodal equality gives nonconstant sphere maps $u_1, \dots, u_\ell$ with

$$u_1(\mathbb{P}^1) \cap K_0 \neq \emptyset, \quad u_\ell(\mathbb{P}^1) \cap K_1 \neq \emptyset, \quad u_i(\mathbb{P}^1) \cap u_{i+1}(\mathbb{P}^1) \neq \emptyset \quad (1 \leq i < \ell).$$

There is at least one such component since $K_0 \cap K_1 = \emptyset$. The components on the path are selected from the global principal sphere and the sphere components of the fixed rooted limit. The joint area estimate (43), after omitting unused components of nonnegative area, gives

$$\sum_{i=1}^{\ell} \int_{\mathbb{P}^1} u_i^* \omega \leq A_{\infty}.$$

The path belongs to the single marked limit supplied by the proposition. Components arising near the moving outer boundary are not needed for its incidence or area estimate. $\square$

5.4. Rescaling and finite-area weighted derivatives

Brody reparametrization [6, Lemma 2.1], in the affine form described by Duval [20, Section 1], turns divergence of B_R into a nonconstant entire limit. For a single finite-area disk, the positive rescaled area cannot persist in shrinking boundary collars; this gives Lemma 2.7.

Lemma 5.2. *Let $f_j : U_j \rightarrow X$ be holomorphic maps, where $U_j \subset \mathbb{C}$ is open and $\overline{\mathbb{D}}_{R_j} \Subset U_j$, and suppose $B_{R_j}(f_j) \rightarrow \infty$. Then after affine source rescaling, a subsequence converges locally smoothly on $\mathbb{C}$ to a nonconstant entire map $u : \mathbb{C} \rightarrow X$. If in addition $\sup_j \text{Area}_{R_j}(f_j) < \infty$, then*

$$\text{Area}(u) \leq \liminf_j \text{Area}_{R_j}(f_j).$$

Proof. Under the additional area hypothesis, first pass to a subsequence for which $\text{Area}_{R_j}(f_j)$ converges to the lower limit of the original area sequence. In either case, choose w_j maximizing $(R_j - |z|)|(f_j)_z(z)|$, and set $d_j := R_j - |w_j|$, $\Lambda_j := |(f_j)_z(w_j)|$. Then

$$d_j \Lambda_j = B_{R_j}(f_j) \rightarrow \infty, \quad |(f_j)_z(z)| \leq 2\Lambda_j \quad (|z - w_j| < d_j/2).$$

Indeed, $|z - w_j| < d_j/2$ implies $R_j - |z| > d_j/2$. Maximality then gives $(d_j/2)|(f_j)_z(z)| \leq d_j \Lambda_j$, which is the second bound. The rescaled map

$$u_j(\zeta) := f_j(w_j + \zeta/\Lambda_j), \quad |\zeta| < \frac{1}{2}d_j \Lambda_j,$$

satisfies $|(u_j)_\zeta|_{\omega} \leq 2$ and $|(u_j)_\zeta(0)|_{\omega} = 1$. The source radii $d_j \Lambda_j/2$ tend to infinity. On every fixed compact source disk, the derivative bound gives equicontinuity, and compactness of X gives a uniformly convergent subsequence. A diagonal selection and Cauchy estimates in local target charts give a smooth limit $u : \mathbb{C} \rightarrow X$ with $|u_\zeta(0)|_{\omega} = 1$; hence u is nonconstant. Under the additional area hypothesis, conformal invariance gives, for each fixed $S > 0$,

$$\text{Area}_S(u) = \lim_j \text{Area}_S(u_j) \leq \lim_j \text{Area}_{R_j}(f_j) = \liminf_j \text{Area}_{R_j}(f_j).$$

Letting $S \rightarrow \infty$ proves the area bound. $\square$

Proof of Lemma 2.7. Suppose $B_R(h) = \infty$. As $s \uparrow R$, the quantities $B_s(h)$ increase to $B_R(h)$: for each fixed $z \in \mathbb{D}_R$, $(s - |z|)|h_z(z)|_{\omega}$ tends to $(R - |z|)|h_z(z)|_{\omega}$ once $s > |z|$, and the suprema increase. Choose $s_j \uparrow R$ with $B_{s_j}(h) \rightarrow \infty$, and apply the point-selection construction of Lemma 5.2 to $h|_{\mathbb{D}_{s_j}}$. These restrictions extend holomorphically past their closed disks. The construction gives centers $w_j \in \mathbb{D}_{s_j}$ and numbers $\Lambda_j = |h_z(w_j)|_{\omega}$ such that

$$(s_j - |w_j|)\Lambda_j \rightarrow \infty, \quad u_j(\zeta) := h(w_j + \zeta/\Lambda_j) \rightarrow u(\zeta)$$

smoothly on compact subsets of $\mathbb{C}$, where $|u_\zeta(0)|_{\omega} = 1$. Because $s_j \leq R$, we have $\Lambda_j \rightarrow \infty$. The derivative of the fixed map h is bounded on each compact subdisk, so $|w_j| \rightarrow R$.

Choose $S > 0$ with $\text{Area}_S(u) > 0$. For all large j , the disk $\mathbb{D}_{S/\Lambda_j}(w_j)$ lies in $\mathbb{D}_{s_j}$, and conformal invariance and smooth convergence give

$$\int_{\mathbb{D}_{S/\Lambda_j}(w_j)} h^* \omega = \text{Area}_S(u_j) \rightarrow \text{Area}_S(u) > 0.$$

On the other hand, $|w_j| \rightarrow R$ and $S/\Lambda_j \rightarrow 0$ place these disks in arbitrarily thin boundary collars of $\mathbb{D}_R$. Since $\text{Area}_R(h) < \infty$, $\int_{\mathbb{D}_R \setminus \overline{\mathbb{D}_s}} h^* \omega \rightarrow 0$ as $s \uparrow R$. This contradicts the preceding positive limit. $\square$

6. APPLICATIONS

The relative theorem recovers the rational connectedness of Fano manifolds [7, 43]: two general points can be joined by a rational curve. Indeed, choose a positive-Ricci metric as in Proposition 3.1. For any quasifibration $q : X \dashrightarrow Y$ with $\dim Y > 0$, restrict to dense open subsets on which q is proper with connected fibers. Generic smoothness [34, III, Corollary 10.7] makes its general full fiber F smooth and compact. Theorem 1.1 gives a rational curve meeting F but not contained in it. Campana’s criterion [7, Proposition 2.7] then gives rational chain connectedness, and smoothing chains in characteristic zero gives rational connectedness [2].

The pointwise construction also recovers Mori’s characterization of projective space by an ample tangent bundle [47]; for an earlier characteristic-zero proof, see [51]. Let X be a connected smooth complex projective n -fold with T_X ample. The case $n = 1$ is immediate. For $n \geq 2$, the determinant $K_X^{-1} = \det T_X$ is ample [32], so X is Fano and Theorem 1.2 supplies a rational curve. The classical two-point degree reduction gives one with $-K_X \cdot C \leq n + 1$ [15, Section 5, Theorem 3, first step]. Ampleness then forces equality and the minimal-curve conditions of [15, Section 5, Corollary (a),(c)]. Mori’s classification argument identifies X with $\mathbb{P}^n$ [15, Section 6]. These remaining algebraic steps take place over $\mathbb{C}$.

ACKNOWLEDGMENTS

We are grateful to Franc Forstnerič for introducing us to Oka geometry and for his encouragement. We also thank Jun-Muk Hwang and Mihai Păun for their helpful suggestions.

S.-Y. Xie acknowledges partial support from the National Key R&D Program of China under Grants No. 2023YFA1010500 and No. 2021YFA1003100, and from the National Natural Science Foundation of China under Grants No. 12288201 and No. 12471081, as well as support from the Xiaomi Young Talents Program.

AI USE DISCLOSURE

The central mathematical ideas and overall approach are due to the authors. AI tools assisted with manuscript preparation and revision. The authors take full responsibility for the paper’s content and the accuracy of its references.

REFERENCES

- [1] A. Alarcón and F. Forstnerič, *Oka-1 manifolds*, Math. Z. **311** (2025), Paper No. 33; doi:10.1007/s00209-025-03833-4.
- [2] C. Araujo and J. Kollár, *Rational curves on varieties*, in K. Böröczky, J. Kollár, and T. Szamuely (eds.), *Higher dimensional varieties and rational points*, Bolyai Society Mathematical Studies **12**, Springer, 2003, 13–68; doi:10.1007/978-3-662-05123-8_3.
- [3] B. Berndtsson and J.-P. Rosay, *Quasi-isometric vector bundles and bounded factorization of holomorphic matrices*, Ann. Inst. Fourier (Grenoble) **53** (2003), no. 3, 885–901; doi:10.5802/aif.1964.
- [4] R. Bhatia and T. Jain, *Higher order derivatives and perturbation bounds for determinants*, Linear Algebra Appl. **431** (2009), no. 11, 2102–2108; doi:10.1016/j.laa.2009.07.003.
- [5] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), 201–248; doi:10.1090/S1056-3911-2012-00574-8.
- [6] R. Brody, *Compact manifolds and hyperbolicity*, Trans. Amer. Math. Soc. **235** (1978), 213–219; doi:10.1090/S0002-9947-1978-0470252-3.
- [7] F. Campana, *Connexité rationnelle des variétés de Fano*, Ann. Sci. École Norm. Sup. (4) **25** (1992), 539–545; doi:10.24033/asens.1658.

- [8] F. Campana, *Algebraicity of foliations on complex projective manifolds, applications*, Electron. Res. Arch. **30** (2022), 1187–1208; doi:10.3934/era.2022063.
- [9] J. Cao and M. Păun, *Remarks on relative canonical bundles and algebraicity criteria for foliations in Kähler context*, preprint, 2025; arXiv:2502.02183v2.
- [10] Y. Chen, J. E. Fornæss, and S.-Y. Xie, *Generating all Ahlfors currents by a single entire curve*, arXiv:2511.12473v1, 2025.
- [11] K. Cho, Y. Miyaoka, and N. I. Shepherd-Barron, *Characterizations of projective space and applications to complex symplectic manifolds*, in *Higher dimensional birational geometry*, Adv. Stud. Pure Math. **35**, Mathematical Society of Japan, 2002, 1–88; doi:10.2969/aspm/03510001.
- [12] O. Debarre, *Variétés de Fano*, Séminaire Bourbaki, Exp. 827, Astérisque **245** (1997), 197–221; published text.
- [13] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, 2001; doi:10.1007/978-1-4757-5406-3.
- [14] J.-P. Demailly, *Complex analytic and differential geometry*, online book, version of June 21, 2012; available online.
- [15] M. Demazure, *Caractérisations de l'espace projectif (conjectures de Hartshorne et de Frankel), d'après Shigefumi Mori*, Séminaire Bourbaki, Exp. 544 (1979/80), Lecture Notes in Math. **842**, Springer, Berlin, 1981, 11–19; published text.
- [16] F. Docquier and H. Grauert, *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, Math. Ann. **140** (1960), 94–123; doi:10.1007/BF01360084.
- [17] Y.-H. Du, B. Guo, and S.-Y. Xie, *Analytic and algebraic Oka-1 approximation for smooth projective morphisms with rationally connected fibers*, arXiv:2609.11883v1, 2026.
- [18] Y.-H. Du, B. Guo, and S.-Y. Xie, *Smooth quartic hypersurfaces of dimension at least three are Oka*, to appear soon (available upon request).
- [19] J. Duval, *Sur le lemme de Brody*, Invent. Math. **173** (2008), no. 2, 305–314; doi:10.1007/s00222-008-0119-9.
- [20] J. Duval, *Around Brody lemma*, in *Hyperbolicity properties of algebraic varieties*, Panoramas et Synthèses **56** (2021), 1–12; published text.
- [21] F. Forstnerič, *Manifolds of holomorphic mappings from strongly pseudoconvex domains*, Asian J. Math. **11** (2007), 113–126; doi:10.4310/AJM.2007.v11.n1.a11.
- [22] F. Forstnerič, *The Oka principle for sections of stratified fiber bundles*, Pure Appl. Math. Q. **6** (2010), no. 3, 843–874; doi:10.4310/PAMQ.2010.v6.n3.a11.
- [23] F. Forstnerič, *Oka manifolds: From Oka to Stein and back*, with an appendix by F. Lárusson, Ann. Fac. Sci. Toulouse Math. (6) **22** (2013), no. 4, 747–809; doi:10.5802/afst.1388.
- [24] F. Forstnerič, *Stein manifolds and holomorphic mappings: The homotopy principle in complex analysis*, second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **56**, Springer, Cham, 2017; doi:10.1007/978-3-319-61058-0.
- [25] F. Forstnerič, *From Stein manifolds to Oka manifolds: the h-principle in complex analysis*, in *Proceedings of the International Congress of Mathematicians 2026*, Vol. 4: Invited Lectures (Sections 5–8), SIAM, 2026, 657–677; doi:10.1137/25M1802808.
- [26] A. Fujiki, *Deformation of uniruled manifolds*, Publ. Res. Inst. Math. Sci. **17** (1981), no. 2, 687–702; doi:10.2977/PRIMS/1195185269.
- [27] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, in *Algebraic geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math. **62**, Part 2, American Mathematical Society, 1997, 45–96; doi:10.1090/pspum/062.2/1492534.
- [28] H. Grauert, *Analytische Faserungen über holomorph-vollständigen Räumen*, Math. Ann. **135** (1958), 263–273; doi:10.1007/BF01351803.
- [29] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347; doi:10.1007/BF01388806.
- [30] A. Grothendieck, *Sur la classification des fibrés holomorphes sur la sphère de Riemann*, Amer. J. Math. **79** (1957), no. 1, 121–138; doi:10.2307/2372388.
- [31] B. Guo and S.-Y. Xie, *Universal holomorphic maps with slow growth I: an algorithm*, Math. Ann. **389** (2024), 3349–3378; doi:10.1007/s00208-023-02719-2.
- [32] R. Hartshorne, *Ample vector bundles*, Publ. Math. IHÉS **29** (1966), 63–94; doi:10.1007/BF02684806.
- [33] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in Math. **156**, Springer-Verlag, Berlin, 1970; doi:10.1007/BFb0067839.
- [34] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, 1977; doi:10.1007/978-1-4757-3849-0.
- [35] D. T. Huynh and S.-Y. Xie, *On Ahlfors currents*, J. Math. Pures Appl. (9) **156** (2021), 307–327; doi:10.1016/j.matpur.2021.05.007.
- [36] S. Ivashkovich and V. Shevchishin, *Gromov compactness theorem for stable curves*, preprint, 1999; arXiv: math/9903047v1 [math.DG].

- [37] S. Ivashkovich and V. Shevchishin, *Gromov compactness theorem for J -complex curves with boundary*, Int. Math. Res. Not. IMRN (2000), no. 22, 1167–1206; doi:10.1155/S1073792800000581.
- [38] E. Jovinelly, B. Lehmann, and E. Riedl, *Optimal bounds in bend-and-break*, Forum Math. Pi **14** (2026), Paper No. e16; doi:10.1017/fmp.2026.10033.
- [39] S. Kaliman and M. Zaidenberg, *Algebraic Gromov’s ellipticity of cubic hypersurfaces*, Proc. Steklov Inst. Math. **329** (2025), 79–87; doi:10.1134/S008154382560067X.
- [40] S. Kebekus, *Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron*, in *Complex geometry*, Springer, 2002, 147–155; doi:10.1007/978-3-642-56202-0_10.
- [41] K. Kodaira, *On Kähler varieties of restricted type*, Proc. Natl. Acad. Sci. USA **40** (1954), no. 5, 313–316; doi:10.1073/pnas.40.5.313.
- [42] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **32**, Springer-Verlag, Berlin, 1996; doi:10.1007/978-3-662-03276-3.
- [43] J. Kollár, Y. Miyaoka, and S. Mori, *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. **36** (1992), no. 3, 765–779; doi:10.4310/jdg/1214453188.
- [44] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics **134**, Cambridge University Press, 1998; doi:10.1017/CBO9780511662560.
- [45] D. McDuff and D. Salamon, *J -holomorphic curves and quantum cohomology*, University Lecture Series **6**, American Mathematical Society, 1994; revised edition, 1995; doi:10.1090/ulect/006.
- [46] Y. Miyaoka and S. Mori, *A numerical criterion for uniruledness*, Ann. of Math. (2) **124** (1986), 65–69; doi:10.2307/1971387.
- [47] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** (1979), no. 3, 593–606; doi:10.2307/1971241.
- [48] S. Mori, *Cone of curves, and Fano 3-folds*, in *Proceedings of the International Congress of Mathematicians (Warszawa, 1983)*, Vol. 1, PWN, Warszawa, and North-Holland, Amsterdam, 1984, pp. 747–752; official proceedings.
- [49] A. Nicolau, *The corona property for bounded analytic functions in some Besov spaces*, Proc. Amer. Math. Soc. **110** (1990), 135–140; doi:10.1090/S0002-9939-1990-1017007-X; author-hosted full text.
- [50] W. Ou, *A characterization of uniruled compact Kähler manifolds*, preprint, 2025; arXiv:2501.18088v1.
- [51] T. Peternell, *Manifolds of semi-positive curvature*, in F. Catanese and C. Ciliberto (eds.), *Transcendental methods in algebraic geometry*, Lecture Notes in Math. **1646**, Springer, Berlin, 1996, 98–142; doi:10.1007/BFb0094303.
- [52] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. (2) **113** (1981), no. 1, 1–24; doi:10.2307/1971131.
- [53] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) **88** (1968), no. 1, 62–105; doi:10.2307/1970556.
- [54] Y.-T. Siu, *Dynamic multiplier ideal sheaves and the construction of rational curves in Fano manifolds*, in M. Passare (ed.), *Complex analysis and digital geometry*, Acta Universitatis Upsaliensis, Skr. Uppsala Univ. C Organ. Hist. **86**, Uppsala Universitet, 2009, 323–360.
- [55] Y.-T. Siu and S.-T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980), no. 2, 189–204; doi:10.1007/BF01390043.
- [56] N. Wiener and P. Masani, *The prediction theory of multivariate stochastic processes. I. The regularity condition*, Acta Math. **98** (1957), 111–150.
- [57] H. Wu and S.-Y. Xie, *Entire curves generating all shapes of Nevanlinna currents*, J. Lond. Math. Soc. (2) **111** (2025), no. 5, Paper No. e70177; doi:10.1112/jlms.70177.
- [58] S.-Y. Xie, *Entire curves producing distinct Nevanlinna currents*, Int. Math. Res. Not. IMRN (2024), no. 10, 8350–8362; doi:10.1093/imrn/rnad255.
- [59] S.-Y. Xie and S. Zhao, *Periods of Oka $K3$ surfaces are dense*, arXiv:2608.24392v1, 2026.
- [60] X. Yang, *RC-positivity and the generalized energy density I: Rigidity*, J. Differential Geom. **128** (2024), no. 3, 1315–1347; doi:10.4310/jdg/1729092462; arXiv:1810.03276v1.
- [61] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), 339–411; doi:10.1002/cpa.3160310304.
- [62] S.-T. Yau, *Survey on partial differential equations in differential geometry*, in S.-T. Yau (ed.), *Seminar on differential geometry*, Ann. of Math. Stud. **102**, Princeton University Press, Princeton, NJ, 1982, 3–71; digital edition: doi:10.1515/9781400881918-002.