# *Universal Holomorphic Maps with Slow Growth II. Functional Analysis Methods*

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ABSTRACT. By means of hypercyclic operator theory, we complement our previous results on hypercyclic holomorphic maps between complex Euclidean spaces having slow growth rates, by showing *abstract abundance* rather than *explicit existence*. Next, we establish that, in the space of holomorphic maps from $\mathbb{C}^n$ to any connected Oka manifold $Y$, equipped with the compact-open topology, there exists a *dense* subset consisting of common *frequently hypercyclic* elements for all nontrivial translation operators. This is new even for $n = 1$ and $Y = \mathbb{C}$.

## 1. INTRODUCTION

On the space of entire holomorphic functions $\mathcal{H}(\mathbb{C})$, equipped with the compact-open topology, the translation operator $\mathsf{T}_a f(\bullet) = f(\bullet + a)$ acts linearly for every $a \in \mathbb{C}$. Birkho  [Bir29] discovered that $\mathcal{H}(\mathbb{C})$ contains *universal* elements $f$, in the sense that a certain countable set of translations $\{\mathsf{T}_{a_i} f\}_{i=1}^{\infty}$ is dense in $\mathcal{H}(\mathbb{C})$. In fact, for any $\mathsf{T}_a$ with $a \neq 0$, Birkho  can make $f \in \mathcal{H}(\mathbb{C})$ a *hypercyclic* element for $\mathsf{T}_a$; that is, the iterated orbit $\{\mathsf{T}_a^i f = \mathsf{T}_{i \cdot a} f\}_{i \geqslant 1}$ is dense. This surprising phenomenon opened the door for a new branch of functional analysis/dynamical systems called linear chaos [GE99, BM09, GEPM11], according to which infinite dimensional linear dynamics may exhibit exotic patterns.

By complex analysis, a universal entire function $f$ must be transcendental, and hence its asymptotic growth rate has to be larger than that of any polynomial. One might expect that the minimal possible growth rate of $f$ shall be considerably larger than this obvious bound. However, a surprising result [DR83] shows the contrary.

***Theorem 1.1 (Duĭos Ruis).*** *For a continuous increasing function $\phi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0}$ growing faster than any polynomials*

$$\lim_{r\to\infty} \phi(r)/r^N = \infty \quad (\forall\, N \geqslant 1), \tag{1.1}$$

*there exists a universal entire function $f$ with slow growth*

$$|f(z)| \leqslant \phi(|z|) \quad (\forall\, z \in \mathbb{C}).$$

The original exposition in [DR83] is not easy to decipher. A breakthrough [CS91] of Chan and Shapiro contributed an alternative proof by functional analysis, and their method has been gradually absorbed, simplified, and became standard in hypercyclic operator theory (cf. [DSW97] and the references therein).

In a similar flavor, Dinh and Sibony [DS20, Problem 9.1] raised a problem of seeking minimal growth rate of universal meromorphic functions $g : \mathbb{C} \to \mathbb{CP}^1$ in terms of the Nevanlinna-Shimizu-Ahlfors characteristic function

$$T_g(r) = \int_1^r \frac{\mathrm{d}t}{t} \int_{\mathbb{D}_t} g^*\omega_{\mathsf{FS}},$$

where $\omega_{\mathsf{FS}}$ is the Fubini-Study form on $\mathbb{CP}^1$ giving unit area $\int_{\mathbb{CP}^1} \omega_{\mathsf{FS}} = 1$, and $\mathbb{D}_t \subset \mathbb{C}$ is the disc centered at the origin with radius $t$. An optimal answer was obtained as the statement below.

***Theorem 1.2*** ([CHX23]). *For any positive continuous nondecreasing function*

$$\psi : \mathbb{R}_{\geqslant 1} \to \mathbb{R}_+ \quad \text{satisfying} \lim_{r\to+\infty} \psi(r) = +\infty,$$

*there exists some universal meromorphic function $g : \mathbb{C} \to \mathbb{CP}^n$ with slow growth*

$$T_g(r) \leqslant \psi(r) \cdot \log r \quad (\forall\, r \geqslant 1).$$

By a constructive algorithm [GX23], Theorems 1.1, 1.2 have been generalized to the higher-dimensional case as follows.

***Definition 1.3.*** Let $X$ be a topological space. Then, a continuous operator $\mathsf{T} : X \to X$ is called *hypercyclic* if some element $x \in X$ has dense orbit

$$\mathsf{orb}(x, \mathsf{T}) := \{x, \mathsf{T}^1(x), \mathsf{T}^2(x), \mathsf{T}^3(x), \ldots\}$$

in $X$. Such an $x$ is called a hypercyclic element for $\mathsf{T}$. The set that consists of all hypercyclic elements for $\mathsf{T}$ is denoted by $\mathsf{HC}(\mathsf{T})$.

***Theorem 1.4*** ([GX23]). *Given any transcendental growth function $\phi$ growing faster than any polynomial, for countably many directions $\{\theta_i\}_{i\geqslant 1}$ in the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$, there exist universal holomorphic maps $F : \mathbb{C}^n \to \mathbb{C}^m$ with slow growth*

$$\|F(z)\| \leqslant \phi(\|z\|) \quad (\forall\, z \in \mathbb{C}^n),$$

*such that $F$ are hypercyclic for all $\mathsf{T}_a$, where $a$ are in some ray $\mathbb{R}_+ \cdot \theta_i$.*

***Theorem 1.5*** ([GX23]). *Let $n \leqslant m$ be two positive integers. Let $\psi : \mathbb{R}_{\geqslant 1} \to \mathbb{R}_+$ be a continuous nondecreasing function tending to infinity. The associated Nevanlinna characteristic function for $F \in \mathsf{Hol}(\mathbb{C}^n, \mathbb{CP}^m)$ is defined by*

$$T_F(r) := \int_1^r \frac{\mathrm{d}t}{t^{2n-1}} \int_{\{\|z\|<t\}} F^*\omega_{\mathsf{FS}} \wedge \alpha^{n-1} \quad (\forall\, r \geqslant 1),$$

*where $\omega_{\mathsf{FS}}$ is the Fubini-Study metric on $\mathbb{CP}^m$, and $\alpha := dd^c\|z\|^2$ is a positive $(1,1)$ form. Then, for countably many given directions $\{\theta_i\}_{i\geqslant 1}$ in the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$, there exist universal entire holomorphic maps $f : \mathbb{C}^n \to \mathbb{CP}^m$ with slow growth*

$$T_f(r) \leqslant \psi(r) \cdot \log r \quad (\forall\, r \geqslant 1),$$

*such that $f$ are hypercyclic for all nontrivial translations $\mathsf{T}_a$, where $a$ are in some ray $\mathbb{R}_+ \cdot \theta_i$.*

The constructive approach of [GX23] has the advantage of writing explicitly universal holomorphic maps $F$ having slow growth, from which delicate information can be read o . For instance, we can show the following result.

***Theorem 1.6*** ([GX23]). *Given any transcendental growth function $\phi$ of the shape (1.1), there exists some universal entire holomorphic function $F \in \mathcal{H}(\mathbb{C})$ with slow growth*

$$|F(z)| \leqslant \phi(|z|) \quad (\forall\, z \in \mathbb{C}),$$

*such that the set $I \subset \mathbb{S}^1$ of hypercyclic directions of $F$ is uncountable.*

However, our constructive method fails to tell the "density" of such $F$ in Theorem 1.4. Our original motivation in this work was to understand Theorem 1.4 by means of hypercyclic operator theory (cf. [BM09, GEPM11]). Indeed, by implementing functional analysis methods (cf. [CS91, DSW97]), we can show that such $F$ are reasonably abundant.

Let $\phi$ be given as (1.1). Consider the set

$$\mathsf{S}_\phi := \{f \in \mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m) : \|f(z)\| \leqslant \phi(\|z\|),\ \forall\, z \in \mathbb{C}^n\}$$

of entire functions having slow growth with respect to $\phi$. Hence, we can rephrase Theorem 1.4 as

$$\mathsf{A}_\phi := \bigcap_{k\geqslant 1} \bigcap_{r\in\mathbb{R}_+} \mathsf{HC}(\mathsf{T}_{r\cdot e^{[k]}}) \cap \mathsf{S}_\phi \neq \emptyset \quad (\text{given } e^{[k]} \in \mathbb{S}^{2n-1} \text{ for } k \geqslant 1).$$

Can we say more about $\mathsf{A}_\phi$? First, it is not everywhere dense, since we can take a holomorphic map $G \in \mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m)$ with $\|G(1)\| > \phi(1) + 2$. Hence, the open neighborhood

$$\{h \in \mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m) : \sup_{z\in\bar{\mathbb{B}}(0,2)} \|h(z) - G(z)\| < 1\}$$

of $G$ has empty intersection with $\mathsf{S}_\phi$. Nevertheless, $\mathsf{A}_\phi$ is reasonably dense near $\mathbf{0} \in \mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m)$.

***Theorem A.*** *For arbitrary countable vectors $\{e^{[k]}\}_{k\geqslant 1}$ in $\mathbb{S}^{2n-1}$ and for $\phi$ as (1.1), there exists a Banach space $\mathbf{B}$ and also a continuous linear injective map $\iota : \mathbf{B} \to \mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m)$ with dense image, such that, for some $G_\delta$-dense subset $G$ of $\mathbf{B}$, it holds that*

$$\iota(G \cap \mathbb{B}(0,1)) \subset \bigcap_{k\geqslant 1} \bigcap_{r\in\mathbb{R}_+} \mathsf{HC}(\mathsf{T}_{r\cdot e^{[k]}}) \cap \mathsf{S}_\phi,$$

*where $\mathbb{B}(0,1) \subset \mathbf{B}$ is the unit ball centered at $0$.*

Moreover, we can generalize Theorem A to an infinite-dimensional case. The insight comes from the tradition in several complex variables: Oka's Jokû-Iko principle (cf. [Oka36, Nog16]). We can imagine that the countably many directions $\{e^{[k]}\}_{k\geqslant 1}$ in $\mathbb{C}^n$ are just the "shadows" or projections of the infinite coordinate directions of $\mathbb{C}^\infty$ onto the finite-dimensional subspace $\mathbb{C}^n \hookrightarrow \mathbb{C}^\infty$; thus, we can guess that a certain analogous statement holds true for "$\mathsf{Hol}(\mathbb{C}^\infty, \mathbb{C}^\infty)$." However, a proper definition of "$\mathsf{Hol}(\mathbb{C}^\infty, \mathbb{C}^\infty)$" requires much e ort (see Section 2.2*2.2*).

***Convention.*** We abuse the notation $\mathsf{S}_\phi$ to denote the subsets of $\mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m)$, or $\mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\ell^2(\mathbb{C}), \ell^2(\mathbb{C}))$ (see (2.3)), which consist of elements $f$ satisfying $\|f(z)\| \leqslant \phi(\|z\|)$ for all possible $z$ according to the context. Let $\mathbb{C}^{\mathbb{N}}_c$ be the subset of $\ell^2(\mathbb{C})$ consisting of elements$(x_i)_{i\geqslant 1}$ such that all but finitely many $x_i$ are equal to 0.

***Theorem B.*** *For arbitrary countably many nonzero vectors $\{a^{[k]}\}_{k\geqslant 1}$ in $\mathbb{C}^{\mathbb{N}}_c$, the set of common hypercyclic elements in $\mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\ell^2(\mathbb{C}), \ell^2(\mathbb{C}))$ for the translation operators $\{\mathsf{T}_{a^{[k]}}\}_{k\geqslant 1}$, having slow growth with respect to $\phi$, is nonempty:*

$$\bigcap_{k\geqslant 1} \mathsf{HC}(\mathsf{T}_{a^{[k]}}) \cap \mathsf{S}_\phi \neq \varnothing.$$

*This set also contains a certain image $\iota(\mathbb{B}(0,1)\cap G)$, where $\iota : \mathbf{B} \hookrightarrow \mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\ell^2(\mathbb{C}), \ell^2(\mathbb{C}))$ is a continuous linear injective map from some Banach space $\mathbf{B}$, and where $\mathbb{B}(0,1) \subset \mathbf{B}$ is the unit ball centered at $0$, and where $G \subset \mathbf{B}$ is some $G_\delta$-dense subset.*

In hypercyclic operator theory, it is known that $\bigcap_{a\in\mathbb{C}\setminus\{0\}} \mathsf{HC}(\mathsf{T}_a) \neq \varnothing$ (cf. [GEPM11, Example 11.21]; see also the discussion [GEPM11, p. 330]). A breakthrough of Costakis and Sambarino shows, moreover, the abundance.

***Theorem 1.7*** ([CS04]). *The set $\bigcap_{a\in\mathbb{C}\setminus\{0\}} \mathsf{HC}(\mathsf{T}_a)$ of common hypercyclic elements for all nontrivial translation operators contains a $G_\delta$-dense subset of $\mathcal{H}(\mathbb{C})$.*

Hence, it is natural to ask whether or not $\bigcap_{a\in\mathbb{C}\setminus\{0\}} \mathsf{HC}(\mathsf{T}_a) \cap \mathsf{S}_\phi \neq \varnothing$ for any $\phi$ satisfying (1.1). The answer is *No* [GX23, Theorem D], by Nevanlinna theory (cf. [NW14, Ru21]).

***Theorem 1.8*** ([GX23]). *There exists some transcendental growth function $\phi$ of the shape (1.1), such that for any entire holomorphic function $F \in \mathcal{H}(\mathbb{C})$ with slow growth*

$$|F(z)| \leqslant \phi(|z|) \quad (\forall\, z \in \mathbb{C}),$$

*the set* $\{a \in \mathbb{S}^1 : F$ *is hypercyclic for* $\mathsf{T}_a\}$ *must have Hausdor dimension zero.*

We can extend Theorem 1.8 as follows.

***Theorem C.*** *For any integer* $n \geqslant 1$, $m \geqslant 1$, *there exists some transcendental growth function* $\hat{\phi}$ *of the shape* (1.1)*, such that for every* $F \in \mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m)$ *with slow growth*

$$\|F(z)\| \leqslant \hat{\phi}(\|z\|) \quad (\forall\, z \in \mathbb{C}^n),$$

*the set*

$$I_F = \{a \in \mathbb{S}^{2n-1} : F \text{ is hypercyclic for } \mathsf{T}_a\}$$

*must have zero Lebesgue measure.*

For $n = 1$, there holds a finer result (see Observation 3.10).

***Question 1.9.*** Denote

$$\mathcal{T} = \{\phi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0} \mid \phi \text{ is a continuous function satisfying (1.1)}\}.$$

Given positive integers $n, m \geqslant 1$, what is the value of

$$\inf_{\phi \in \mathcal{T}} (\sup\{\dim_{\mathcal{H}} I_F : F \in \mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m) \cap \mathsf{S}_\phi\})?$$

Here, $\dim_{\mathcal{H}}$ stands for the Hausdor dimension.

***Question 1.10.*** Let $\phi$ be given as (1.1). Find some su cient (respectively, necessary) condition on an uncountable direction, set $I \subset \mathbb{S}^{2n-1}$ so (respectively, such) that $\bigcap_{a \in I} \mathsf{HC}(\mathsf{T}_a) \cap \mathsf{S}_\phi \neq \varnothing$ (compared with Theorem 1.4).

***Definition 1.11*** ([For09]). A complex manifold $Y$ is an Oka manifold if every holomorphic map from a neighbourhood of a compact convex set $K$ in a Euclidean space $\mathbb{C}^n$ (for any $n \in \mathbb{Z}_+$) to $Y$ is a uniform limit on $K$ of entire maps $\mathbb{C}^n \to Y$.

Known examples of Oka manifolds include $\mathbb{C}^n$, $\mathbb{CP}^n$, complex tori, complex homogeneous manifolds [Gra57], elliptic manifolds [Gro89], hyperquadrics [KK08], smooth toric varieties [Lár11], "ball complements" [Kus23], and so on [For17, For23].

Zajac [Zaj16] gave a full characterization of hypercyclic composition operators acting on holomorphic functions on Stein manifolds. Kusakabe [Kus17] considered universal holomorphic maps from Stein manifolds to Oka manifolds rather than to $\mathbb{C}$. By the argument of [BG17], one can show that the $G_\delta$-density of common hypercyclic elements in the space $\mathsf{Hol}(\mathbb{C}^n, Y)$ of holomorphic maps from $\mathbb{C}^n$ to any connected Oka manifold $Y$ with respect to translation operators $\mathsf{T}_a : f(\bullet)$ maps to $f(\bullet + a)$ for all $a \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ (see Theorem 1.13).

Next, we establish the abundance of common frequently hypercyclic elements in $\mathsf{Hol}(\mathbb{C}^n, Y)$ with respect to $\mathsf{T}_a : f(\bullet) \mapsto f(\bullet + a)$ for all $a \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. A source of inspiration is [Bay16], in which Bayart established the $G_\delta$-genericity of hypercyclic elements for high-dimensional families of operators in the framework of $F$-spaces enjoying linear structure. Note that most Oka manifolds have no linear structure, while satisfying various equivalent Runge-type approximation properties (cf. [For06, For17]).

***Definition 1.12*** *(*[BM09, GEPM11]*)***.** For a subset $A$ of natural integers $\mathbb{N}$, the lower density of $A$ is defined as

$$\underline{\mathrm{dens}}(A) := \liminf_{N\to+\infty} \frac{\#\{n \leqslant N : n \in A\}}{N}.$$

We call $x \in \mathsf{HC}(\mathsf{T})$ a *frequently hypercyclic element* for $\mathsf{T}$, if for any nonempty open subset $U$ of the ambient space, the set $\{n \in \mathbb{N} : \mathsf{T}^n x \in U\}$ has positive lower density. We denote by $\mathsf{FHC}(\mathsf{T})$ the set of frequently hypercyclic elements for $\mathsf{T}$.

The notion of lower density goes back at least to Voronin Universality Theorem about the Riemann zeta-function [Vor75].

***Theorem D.*** *Let $Y$ be a connected Oka manifold, and $n \geqslant 1$ an integer. Then, in $\mathsf{Hol}(\mathbb{C}^n, Y)$ with the compact-open topology, $\bigcap_{b\in\mathbb{C}^n\setminus\{\mathbf{0}\}} \mathsf{FHC}(\mathsf{T}_b)$ is a dense subset.*

This is a surprising result, at least to the authors. The key ingredient in the proof of Theorem D is to arrange many disjoint closed hypercubes densely in some bounded domains in $\mathbb{C}^n$, subject to a certain equidistribution criterion, such that the union of these closed hypercubes is polynomially convex. This is already an interesting (and di cult) problem on its own.

Note that a certain configuration of hypercubes also appeared in [BG17], by which Bayart and Gauthier generalized Theorem 1.7 [CS04] in higher dimensions as follows.

***Theorem 1.13*** *(*[BG17]*)***.** *Let $Y$ be a connected Oka manifold, and $n \geqslant 1$ an integer. Then, in $\mathsf{Hol}(\mathbb{C}^n, Y)$ with the compact-open topology,*

$$\bigcap_{a\in\mathbb{C}^n\setminus\{\mathbf{0}\}} \mathsf{HC}(\mathsf{T}_a) \quad \text{contains a } G_\delta\text{-dense subset.}$$

The purpose of using hypercubes rather than balls is to simplify the argument of polynomial convexity (see Lemma 4.5). In fact, the use of unions of hypercubes appeared at the very beginning of several complex variables (see, e.g., [Oka36]). The original statement of [BG17] only concerns $Y = \mathbb{C}$, while the proof actually works for any connected Oka manifold $Y$, since $\mathsf{Hol}(\mathbb{C}^n, Y)$ is separable (see Remark 4.9), and since Runge's approximation property holds for any holomorphic map from a neighborhood of a polynomially convex set $K$ in $\mathbb{C}^n$ to $Y$ (see Remark 4.3), thanks to the fundamental work of Forstnerič [For06] asserting that each Oka manifold $Y$ satisfies the following *BOPA property* ([For17, p. 258]).

> *Every continuous map $f_0 : X \to Y$ from a Stein space $X$ that is holomorphic on a neighborhood of a compact $\mathcal{O}(X)$-convex subset $K \subset X$ can be deformed to a holomorphic map $f_1 : X \to Y$ by a homotopy of maps that are holomorphic near $K$ and arbitrarily close to $f_0$ on $K$.*

To prove Theorem D, we have to manipulate hypercubes carefully on every $\mathsf{Q}(0, R+N) \setminus \bar{\mathsf{Q}}(0, R)$ for some $N > 0$ independent of $R \gg 1$, while for Theorem 1.13, one has more flexibility about placing hypercubes outside $\bar{\mathsf{Q}}(0, R)$ for every $R \gg 1$. Therefore, we cannot directly use the construction of [BG17] for Theorem D, while we can provide an alternative proof of Theorem 1.13 by our hypercubes arrangement (see Appendix A).

***Question 1.14.*** In spirit of [DS20, Problem 9.1] and [GX23, Questions 5.1–5.3], we can ask the corresponding questions for frequently hypercyclic holomorphic maps from some source space $X$ with mild symmetry to a connected Oka manifold $Y$ (cf. [Kus17, Theorem 1.4], [Xie23, Remark 3.3]).

(1) Find the minimal growth rate among entire (respectively, meromorphic) functions in $\mathsf{FHC}(\mathsf{T}_a)$ for $X = \mathbb{C}^n, Y = \mathbb{C}^m$ (respectively, $Y = \mathbb{CP}^m$), where $a \in \mathbb{C}^n \setminus \{0\}$. Note that the basic case $X = Y = \mathbb{C}$ of this problem was completely solved by [BBGE10].

(2) Determine the minimal growth rate among Nevanlinna characteristic functions $T_f(r)$ (cf. [Ru21]) associated with frequently hypercyclic (with respect to $\mathsf{T}_1$) entire curves $f : \mathbb{C} \to Y$ in a compact connected Oka manifold $Y$.

(3) Seek the minimal growth rate of common hypercyclic (respectively, frequently hypercyclic) elements for all nontrivial translation operators in the above settings (1) and (2).

Here is the structure of this paper. In Section 2, we provide some preparations. In Section 3 and 4, we establish Theorems A, B, C, and D, respectively.

## 2. Preparations

***2.1. Hypercyclic operator theory.*** We refer interested readers to the helpful expositions [BM09, GEPM11] for learning this subject.

***Definition 2.1.*** Let $X$ be a metric space. A continuous operator $\mathsf{T} : X \to X$ is called *topologically transitive*, if for any two nonempty open subsets $U$, $V$ of $X$, there exists some $n \geqslant 1$ such that $\mathsf{T}^n(U) \cap V \neq \emptyset$.

The equivalence between hypercyclicity and topological transitivity for continuous operators $\mathsf{T}$ on a separable Fréchet space is known as the *Birkho transitivity theorem* (cf. e.g., [GEPM11, p. 10]). Moreover, from the proof, one sees the following result.

***Proposition 2.2.*** *On a separable Fréchet space, any countable family of hypercyclic operators* $\{\mathsf{T}_n\}_{n\in\mathbb{N}}$ *has a dense set of common hypercyclic elements, that is,* $\bigcap_{n\in\mathbb{N}} \mathsf{HC}(\mathsf{T}_n)$ *is a* $G_\delta$*-dense subset.*

***Definition 2.3.*** Let $X$, $Y$ be two metric spaces, and $S : Y \to Y$, $T : X \to X$ be continuous maps. We say that $T$ is *quasi-conjugate* to $S$ if there exists a continuous map $\phi : Y \to X$ with dense image such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow{T} & X \end{array}.$$

***Proposition 2.4 (cf.** [GEPM11, Propositions 1.13, 1.19]**).*** *The following are preserved under quasi-conjugacy:*

(1) *topological transitivity;*
(2) *dense orbit.*

***Definition 2.5.*** A continuous operator $\mathsf{T} : X \to X$ is called *mixing* if, for any nonempty open subsets $U, V$ of $X$, there exists some $N \geqslant 1$ such that $\mathsf{T}^n(U) \cap V \neq \varnothing$ for all $n \geqslant N$.

***2.2. Analytic maps between infinite-dimensional complex Euclidean spaces.*** For our purpose, we desire a separable "$\mathsf{Hol}(\mathbb{C}^\infty, \mathbb{C}^\infty)$" containing all $\mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m)$ as subspaces. However, it is known in the literature [Din99] that the space of "holomorphic functions" on an infinite-dimensional space cannot be separable. Notice that in the finite-dimensional case, holomorphic maps $\mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m)$ are nothing but the analytic maps $\mathcal{A}(\mathbb{C}^n, \mathbb{C}^m)$. Hence, instead of "$\mathsf{Hol}(\mathbb{C}^\infty, \mathbb{C}^\infty)$", we shall consider some analytic maps "$\mathcal{A}(\mathbb{C}^\infty, \mathbb{C}^\infty)$" between infinite-dimensional complex spaces.

Denote the set of finite multi-indices by

$$\mathcal{F} = \{(i_j)_{j \in \mathbb{N}} : \text{every } i_j \in \mathbb{Z}_{\geqslant 0}, \text{ all but finite } i_j = 0\}, \tag{2.1}$$

which is countable. Each

$$K = (i_1, \ldots, i_n, 0, 0, 0, \ldots) \in \mathcal{F}$$

corresponds to a monomial

$$Z^K := z_1^{i_1} \cdots z_n^{i_n}. \tag{2.2}$$

On the infinite-dimensional $\mathbb{C}$-linear space

$$\ell^2(\mathbb{C}) = \Big\{(a_i)_{i \in \mathbb{N}} : \sum_{i \geqslant 1} |a_i|^2 < +\infty\Big\}$$

equipped with the weak topology, we say a continuous function $f : \ell^2(\mathbb{C}) \to \mathbb{C}$ is analytic if it can be written as a power series $\sum_{I \in \mathcal{F}} a_I \cdot Z^I$, $a_I \in \mathbb{C}$, which is uniformly absolutely convergent on closed balls $\bar{\mathbb{B}}(0, R)$, for all $R > 0$. The space of analytic functions on $\ell^2(\mathbb{C})$ is denoted by $\mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C})$. Also, we define analytic maps from $\ell^2(\mathbb{C})$ to $\mathbb{C}^m = \bigoplus_{d=1}^m \mathbb{C}$ as

$$\mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C}^m) := \bigoplus_{d=1}^{m} \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C})$$

by functoriality. The indicator $\mathsf{w}$ on $\mathcal{A}^{\mathsf{w}}$ emphasizes that $f : \ell^2(\mathbb{C}) \to \mathbb{C}$ is a continuous function with respect to the weak-topology of $\ell^2(\mathbb{C})$. We endow $\mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C}^m)$ with the weak-compact-open topology; that is, open sets are generated by

$$\mathcal{U}^{\mathsf{w}}(K,V) := \{f \in \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C}^m) : f(K) \subset V\},$$

for all weakly compact subsets $K$ in $\ell^2(\mathbb{C})$ and for all open subsets $V$ in $\mathbb{C}^m$.

The reason for considering the weak topology on the source space $\ell^2(\mathbb{C})$ is the following statement.

***Proposition 2.6.*** *The space $\mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C}^m)$ is separable and metrizable.*

*Proof.* Fix an increasing positive sequence $\{R_n\}_{n\geqslant 1} \nearrow +\infty$. Define

$$d(f,g) := \sum_{n=1}^{+\infty} \frac{1}{2^n} \min\{\rho_n(f-g),1\} \quad (\forall\, f,g \in \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C}^m)),$$

where

$$\rho_n(f) := \sup_{z\in\bar{\mathbb{B}}(0,R_n)} \|f(z)\|_{\mathbb{C}^m} \quad (\forall\, f \in \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C}^m)).$$

Denote

$$N_{f,R,\varepsilon} := \{h \in \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C}^m)) : \sup_{z\in\bar{\mathbb{B}}(0,R)} \|h(z)-f(z)\|_{\mathbb{C}^m} < \varepsilon\}$$
$$(\forall\, f \in \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C}^m);\ \forall\, R,\varepsilon > 0).$$

Since $\bar{\mathbb{B}}(0,R)$ is weakly compact (cf. [Bre11, Theorem 3.17]), the sets

$$\{N_{f,R_n,1/m}\}_{n,m\in\mathbb{Z}_+}$$

form a countable open neighborhood basis for each $f$. Hence, $d(\cdot,\cdot)$ is a metric on $\mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C}^m)$ inducing the same weak-compact-open topology.

The separability of $\mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}),\mathbb{C}^m)$ relies essentially on the fact that $\mathbb{C}$ contains some countable and dense subfield, say, $\mathbb{Q}_c := \mathbb{Q}(\sqrt{-1})$.

First of all, $\mathsf{Hol}(\mathbb{C}^n,\mathbb{C}^m)$ is clearly separable, since there is an obvious countable dense subset

$$\mathcal{P}_{n,m} := \{(P_1(z),\dots,P_m(z)) : P_i(z) \in \mathbb{Q}_c[z_1,\dots,z_n],\ 1 \leqslant i \leqslant m\}.$$

For $F = (\sum_{I\in\mathcal{F}} a_I^{[i]} \cdot Z^I)_{1\leqslant i\leqslant m} \in \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}), \mathbb{C}^m)$ and a given neighborhood $N_{F,R,\varepsilon}$, since each component of $F$ is uniformly absolutely convergent on closed balls $\bar{\mathbb{B}}(0,R)$, we can find a finite subset $A \subset \mathcal{F}$ such that

$$\sup_{z\in\bar{\mathbb{B}}(0,R)} \Big| \sum_{I\in\mathcal{F}\setminus A} a_I^{[k]} \cdot Z^I \Big| < \frac{\varepsilon}{2\sqrt{m}}, \qquad (1 \leqslant k \leqslant m).$$

Set $G := (\sum_{I\in A} a_I^{[i]} \cdot Z^I)_{1\leqslant i\leqslant m}$. Then, it is clear that

$$\sup_{z\in\bar{\mathbb{B}}(0,R)} \|G - F\|_{\mathbb{C}^m} \leqslant \sup_{z\in\bar{\mathbb{B}}(0,R)} \Big( \sum_{k=1}^m \Big| \sum_{I\in\mathcal{F}\setminus A} a_I^{[k]} \cdot Z^I \Big|^2 \Big)^{1/2} < \frac{\varepsilon}{2}.$$

Since $A$ is finite, we can find some $N$ such that $G$ only involves the variables $z_1, \ldots, z_N$; that is, we can regard $G \in \mathsf{Hol}(\mathbb{C}^N, \mathbb{C}^m)$. Therefore, there exists some $P \in \mathcal{P}_{N,m}$ such that

$$\sup_{z\in\bar{\mathbb{B}}(0,R)} \|G - P\|_{\mathbb{C}^m} < \frac{\varepsilon}{2}.$$

Therefore, $P \in N_{F,R,\varepsilon}$. Thus, $\bigcup_{1\leqslant i<+\infty} \mathcal{P}_{i,m}$ is a countable and dense subset of $\mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}), \mathbb{C}^m)$. ◻

In the space

$$\text{(2.3)} \quad \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}), \ell^2(\mathbb{C})) := \Big\{ (f_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}), \mathbb{C}) : \forall\, z \in \mathbb{C},\ (f_i(z))_{i\in\mathbb{N}} \in \ell^2(\mathbb{C}) \Big\},$$

endowed with the weak-compact-open topology, introduce $\mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\ell^2(\mathbb{C}), \ell^2(\mathbb{C}))$, the closure of $\bigcup_{1\leqslant m<+\infty} \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}), \mathbb{C}^m)$, which is metrizable and separable.

Recall a classical result that any compact linear operator can be approximated by a sequence of finite-rank operators, and that any limit of a sequence of finite-rank operators must be compact ([SS05]). Here, elements in $\mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}), \mathbb{C}^m)$ shall be compared with finite-rank operators, and elements in $\mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\ell^2(\mathbb{C}), \ell^2(\mathbb{C}))$ shall be compared with compact linear operators. Hence, we can show that $\mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\ell^2(\mathbb{C}), \ell^2(\mathbb{C}))$ is a proper subspace of $\mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}), \ell^2(\mathbb{C}))$.

Likewise, we can define

$$\mathcal{A}_{\mathsf{fin}}(\mathbb{C}^n, \ell^2(\mathbb{C}))$$

to be the closure of $\bigcup_{1\leqslant m<+\infty} \mathcal{A}(\mathbb{C}^n, \mathbb{C}^m)$ in $\mathcal{A}(\mathbb{C}^n, \ell^2(\mathbb{C}))$, which is also separable and metrizable. Let $\Lambda$, $J$ be two index sets counting the dimension of the source space

$$\text{(2.4)} \qquad \mathbf{F} = \begin{cases} \mathbb{C}^n, & \text{if } \Lambda = \{1, \ldots, n\}, \\ \ell^2(\mathbb{C}), & \text{if } \Lambda = \mathbb{N}, \end{cases}$$

and the target space

$$\mathbf{G} = \begin{cases} \mathbb{C}^m, & \text{if } J = \{1, \dots, m\}, \\ \ell^2(\mathbb{C}), & \text{if } J = \mathbb{N}. \end{cases}$$

As the weak topology equals the strong topology on every finite-dimensional space, we have

$$\mathcal{A}(\mathbb{C}^n, \mathbb{C}^m) = \mathcal{A}^w(\mathbb{C}^n, \mathbb{C}^m), \quad \mathcal{A}_{\mathsf{fin}}(\mathbb{C}^n, \ell^2(\mathbb{C})) = \mathcal{A}^w_{\mathsf{fin}}(\mathbb{C}^n, \ell^2(\mathbb{C})).$$

If the target space is finite dimensional, then

$$\mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\ell^2(\mathbb{C}), \mathbb{C}^m) = \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}), \mathbb{C}^m).$$

Therefore, we adopt the following notation:

$$(2.5) \quad \mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\mathbf{F}, \mathbf{G}) := \begin{cases} \mathcal{A}(\mathbb{C}^n, \mathbb{C}^m) \\ \quad = \mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m), & \Lambda = \{1, \dots, n\},\ J = \{1, \dots, m\}, \\ \mathcal{A}_{\mathsf{fin}}(\mathbb{C}^n, \ell^2(\mathbb{C})), & \Lambda = \{1, \dots, n\},\ J = \mathbb{N}, \\ \mathcal{A}^{\mathsf{w}}(\ell^2(\mathbb{C}), \mathbb{C}^m), & \Lambda = \mathbb{N},\ J = \{1, \dots, m\}, \\ \mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\ell^2(\mathbb{C}), \ell^2(\mathbb{C})), & \Lambda = \mathbb{N},\ J = \mathbb{N}. \end{cases}$$

In summary, we have the following result.

***Proposition 2.7.*** *The space $\mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\mathbf{F}, \mathbf{G})$ is separable and metrizable.*

## 3. Proofs of Theorems A, B, C

***3.1. A key lemma.*** The engine is Lemma 3.4 below, which is an infinite-dimensional version of Theorem 3.2 that is due to Desch, Schappacher, and Webb [DSW97].

***Definition 3.1.*** Let $I$ be a countable index set and $v = (v_i)_{i \in I}$ a sequence of positive numbers that is indexed by $I$. The *weighted $\ell^1(v)$ space* over $\mathbb{C}$ (respectively, $\mathbb{R}$) is

$$(3.1) \qquad \ell^1(v) = \Big\{ (\alpha_i)_{i \in I} : \alpha_i \in \mathbb{C} \text{ (respectively, } \mathbb{R}), \ \sum_{i \in I} v_i |\alpha_i| < \infty \Big\}$$

with the norm

$$\|(\alpha_i)_{i \in I}\| = \sum_{i \in I} v_i |\alpha_i|.$$

For $I = \mathbb{N}$, define the backward shift operator

$$B : \ell^1(v) \to \ell^1(v), \quad (\alpha_i)_{i \in \mathbb{N}} \mapsto (\alpha_{i+1})_{i \in \mathbb{N}}.$$

***Theorem 3.2*** *([DSW97]). If* $\sup_{i\in\mathbb{N}} v_i/(v_{i+1}) < +\infty$, *then* $e^B := \sum_{k=0}^{+\infty} B^k/k!$ *is a mixing operator on* $\ell^1(v)$.

The original statement in [DSW97, Theorem 5.2] only concerns the hypercyclicity of $e^B$, while the proof actually gives the "mixing" (see Definition 2.5), cf. [GEPM11, Theorem 8.1].

Let $\Lambda$ be a finite set $\{1, 2, \ldots, n\}$ or $\mathbb{N}$ (see (2.4)). Set the multi-index set $\mathcal{I}_c^\Lambda$ associated with $\Lambda$ as

$$\mathcal{I}_c^\Lambda = \begin{cases} \mathbb{Z}_{\geqslant 0}^n, & \text{if } \Lambda = \{1, \ldots, n\}, \\ \mathcal{F}, & \text{if } \Lambda = \mathbb{N}, \end{cases}$$

where

$$\mathcal{F} = \{(i_j)_{j\in\mathbb{N}} : \text{every } i_j \in \mathbb{Z}_{\geqslant 0} \mid \text{all but finitely many } i_j = 0\}.$$

In short, we write multi-indices $(k_i)_{i\in\Lambda} \in \mathcal{I}_c^\Lambda$ as $K \in \mathcal{I}_c^\Lambda$.

***Definition 3.3.*** Let $v = (v_K)_{K\in\mathcal{I}_c^\Lambda}$ be a positive weight. For any $s \geqslant 1$, and $K = (k_1, \ldots, k_{s-1}, k_s, k_{s+1}, \ldots)$, define

$$K_s^{\pm 1} := (k_1, \ldots, k_{s-1}, k_s \pm 1, k_{s+1}, \ldots).$$

We call

$$B_s : \ell^1(v) \to \ell^1(v), \quad (\alpha_K)_{K\in\mathcal{I}_c^\Lambda} \mapsto (\alpha_{K_s^{+1}})_{K\in\mathcal{I}_c^\Lambda}$$

*the backward shift operator with respect to the* $s$*-th index.*

Introduce a partial order "$\geqslant$" on $\mathcal{I}_c^\Lambda$ by declaring $K \geqslant K'$ if and only if each index factor of $K$ is no less than the corresponding one of $K'$. We demand

$$\sup_{K\geqslant K'} \frac{v_{K'}}{v_K} \leqslant 1 \tag{3.2}$$

to ensure that for any $a = (a_i)_{i\in\Lambda} \in \mathbb{C}_c^\Lambda$ (recall: $\mathbb{C}_c^\Lambda$ is the subset of $\mathbb{C}^\Lambda$ consisting of elements $(x_i)_{i\geqslant 1}$ such that all but finitely many $x_i = 0$), the finite sum $\sum_{i\in\Lambda} a_i B_i$ and its "exponential"

$$e^{\sum_{i\in\Lambda} a_i B_i} = \sum_{k=0}^{+\infty} \frac{\left(\sum_{i\in\Lambda} a_i B_i\right)^k}{k!}$$

are continuous on $\ell^1(v)$. Since

$$\|B_s((\alpha_K)_{K\in\mathcal{I}_c^\Lambda})\|_{\ell^1(v)} = \sum_{K\in\mathcal{I}_c^\Lambda} v_K |\alpha_{K_s^{+1}}|$$

$$\begin{aligned}&\leqslant \sum_{K\in\mathcal{I}_c^\Lambda} v_{K_s^{+1}} |\alpha_{K_s^{+1}}| \quad \text{by (3.2)}\\ &\leqslant \sum_{K\in\mathcal{I}_c^\Lambda} v_K |\alpha_K|\\ &= \|(\alpha_K)_{K\in\mathcal{I}_c^\Lambda}\|_{\ell^1(v)},\end{aligned}$$

we conclude $\|B_s\|_{\ell^1(v)\to\ell^1(v)} \leqslant 1$. Hence,

$$\begin{aligned}\|e^{\sum_{i\in\Lambda} a_i B_i}\|_{\ell^1(v)\to\ell^1(v)} &\leqslant \prod_{i\in\Lambda}\sum_{k=0}^{+\infty} \frac{\|a_i B_i\|^k_{\ell^1(v)\to\ell^1(v)}}{k!}\\ &\leqslant \prod_{i\in\Lambda}\sum_{k=0}^{+\infty} \frac{|a_i|^k}{k!} = e^{\sum_{i\in\Lambda}|a_i|} < +\infty.\end{aligned}$$

***Lemma 3.4.*** *For any positive weight $v = (v_K)_{K\in\mathcal{I}_c^\Lambda}$ satisfying (3.2), for any nonempty subset $J \subset \mathbb{N}$, and for any $(a_i)_{i\in\Lambda} \in \mathbb{C}_c^\Lambda \setminus \{\mathbf{0}\}$, the operator $T = \bigoplus_J e^{\sum_{i\in\Lambda} a_i B_i}$ is topologically transitive on $(\bigoplus_J \ell^1(v))_{\ell^2}$ (see (3.3)).*

To prove this lemma, we need the following result.

***Proposition 3.5*** *([GEPM11]–**Proposition 2.41**). For every $n \geqslant 1$, let $\mathsf{T}_n$ be a continuous operator on a separable Banach space $X_n$, with $\sup_{n\in\mathbb{N}} \|\mathsf{T}_n\| < \infty$. Let $1 \leqslant p < \infty$. Define*

$$(3.3)\qquad \Big(\bigoplus_{i=1}^{+\infty} X_i\Big)_{\ell^p} := \Big\{X = (x_1, x_2, x_3, \dots) : x_i \in X_i,\ \|X\|_{\ell^p} := \Big(\sum_{i\geqslant 1} \|x_i\|^p_{X_i}\Big)^{1/p} < \infty\Big\}.$$

*Then, $\bigoplus_{n=1}^\infty \mathsf{T}_n$ is mixing on $(\bigoplus_{n=1}^\infty X_n)_{\ell^p}$ if and only if each operator $\mathsf{T}_n$ is mixing.*

*Proof of Lemma 3.4.* First, we consider a basic case $a = (1, 0, 0, \dots)$. Rewrite

$$(3.4)\qquad \ell^1(v) = \Big(\bigoplus_{(k_2,\dots)\in\mathcal{I}_c^{\Lambda\setminus\{1\}}} \ell^1(v_{\cdot,k_2,\dots})\Big)_{\ell^1}.$$

By Theorem 3.2, $e^{B_1}$ is mixing on each factor $\ell^1(v_{\cdot,k_2,\dots})$ in (3.4). Hence, by Proposition 3.5, $e^{B_1}$ is mixing on the total space $\ell^1(v)$. Using Proposition 3.5 again, we see that the operator $\bigoplus_J e^{B_1}$ is mixing on $(\bigoplus_J \ell^1(v))_{\ell^2}$. By symmetry, for other basic cases $a = (0, \dots, 0, 1, 0, 0, \dots)$, the same statement also holds true.

Next, we deal with general cases $a = (a_i)_{i\in\Lambda} \in \mathbb{C}_c^\Lambda \setminus \{\mathbf{0}\}$. Without loss of generality, we can assume that $a_1 \neq 0$. Our strategy is to construct an auxiliary space $\ell^1(w)$ with positive weight $w = (w_K)_{K\in\mathcal{I}_c^\Lambda}$ satisfying

$$\sup_{K\in\mathcal{I}_c^\Lambda} \frac{w_K}{w_{K_1^{+1}}} < 1,$$

and a continuous map $F : \ell^1(w) \to \ell^1(v)$ with dense image such that the following diagram commutes:

$$\begin{tikzcd} \ell^1(w) \arrow[r, "e^{B_1}"] \arrow[d, "F"] & \ell^1(w) \arrow[d, "F"] \\ \ell^1(v) \arrow[r, "e^{\sum_{i\in\Lambda} a_i B_i}"] & \ell^1(v). \end{tikzcd}$$

Then, by taking the direct sum (recall (2.5)), we obtain that

$$\bigoplus_J e^{\sum_{i\in\Lambda} a_i B_i} : \Big(\bigoplus_J \ell^1(v)\Big)_{\ell^2} \longrightarrow \Big(\bigoplus_J \ell^1(v)\Big)_{\ell^2}$$

is quasi-conjugate to

$$\bigoplus_J e^{B_1} : \Big(\bigoplus_J \ell^1(w)\Big)_{\ell^2} \longrightarrow \Big(\bigoplus_J \ell^1(w)\Big)_{\ell^2}.$$

By the preceding argument, $\bigoplus_J e^{B_1}$ is mixing on $(\bigoplus_J \ell^1(w))_{\ell^2}$. Hence, by Proposition 2.4, we conclude that

$$\bigoplus_J e^{\sum_{i\in\Lambda} a_i B_i} : \Big(\bigoplus_J \ell^1(v)\Big)_{\ell^2} \longrightarrow \Big(\bigoplus_J \ell^1(v)\Big)_{\ell^2}$$

is topologically transitive.

Now, we construct such $w = (w_K)_{K\in\mathcal{I}_c^\Lambda}$ and $F : \ell^1(w) \to \ell^1(v)$.

The backward shift operator $B_s$ can therefore be extended to the larger space $\mathbb{C}^{\mathcal{I}_c^\Lambda} \supset \ell^1(v)$ naturally:

$$B_s : \mathbb{C}^{\mathcal{I}_c^\Lambda} \to \mathbb{C}^{\mathcal{I}_c^\Lambda}, \quad (\alpha_K)_{K\in\mathcal{I}_c^\Lambda} \mapsto (\alpha_{K_s^{+1}})_{K\in\mathcal{I}_c^\Lambda}.$$

Define a linear isomorphism

$$\varphi_z : \mathbb{C}^{\mathcal{I}_c^\Lambda} \to \mathbb{C}[[z_i : i\in\Lambda]], \quad (\alpha_K)_{K\in\mathcal{I}_c^\Lambda} \mapsto \sum_{K\in\mathcal{I}_c^\Lambda} \alpha_K \frac{Z^K}{K!}, \tag{3.5}$$

where $Z^K$ is defined as (2.2), and where $K!$ is understood to be the product of the factorials $i_1!\cdots i_n!$ for $K = (i_1,\ldots,i_n,0,0,\ldots)$, with the convention that $Z^K/K! = 0$ if some $i_j < 0$. Computing

$$\varphi_z(B_s((\alpha_K)_{K\in\mathcal{I}_c^\Lambda})) = \sum_{K\in\mathcal{I}_c^\Lambda} \alpha_{K_s^{+1}} \frac{Z^K}{K!} = \sum_{K\in\mathcal{I}_c^\Lambda} \alpha_K \frac{Z^{K_s^{-1}}}{K_s^{-1}!}$$

$$= \frac{\partial}{\partial z_s}\Big( \sum_{K \in \mathcal{I}_c^\Lambda} \alpha_K \frac{Z^K}{K!} \Big) = \frac{\partial}{\partial z_s} (\varphi_z((\alpha_K)_{K \in \mathcal{I}_c^\Lambda})),$$

we obtain the following statement.

***Observation 3.6.*** The diagram

$$\begin{array}{ccc} \mathbb{C}^{\mathcal{I}_c^\Lambda} & \xrightarrow{\sum_{i\in\Lambda} a_i B_i} & \mathbb{C}^{\mathcal{I}_c^\Lambda} \\ \downarrow{\scriptstyle \varphi_z} & & \downarrow{\scriptstyle \varphi_z} \\ \mathbb{C}[[z_i : i \in \Lambda]] & \xrightarrow{\sum_{i\in\Lambda} a_i \frac{\partial}{\partial z_i}} & \mathbb{C}[[z_i : i \in \Lambda]]. \end{array} \tag{3.6}$$

is commutative.

Changing coordinate system linearly by the law

$$\begin{cases} z_1 = \phi_1(u) = a_1 u_1, \\ z_i = \phi_i(u) = a_i u_1 + u_i, \quad \text{if } i \neq 1, \end{cases}$$

$\phi = (\phi_1, \phi_2, \phi_3, \dots) : \mathbb{C}^\Lambda \to \mathbb{C}^\Lambda$ induces an isomorphism between coordinate rings

$$\begin{aligned} \phi^* : \mathbb{C}[[z_i : i \in \Lambda]] &\to \mathbb{C}[[u_i : i \in \Lambda]] \\ f(z) &\mapsto f \circ \phi(u). \end{aligned}$$

By the chain rule, we have

$$\frac{\partial}{\partial u_1} = \sum_{i\in\Lambda} \frac{\partial z_i}{\partial u_1} \frac{\partial}{\partial z_i} = \sum_{i \in \Lambda} a_i \frac{\partial}{\partial z_i}.$$

Combining this with (3.6), we have the following commutative diagram:

$$\begin{array}{ccccccc} \mathbb{C}^{\mathcal{I}_c^\Lambda} & \xrightarrow{\varphi_z} & \mathbb{C}[[z_i : i \in \Lambda]] & \xrightarrow{\phi^*} & \mathbb{C}[[u_i : i \in \Lambda]] & \xleftarrow{\varphi_u} & \mathbb{C}^{\mathcal{I}_c^\Lambda} \\ \downarrow{\scriptstyle \sum_{i\in\Lambda} a_i B_i} & & \downarrow{\scriptstyle \sum_{i\in\Lambda} a_i \frac{\partial}{\partial z_i}} & & \downarrow{\scriptstyle \frac{\partial}{\partial u_1}} & & \downarrow{\scriptstyle B_1} \\ \mathbb{C}^{\mathcal{I}_c^\Lambda} & \xrightarrow{\varphi_z} & \mathbb{C}[[z_i : i \in \Lambda]] & \xrightarrow{\phi^*} & \mathbb{C}[[u_i : i \in \Lambda]] & \xleftarrow{\varphi_u} & \mathbb{C}^{\mathcal{I}_c^\Lambda}. \end{array}$$

Fix the invertible linear map $F = \varphi_z^{-1} \circ \phi^{*-1} \circ \varphi_u : \mathbb{C}^{\mathcal{I}_c^\Lambda} \to \mathbb{C}^{\mathcal{I}_c^\Lambda}$. Using the next lemma, we can choose $w = (w_K)_{K \in I}$ such that, by restriction, $F : \ell^1(w) \to \ell^1(v)$ is continuous.

***Lemma 3.7.*** *Let $\mathbb{F}$ be $\mathbb{C}$ or $\mathbb{R}$. Let $G : \mathbb{F}^I \to \mathbb{F}^I$ be a linear map such that for each unit coordinate vector $e_i = (\delta_{ij})_{j\in I}$, $G(e_i) = \sum_{j\in I} a_i^j e_j$ is a finite sum, where $\delta_{ij}$ is the Kronecker symbol. Then, for any positive weight $v = (v_i)_{i\in I}$, one can find some positive weight $w = (w_j)_{j\in I}$ such that $G : \ell^1(w) \to \ell^1(v)$ is a continuous map with operator norm $\|G\|_{\ell^1(w)\to\ell^1(v)} \leqslant 1$.*

*Proof.* For each $i \in I$, write $G(e_i) = \sum_{j\in J_i} a_i^j e_j$, where $J_i$ is a finite index set. Choose

$$w_i > \sum_{j\in J_i} |a_i^j| v_j. \tag{3.7}$$

For any $\alpha = \sum_{i\in I} \alpha_i e_i \in \ell^1(w)$, computing

$$G(\alpha) = \sum_{i\in I} \alpha_i G(e_i) = \sum_{i\in I} \alpha_i \sum_{j\in J_i} a_i^j e_j,$$

we conclude that

$$\|G(\alpha)\|_{\ell^1(v)} \leqslant \sum_{i\in I} |\alpha_i| \sum_{j\in J_i} |a_i^j| v_j \leqslant \sum_{i\in I} |\alpha_i| w_i = \|\alpha\|_{\ell^1(w)}. \qquad \square$$

Besides the requirement (3.7), we also demand that $(w_K)_{K\in\mathcal{I}_c^\Lambda}$ is increasing with respect to the first index (this can be done by selecting $w_K$ inductively in lexicographical order, or Lemma 3.8). Whence we have

$$\sup_{K\in\mathcal{I}_c^\Lambda} \frac{w_K}{w_{K_1^{+1}}} < \infty,$$

so $e^{B_1}$ is a well-defined continuous linear operator on $\ell^1(w)$. It is clear that $F$ has dense image in $\ell^1(v)$. This concludes the proof. $\square$

Recall (3.5). For a positive weight $v = (v_K)_{K\in\mathcal{I}_c^\Lambda}$, the image of $\ell^1(v) \subset \mathbb{C}^{\mathbb{N}_c^\Lambda}$ under $\varphi_z$ is

$$\varphi_z(\ell^1(v)) = \Big\{ \sum_{K\in\mathcal{I}_c^\Lambda} \alpha_K \frac{Z^K}{K!} : \sum_{K\in\mathcal{I}_c^\Lambda} |\alpha_K| v_K < +\infty \Big\}.$$

***3.2. Controlling growth rates.*** If $(v_K)_{K\in\mathcal{I}_c^\Lambda}$ grows rapidly to some order with respect to $K$, then any sequence $(\alpha_K)_{K\in\mathcal{I}_c^\Lambda}$ in $\ell^1(v)$ shall shrink reasonably fast. Hence, the corresponding power series $\sum_{K\in\mathcal{I}_c^\Lambda} \alpha_K (Z^K/K!)$ could converge, and would have slow growth rate. This insight can be achieved by careful manipulation of a standard technique in hypercyclic operator theory (cf. [CS91, DSW97, BM09, GCMGPR10, GEPM11]).

For convenience later, we desire the following result.

**Lemma 3.8.** *There is an enumeration $\iota : \mathbb{N} \xrightarrow{\sim} \mathcal{I}_c^\Lambda$ such that, if $K \geqslant K'$, then*

$$\iota^{-1}(K) \geqslant \iota^{-1}(K'). \tag{3.8}$$

*Proof.* When $\Lambda = \{1, \dots, n\}$, such an enumeration can easily be obtained, e.g., by using *degree-lexicographic order* (cf. e.g., [Eis95, p. 48]). However, when $\Lambda = \mathbb{N}$, this requires much more e ort. Fortunately, we discover an enumeration on $\mathcal{F}$ (see (2.1)) encoded in the structure of natural numbers as follows.

Let $p_i$ be the $i$-th prime number for every $i \geqslant 1$. Each positive integer $M$ can be factorized as $M = p_1^{k_{M,1}} \cdots p_n^{k_{M,n}}$ for some $n \geqslant 1$ and $k_{M,1}, \dots, k_{M,n} \geqslant 0$. Thus, we can define a map $\iota : \mathbb{N} \to \mathcal{F}$ sending each $M$ to $(k_{M,1}, \dots, k_{M,n}, 0, 0, \dots)$. By the existence and uniqueness of prime decomposition, $\iota$ is uniquely defined and is bijective. Whence we have that (3.8) holds trivially. □

**Lemma 3.9.** *Let $\phi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0}$ be a continuous function satisfying (1.1). Fix a basis*

$$\{P_K(z) = Z^K/K!\}_{K \in \mathcal{I}_c^\Lambda}$$

*of $\mathbb{C}[[z_i : i \in \Lambda]]$. Then, one can choose $v = (v_K)_{K \in \mathcal{I}_c^\Lambda}$ such that (3.2) and the following considerations hold simultaneously:*

(1) *The direct sum of maps (recall (2.5))*

$$\bigoplus_J \varphi_x : \Big(\bigoplus_J \ell^1(v)\Big)_{\ell^2} \to \mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\mathbf{F}, \mathbf{G})$$
$$((\alpha^j_{K_j})_{K_j \in \mathcal{I}_c^\Lambda})_{j \in J} \mapsto \Big(\sum_{K_j \in \mathcal{I}_c^\Lambda} \alpha^j_{K_j} P_{K_j}(x)\Big)_{j \in J}$$

*is a well-defined continuous linear map.*

(2) *This map $\bigoplus_J \varphi_x : (\bigoplus_J \ell^1(v))_{\ell^2} \to \mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\mathbf{F}, \mathbf{G})$ has dense image.*

(3) *For any $\alpha \in (\bigoplus_J \ell^1(v))_{\ell^2}$, one has*

$$\Big\| \bigoplus_J \varphi_x(\alpha)(x) \Big\|_G \leqslant \|\alpha\|_{(\bigoplus_J \ell^1(v))_{\ell^2}} \cdot \phi(\|x\|) \quad (\forall\, x \in F).$$

*Proof.*
*Step* 1. Assume for the moment $\mathbf{G} = \mathbb{C}$. With the enumeration $\iota : \mathbb{N} \xrightarrow{\sim} \mathcal{I}_c^\Lambda$ defined above, we relabel $\hat{P}_k(x) := P_{\iota(k)}(x)$ and select $\hat{v}_k := v_{\iota(k)}$ inductively for $k = 1, 2, 3, \dots$ as follows.

Since $\phi$ grows faster than any polynomial, we can find a sequence of positive radii $\{R_k\}_{k \in \mathbb{N}}$ such that

$$\phi(\|x\|) \geqslant 2^k |\hat{P}_k(x)| \quad (\forall\, \|x\| \geqslant R_k). \tag{3.9}$$

We now choose an increasing sequence $(\hat{v}_k)_{k\geqslant 1}$ inductively (hence, (3.2) is satisfied; see Lemma 3.8) obeying the extra law

$$\hat{v}_k \geqslant \max\left\{\frac{1}{\phi(0)}\sup_{\|x\|\leqslant R_k}|\hat{P}_k(x)|, \frac{1}{2^k}, \hat{v}_{k-1}\right\} \qquad (k=1,2,3,\ldots;\ \hat{v}_0 := 1). \tag{3.10}$$

Then, for any $\alpha\in\ell^1(v)$, we have

$$\begin{aligned}
&|\varphi_x(\alpha)(x)| \\
&\quad\leqslant \sum_{m:\|x\|\leqslant R_m}|\alpha_m|\cdot|\hat{P}_m(x)| + \sum_{m:R_m<\|x\|}|\alpha_m|\cdot|\hat{P}_m(x)| \\
&\quad\leqslant \sum_{m:\|x\|\leqslant R_m}|\alpha_m|\cdot\hat{v}_m\phi(0) + \sum_{m:R_m<\|x\|}|\alpha_m|\cdot\frac{1}{2^m}\phi(\|x\|) \qquad \text{by (3.10), (3.9)} \\
&\quad\leqslant \sum_{m:\|x\|\leqslant R_m}|\alpha_m|\cdot\hat{v}_m\phi(0) + \sum_{m:R_m<\|x\|}|\alpha_m|\cdot\hat{v}_m\cdot\phi(\|x\|) \qquad \text{by (3.10)} \\
&\quad\leqslant \sum_m|\alpha_m|\cdot\hat{v}_m\cdot\phi(\|x\|) \\
&\quad= \|\alpha\|_{\ell^1(v)}\cdot\phi(\|x\|).
\end{aligned}$$

This ensures that the image of $\varphi$ is contained in $\mathcal{A}^{\mathrm{w}}(\mathbf{F},\mathbb{C})$, since the power series converges as follows:

$$|\varphi_x(\alpha)(x)| \leqslant \|\alpha\|_{\ell^1(v)}\cdot\phi(\|x\|) < +\infty \qquad (x\in F).$$

Moreover, we can see that $\varphi_x:\ell^1(v)\to\mathcal{A}^{\mathrm{w}}(\mathbf{F},\mathbb{C})$ is continuous by the estimate

$$\begin{aligned}
|\varphi_x(\alpha_1)(x)-\varphi_x(\alpha_2)(x)| &= |\varphi_x(\alpha_1-\alpha_2)(x)| \\
&\leqslant \|\alpha_1-\alpha_2\|_{\ell^1(v)}\cdot\phi(R) \\
&\qquad (\forall\ \alpha_1,\alpha_2\in\ell^1(v),\ \forall\ \|x\|\leqslant R).
\end{aligned}$$

Since the set of all polynomials constitutes a dense subset of $\mathcal{A}^{\mathrm{w}}(\mathbf{F},\mathbb{C})$, we conclude that $\varphi_x$ has dense image.

*Step* 2 For general $\mathbf{G}$, we choose $v_k$ inductively as (3.9), (3.10). Likewise, our goal is to show

$$\Big\|\bigoplus_J\varphi_x(\alpha)(x)\Big\|_{\mathbf{G}} \leqslant \|\alpha\|_{(\bigoplus_J\ell^1(v))_{\ell^2}}\cdot\phi(\|x\|) \qquad (\forall\ x\in F). \tag{3.11}$$

Repeating the same argument as in *Step* 1, we can obtain for each component of $\alpha = ((\alpha^j_{K_j})_{K_j \in \mathcal{I}^\Lambda_c})_{j\in J}$ an estimate

(3.12)
$$\Big| \sum_{K_j \in \mathcal{I}^\Lambda_c} \alpha^j_{K_j} P_{K_j}(x) \Big| \leqslant \|(\alpha^j_{K_j})_{K_j\in\mathcal{I}^\Lambda_c}\|_{\ell^1(v)} \cdot \phi(\|x\|) \quad (\forall\, x \in F,\ \forall\, j \in J).$$

Hence the desired estimate (3.11) holds:

$$\begin{aligned}
\Big\| \bigoplus_J \varphi_x(\alpha)(x) \Big\|_{\mathbf{G}} &= \Big( \sum_{j\in J} \Big| \sum_{K_j\in\mathcal{I}^\Lambda_c} \alpha^j_{K_j} P_{K_j}(x) \Big|^2 \Big)^{1/2} \\
&\leqslant \Big( \sum_{j\in J} (\|(\alpha^j_{K_j})_{K_j\in\mathcal{I}^\Lambda_c}\|_{\ell^1(v)} \cdot \phi(\|x\|))^2 \Big)^{1/2} \quad \text{by (3.12)} \\
&= \Big( \sum_{j\in J} \|(\alpha^j_{K_j})_{K_j\in\mathcal{I}^\Lambda_c}\|^2_{\ell^1(v)} \Big)^{1/2} \cdot \phi(\|x\|) \\
&= \|\alpha\|_{(\bigoplus_J \ell^1(v))_{\ell^2}} \cdot \phi(\|x\|) \quad (\forall\, x \in F).
\end{aligned}$$

Therefore, $\bigoplus_J \varphi_x : (\bigoplus_J \ell^1(v))_{\ell^2} \to \mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\mathbf{F},\mathbf{G})$ is continuous and has dense image. □

***3.3. A uniform proof for Theorems A, B.*** Take a basis

$$\Big\{ P_K(z) = \frac{Z^K}{K!} \Big\}_{K\in\mathcal{I}^\Lambda_c} \qquad \text{of } \mathbb{C}[z_i : i \in \Lambda].$$

Choose a positive weight $v = (v_K)_{K\in\mathcal{I}^\Lambda_c}$ as in Lemma 3.9. Notice that for $a = (a_1,\dots,a_n,0,0,\dots)$, we have

$$\mathsf{T}_a\Big(\sum_K \alpha_K Z^K\Big) = e^{\sum_{i=1}^n a_i \partial_i}\Big(\sum_K \alpha_K Z^K\Big)$$

by the Taylor expansion. Combining with (3.6) and Lemma 3.9, we obtain the following quasi-conjugate relation for every $a^{[k]} = (a^{[k]}_i)_{i\in\Lambda} \in \mathbb{C}^\Lambda_c \setminus \{\mathbf{0}\}$, $k \geqslant 1$:

$$\begin{array}{ccc}
(\bigoplus_J \ell^1(v))_{\ell^2} & \xrightarrow{\ \bigoplus_J e^{\sum_{i\in\Lambda} a^{[k]}_i B_i}\ } & (\bigoplus_J \ell^1(v))_{\ell^2} \\
\Big\downarrow {\scriptstyle \bigoplus_J \varphi_z} & & \Big\downarrow {\scriptstyle \bigoplus_J \varphi_z} \\
\mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\mathbf{F},\mathbf{G}) & \xrightarrow{\ \mathsf{T}_{a^{[k]}}\ } & \mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\mathbf{F},\mathbf{G}).
\end{array}$$

Condition (3.2), guaranteed by Lemma 3.9, ensures that each $e^{\sum_{i\in\Lambda} a^{[k]}_i B_i}$ is continuous. Then, Lemma 3.4 applies to show that the operator

$$\bigoplus_J e^{\sum_{i\in\Lambda} a^{[k]}_i B_i} : (\bigoplus_J \ell^1(v))_{\ell^2} \to (\bigoplus_J \ell^1(v))_{\ell^2}$$

is hypercyclic. It follows from Proposition 2.2 that $\bigcap_{k\in\mathbb{N}} \mathsf{HC}(\bigoplus_J e^{\sum_{i\in\Lambda} a_i^{[k]} B_i})$ is a $G_\delta$-dense subset for any countable $\{a^{[k]}\}_{k\in\mathbb{N}} \subset \mathbb{C}_c^\Lambda$. Let $\mathbb{B}(0,1)$ be the closed unit ball of $(\bigoplus_J \ell^1(v))_{\ell^2}$. By Lemma 3.9, for any $\alpha \in \bigcap_{k\in\mathbb{N}} \mathsf{HC}(\bigoplus_J e^{\sum_{i\in\Lambda} a_i^{[k]} B_i}) \cap \mathbb{B}(0,1)$, we have

$$\Big\| \bigoplus_J \varphi_x(\alpha)(z) \Big\|_{\mathbf{G}} \leqslant \phi(\|z\|) \quad (\forall\, z \in F).$$

Since $\bigoplus_J \varphi_x$ is continuous and the orbit of $\alpha$ under each operator $e^{\sum_{i=1}^n a_i^{[k]} B_i}$ is dense, by Proposition 2.4, the orbit of $\bigoplus_J \varphi_x(\alpha)(z)$ under each operator $\mathsf{T}_{a^{[k]}}$ must be also dense in $\mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\mathbf{F},\mathbf{G})$. Hence, $\bigoplus_J \varphi_x(\alpha)(z)$ is a common hypercyclic element for all $\{\mathsf{T}_{a^{[k]}}\}_{k=1}^{+\infty}$. Summarizing, we conclude

$$\begin{aligned}\Big(\bigoplus_J \varphi_x\Big)\Big(\bigcap_{k\in\mathbb{N}} \mathsf{HC}\Big(\bigoplus_J e^{\sum_{i\in\Lambda} a_i^{[k]} B_i}\Big) \cap \mathbb{B}(0,1)\Big) \\ \subset \bigcap_{k\in\mathbb{N}} \mathsf{HC}(\mathsf{T}_{a^{[k]}}) \cap \mathsf{S}_\phi \subset \mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\mathbf{F},\mathbf{G}).\end{aligned}$$

For the finite-dimensional case where $\mathcal{A}^{\mathsf{w}}_{\mathsf{fin}}(\mathbf{F},\mathbf{G}) = \mathsf{Hol}(\mathbb{C}^n,\mathbb{C}^m)$, applying Lemma 4.1, we obtain

$$\begin{aligned}\Big(\bigoplus_J \varphi_x\Big)\Big(\bigcap_{k\in\mathbb{N}} \mathsf{HC}\Big(\bigoplus_J e^{\sum_{i\in\Lambda} a_i^{[k]} B_i}\Big) \cap \mathbb{B}(0,1)\Big) \subset \\ \subset \bigcap_{r\in\mathbb{R}_+} \bigcap_{k\in\mathbb{N}} \mathsf{HC}(\mathsf{T}_{r\cdot a^{[k]}}) \cap \mathsf{S}_\phi \subset \mathsf{Hol}(\mathbb{C}^n,\mathbb{C}^m).\end{aligned}$$

### 3.4. Proof of Theorem C.

***Observation 3.10.*** For any integer $m \geqslant 1$, there exists some transcendental growth function $\phi$ of the shape (1.1), such that for every $F \in \mathsf{Hol}(\mathbb{C},\mathbb{C}^m)$ with slow growth

$$\|F(z)\| \leqslant \phi(\|z\|) \quad (\forall\, z \in \mathbb{C}),$$

the set

$$\{a \in \mathbb{S}^1 : F \text{ is hypercyclic for } \mathsf{T}_a\}$$

has zero Hausdor dimension.

*Proof.* Take one $\phi$ satisfying Theorem 1.8. Let $F_1$ be the first coordinate factor of $F = (F_1,\ldots,F_m)$. Note that if $a \in \mathbb{S}^1$ is a hypercyclic direction for $F$, it must be also for $F_1 \in \mathcal{H}(\mathbb{C})$. By Theorem 1.8, we conclude the proof. ◻

*Proof of Theorem C.* We only need to verify the case that $n \geqslant 2$. Take a growth function $\phi$ of the shape (1.1) such that Theorem 1.8 holds true. Define $\hat{\phi}(r) = \phi(r/\sqrt{n})$. We claim that, for any $F \in \mathsf{S}_{\hat{\phi}} \cap \mathsf{Hol}(\mathbb{C}^n,\mathbb{C}^m)$, the set

$$I_F := \{e \in \mathbb{S}^{2n-1} : F \text{ is hypercyclic for } \mathsf{T}_e\} \tag{3.13}$$

of hypercyclic directions of $F$ has Lebesgue measure zero.

First, we show that $I_F$ is Lebesgue measurable. Take a dense countable subset $\{\psi_j\}_{j\geqslant 1}$ of $\mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m)$ with respect to the compact-open topology. Define

$$I(j,k) := \left\{ e \in \mathbb{S}^{2n-1} \mid \exists\, n \in \mathbb{Z}_+ : \sup_{|z|\leqslant k} |\mathsf{T}_{n\cdot e} F(z) - \psi_j(z)| < \frac{1}{k} \right\} \quad (\forall\, k, j \in \mathbb{Z}_+),$$

which are clearly open. Whence we have that $I_F = \bigcap_{j\geqslant 1} \bigcap_{k\geqslant 1} I(j,k)$ is measurable.

Next, we prove that $I_F$ has zero Lebesgue measure. For any

$$a = (a_1, \ldots, a_n) \in I_F \subset \mathbb{S}^{2n-1} \subset \mathbb{C}^n,$$

find some $|a_j| = \max_{1\leqslant i\leqslant n} |a_i|$. Then, define

$$f_a(z) := F\left(\frac{a_1}{a_j} z, \ldots, \frac{a_{j-1}}{a_j} z, z, \frac{a_{j+1}}{a_j} z, \ldots, \frac{a_n}{a_j} z\right), \tag{3.14}$$

which is clearly an element in $\mathsf{S}_\phi \cap \mathsf{Hol}(\mathbb{C}, \mathbb{C}^m)$, since $F \in \mathsf{S}_{\hat{\phi}}$ guarantees that

$$\|f_a(z)\| \leqslant \hat{\phi}\left(\left(\sum_{i=1}^{n} \left|\frac{a_i}{a_j}\right|^2 |z|^2\right)^{1/2}\right) \leqslant \hat{\phi}(\sqrt{n} \cdot |z|) = \phi(|z|).$$

Mimicking (3.13), we denote $I_{f_a} \subset \mathbb{S}^1$ the set of hypercyclic directions for $f_a$. We claim that $f_a \in \mathsf{HC}(\mathsf{T}_{a_j}) \cap \mathsf{Hol}(\mathbb{C}, \mathbb{C}^m)$. Indeed, direct computation shows

$$\begin{aligned} \mathsf{T}_{n\cdot a_j} f_a(z) &= F\left(\frac{a_1}{a_j} z + n \cdot a_1, \ldots, z + n \cdot a_j, \ldots, \frac{a_n}{a_j} z + n \cdot a_n\right) \\ &= \mathsf{T}_{n\cdot a} F\left(\frac{a_1}{a_j} z, \ldots, z, \ldots, \frac{a_n}{a_j} z\right). \end{aligned}$$

For any $g \in \mathsf{Hol}(\mathbb{C}, \mathbb{C}^m)$, we can regard it as an element $\hat{g}$ in $\mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m)$ by setting $\hat{g}(z_1, \ldots, z_n) = g(z_j)$. For any compact disc $K$ of $\mathbb{C}$ centered at zero, and any $\varepsilon > 0$, since $a \in I_F$, there is some $n \gg 1$ such that $\mathsf{T}_{n\cdot a} F(\cdot)$ approximates $\hat{g}(\cdot)$ uniformly on $K \times \cdots \times K$ ($n$ times) having error less than $\varepsilon$. Therefore, for any $z \in K$,

$$\begin{aligned} \mathsf{T}_{n\cdot a_j} f_a(z) &= \mathsf{T}_{n\cdot a} F\left(\frac{a_1}{a_j} z, \ldots, z, \ldots, \frac{a_n}{a_j} z\right) \\ &\approx \hat{g}\left(\frac{a_1}{a_j} z, \ldots, z, \ldots, \frac{a_n}{a_j} z\right) = g(z), \end{aligned}$$

that is, $\mathsf{T}_{n\cdot a_j} f_a$ can approximate $g$ on $K$ up to error $\varepsilon$.

Summarizing, $f_a \in \mathsf{S}_\phi \cap \mathsf{Hol}(\mathbb{C}, \mathbb{C}^m) \cap \mathsf{HC}(\mathsf{T}_{a_j})$, whence $a_j/|a_j| \in I_{f_a}$ by Lemma 4.1. Now we look at the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^{2n-1} \to \mathbb{CP}^{n-1}$ induced by

$$\mathbb{CP}^{n-1} := \frac{\mathbb{C}^n \setminus \{0\}}{\mathbb{C} \setminus \{0\}} \cong \frac{\mathbb{S}^{2n-1}}{\mathbb{S}^1}.$$

The $\mathbb{S}^1$ fiber of $[a] \in \mathbb{CP}^{n-1}$ is given by the image of

$$\begin{aligned} i_a : \mathbb{S}^1 &\hookrightarrow \mathbb{S}^{2n-1} \\ e^{i\theta} &\mapsto (a_1 e^{i\theta}, \ldots, a_n e^{i\theta}). \end{aligned}$$

Repeating the above argument, if $(a_1 e^{i\theta}, \ldots, a_n e^{i\theta}) \in I_F$, we set the following as in (3.14):

$$\begin{aligned} f_{e^{i\theta}a}(z) &:= F\left(\frac{a_1 e^{i\theta}}{a_j e^{i\theta}} z, \ldots, \frac{a_{j-1} e^{i\theta}}{a_j e^{i\theta}} z, z, \frac{a_{j+1} e^{i\theta}}{a_j e^{i\theta}} z, \ldots, \frac{a_n e^{i\theta}}{a_j e^{i\theta}} z\right) \\ &= F\left(\frac{a_1}{a_j} z, \ldots, \frac{a_{j-1}}{a_j} z, z, \frac{a_{j+1}}{a_j} z, \ldots, \frac{a_n}{a_j} z\right) = f_a(z). \end{aligned}$$

Then,

$$\frac{a_j e^{i\theta}}{|a_j e^{i\theta}|} = \frac{a_j}{|a_j|} \cdot e^{i\theta} \in I_{f_{e^{i\theta}a}} = I_{f_a}.$$

By Observation 3.10, $I_{f_a}$ has Hausdor dimension zero, and hence has Lebesgue measure zero.

Since the same statement holds for the intersection of $I_F$ with the Hopf fiber $\mathbb{S}^1$ above any $[a] \in \mathbb{CP}^{n-1}$, by the Fubini theorem, we conclude that $I_F \subset \mathbb{S}^{2n-1}$ must have 0 Lebesgue measure. □

## 4. PROOF OF THEOREM D

***4.1. Plan.*** Regard $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ by reading each

$$(z_m = x_{2m-1} + \sqrt{-1} \cdot x_{2m})_{1 \leqslant m \leqslant n} \in \mathbb{C}^n$$

as $(x_j)_{1 \leqslant j \leqslant 2n} \in \mathbb{R}^{2n}$. Denote the hypercubes

$$\begin{aligned} &\mathsf{Q}(0, R) := \{(x_1, \ldots, x_{2n}) : |x_i| < R,\ i = 1, \ldots, 2n\} \subset \mathbb{R}^{2n} = \mathbb{C}^n \\ &\mathsf{Q}(a, R) := a + \mathsf{Q}(0, R) \quad (\forall\, R > 0,\ a \in \mathbb{R}^{2n}). \end{aligned}$$

We will prove Theorem D as a corollary of Lemma 4.1 (2) and Theorem 4.2 below.

***Lemma 4.1.*** *Let $Y$ be a connected Oka manifold. For any $\theta \in \partial \mathsf{Q}(0,1)$ and $b \in \mathbb{R}_+ \cdot \theta$, in $\mathsf{Hol}(\mathbb{C}^n, Y)$ with the compact-open topology, one has*

(1) $\mathsf{HC}(\mathsf{T}_\theta) = \mathsf{HC}(\mathsf{T}_b)$ (*cf.* [GX23, Lemma 1.5]);

(2) $\mathsf{FHC}(\mathsf{T}_\theta) = \mathsf{FHC}(\mathsf{T}_b)$.

Note that (1) is reminiscent of [CS04, Theorem 8]. We will show (2) (see also a similar result in [BG06]) in Section 4.2***4.2***.

***Theorem 4.2.*** *Let $Y$ be a connected Oka manifold.*

*Then, $\bigcap_{\theta \in \partial \mathsf{Q}(0,1)} \mathsf{FHC}(\mathsf{T}_\theta)$ contains a dense subset of $\mathsf{Hol}(\mathbb{C}^n, Y)$ with respect to the compact-open topology.*

The proof will be reached in Section 4.4***4.4***, by manipulating a certain sophisticated configuration of closed hypercubes whose union is polynomially convex.

### ***4.2. Proof of Lemma 4.1 (2).***

By the argument of [CS04, Theorem 8], (2) follows with a minor adaptation (cf. [GX23, Lemma 1.5]). For readers' convenience, we provide the details here. (Note that this result fits into a very general property of semigroup of operators; see [CMP07].)

Fix a complete distance $d_Y$ on $Y$. Suppose $f \in \mathsf{FHC}(\mathsf{T}_\theta)$. Our goal is to show that, for any $g \in \mathsf{Hol}(\mathbb{C}^n, Y)$ and $r > 0$, we can find some set $A \subset \mathbb{N}$ with positive lower density such that

$$\max_{z \in \bar{\mathsf{Q}}(0,r)} d_Y(\mathsf{T}_{m \cdot b} f(z), g(z)) < \frac{1}{r} \quad (\forall\, m \in A). \tag{4.1}$$

Since $g$ is uniformly continuous on $\bar{\mathsf{Q}}(0, r+1)$, we can take a positive number $\delta \ll 1$ such that

$$d_Y(g(z_1), g(z_2)) < \frac{1}{3r} \tag{4.2}$$
$$(\forall\, z_1, z_2 \in \bar{\mathsf{Q}}(0, r+\delta), \text{ with } \|z_1 - z_2\| \leqslant \sqrt{n}\delta).$$

Choose su ciently many residue classes $\{[c_i]\}_{i=1}^k$ in $\mathbb{R} \cdot b / \mathbb{Z} \cdot b$ such that

$$\{[c_i]\}_{i=1}^k \quad \text{are } \delta\text{-dense} \tag{4.3}$$

in the sense that any vector $c \in \mathbb{R} \cdot b$ can be approximated by a representative $c_i'$ of some class $[c_i]$ within tiny distance $c \in \mathsf{Q}(c_i', \delta)$. This can be achieved by requiring $k \gg 1$ su ciently large.

Next, for $i = 1, 2, \ldots, k$, we subsequently select some representatives $\hat{c}_i \in \mathbb{R} \cdot b$ of $[c_i]$ far away from each other so that the closed hypercubes $\bar{\mathsf{Q}}(\hat{c}_i, r+\delta)$ are pairwise disjoint. Therefore, $\bigcup_{i=1}^k \bar{\mathsf{Q}}(\hat{c}_i, r+\delta)$ is polynomially convex (see Lemma 4.5).

Now we define a holomorphic map $g_2$ on a neighborhood of $\bigcup_{i=1}^k \bar{\mathsf{Q}}(\hat{c}_i, r+\delta)$ by translations of $g$

$$g_2 \restriction_{\bar{\mathsf{Q}}(\hat{c}_i, r+\delta)} (\bullet) := g(\bullet - \hat{c}_i) \quad (i = 1, 2, \ldots, k). \tag{4.4}$$

Since $Y$ is Oka, by the BOPA property [For17, p. 258], we can "extend" $g_2$ to a holomorphic map $\hat{g}$ from a neighborhood of a large hypercube $\bar{\mathsf{Q}}(0,R)$ with

$$R > \max_{1 \leqslant i \leqslant k} \{\|\hat{c}_i\|\} + r + \delta \tag{4.5}$$

to $Y$ within small error (see Remark 4.3)

$$d_Y(\hat{g}(z), g_2(z)) < \frac{1}{3r} \quad \Big(\forall\, z \in \bigcup_{i=1}^{k} \bar{\mathsf{Q}}(\hat{c}_i, r + \delta)\Big). \tag{4.6}$$

As $f \in \mathsf{FHC}(\mathsf{T}_\theta)$, there is some $\{n_j\}_{j=1}^{+\infty} \subset \mathbb{N}$ with positive lower density such that

$$\sup_{z \in \bar{\mathsf{Q}}(0,R)} d_Y(\mathsf{T}_{n_j \cdot \theta} f(z) \hat{g}(z)) < \frac{1}{3r} \quad (\forall\, j \geqslant 1). \tag{4.7}$$

For each $n_j$, at least one vector in $\{n_j \cdot \theta + \hat{c}_i\}_{i=1}^{k}$ is near to $0 + \mathbb{Z} \cdot b$ within the uniform distance $\delta$, because the segment between $(-n_j - \delta) \cdot \theta$ and $(-n_j + \delta) \cdot \theta$ cannot avoid all vectors in $\{\hat{c}_i + \mathbb{Z} \cdot b\}_{1 \leqslant i \leqslant k}$ by our construction (4.3). Hence, we can rewrite

$$n_j \cdot \theta + \hat{c}_{i_j} = m_j \cdot b + \delta'_j \tag{4.8}$$

for one $i_j \in \{1, 2, \ldots, k\}$, $m_j \in \mathbb{Z}$, and some residue

$$\|\delta'_j\| < \delta \|\theta\| \leqslant \sqrt{n}\delta. \tag{4.9}$$

Summarizing (4.2), (4.4), (4.6), (4.7), for any $z$ in $\bar{\mathsf{Q}}(0,r)$, we obtain

$$\begin{aligned}
\mathsf{T}_b^{m_j} f(z) &= f(m_j \cdot b + z) \\
&= f(n_j \cdot a + \hat{c}_{i_j} - \delta'_j + z) && \text{by (4.8)} \\
&\approx \hat{g}(\hat{c}_{i_j} - \delta'_j + z) && \text{by (4.5), (4.7)} \\
&\approx g_2(\hat{c}_{i_j} - \delta'_j + z) && \text{by (4.6)} \\
&= g(-\delta'_j + z) && \text{by (4.4)} \\
&\approx g(z) && \text{by (4.9), (4.2).}
\end{aligned}$$

Thus, the estimate (4.1) is guaranteed by adding up small di erences in the above three "≈."

Lastly, it is clear that for su ciently large $s$ one has $\{m_j\}_{j \geqslant s} \subset \mathbb{N}$. We claim that this set has positive lower density. A delicate place needs some caution; namely, di erent $n_j$ and $n_l$ might contribute the same $m_j = m_l$. However, noting that $\{\|c_i\| + \delta\}_{i=1}^k$ is uniformly bounded, the repetition multiplicity of each element in the sequence $\{m_j\}_{j\geqslant s}$ is uniformly bounded from above by some constant $O(1) < \infty$. Thus, we conclude the proof by (4.8) and the positive lower density of $\{n_j\}_{j\geqslant 1}$.

***Remark 4.3.*** Now we explain how to use the BOPA property in our setting. Let $F : \bigcup_i \bar{\mathsf{Q}}_i \to Y$ be a holomorphic map defined near some disjoint hypercubes $\{\bar{\mathsf{Q}}_i\}_i$. By the connectedness of $Y$, we can extend $F$ to a continuous map $\hat{F}$ from $\mathbb{C}^n$ to $Y$. Hence, by the BOPA property, we can deform $\hat{F}$ to some holomorphic map which approximates $F$ very closely on $\bigcup_i \bar{\mathsf{Q}}_i$.

Here is some extra detail. Every hypercube $\bar{\mathsf{Q}}_i$ is contractible. Hence, we can first continuously extend $F$ to the union of some disjoint neighborhoods $V_i \supset \bar{\mathsf{Q}}_i$ such that the image of each $F(\partial V_i)$ is some point $y_i \in Y$. As $Y$ is connected, we can find a continuous path $\gamma : \mathbb{R} \to Y$ passing through these points $\{y_i\}_i$, say $\gamma(i) = y_i$ for each enumeration $i$. Now, we define a continuous map $G : \bigcup_i \partial V_i \to \mathbb{R}$ mapping each $\partial V_i$ to the constant $i$. By Tietze's extension theorem, we can extend $G$ to some continuous map $\hat{G} : \mathbb{C}^n \to \mathbb{R}$. Thus, we can write a continuous extension of $F$:

$$\hat{F}(z) = \begin{cases} F(z), & \forall\, z \in \bigcup_i V_i, \\ \gamma \circ \hat{G}(z), & \forall\, z \notin \bigcup_i V_i. \end{cases}$$

### *4.3. Hypercubes arrangement.*

***Definition 4.4.*** For a compact subset $K \subset \mathbb{C}^n$, the polynomially convex hull of $K$ is

$$\hat{K} := \{w \in \mathbb{C}^n : |P(w)| \leqslant \sup_{z\in K} |P(z)|,\ \forall\, P \in \mathbb{C}[z_1, \ldots, z_n]\}.$$

Note that $K$ is called polynomially convex if $K = \hat{K}$.

***Lemma 4.5*** **(**[BG17]**–*****Lemma 2.3*****).** *For two polynomially convex compact subsets $X, Y \subset \mathbb{C}^n = \mathbb{R}^{2n}$, if they can be separated by some real a ne coordinate hyperplane $\{x \in \mathbb{R}^{2n} : x_i = a\}$, for some $1 \leqslant i \leqslant 2n$, $a \in \mathbb{R}$, then $X \cup Y$ is polynomially convex.*

The proof of the above lemma is straightforward.[1] Now, we use Lemma 4.5 to provide our key construction.

***Lemma 4.6.*** *Let $0 < \delta < \frac{1}{2}$ and $k \in \mathbb{N}$ be given. Then, one can find positive integers $N_0 = N_0(k, \delta)$, $R_0 = R_0(k, \delta)$, such that, for any integer $N \geqslant N_0$, $R \geqslant R_0$, the domain $\mathsf{Q}(0, R + N) \setminus \bar{\mathsf{Q}}(0, R)$ contains a family of pairwise disjoint closed hypercubes $\{\bar{\mathsf{Q}}(a_i, k + \delta)\}_{i\in\Gamma_{R,k,\delta}}$ with the following two properties.*

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[1]We thank an anonymous referee for pointing out that such a lemma can be traced back at least to [Kal65] as a corollary of Kallin's lemma.

(1) *For any* $\theta \in \partial \mathsf{Q}(0,1)$, *for some* $m \in \mathbb{N}$, *the hypercube* $\mathsf{Q}(m \cdot \theta, k)$ *is contained in one of* $\{\bar{\mathsf{Q}}(a_i, k+\delta)\}_{i\in\Gamma_{R,k,\delta}}$.

(2) *For any* $L \in (0, R)$, *the union of the disjoint hypercubes*

$$\bigcup_{i\in\Gamma_{R,k,\delta}} \bar{\mathsf{Q}}(a_i, k+\delta) \cup \bar{\mathsf{Q}}(0, L)$$

*is polynomially convex.*

For $m = 1, \ldots, 2n$, consider the closed cones

$$\mathsf{F}_{2m-1} := \{(x_1, x_2, \ldots, x_{2n}) : x_m \geqslant 0,\ |x_j| \leqslant |x_m|,\ \forall\, j \neq m\},$$
$$\mathsf{F}_{2m} := \{(x_1, x_2, \ldots, x_{2n}) : x_m \leqslant 0,\ |x_j| \leqslant |x_m|,\ \forall\, j \neq m\}.$$

We can use these cones to partition $\partial\mathsf{Q}(0,1)$ as the union of

$$\{\partial \mathsf{Q}(0,1) \cap \mathsf{F}_s\}_{1\leqslant s\leqslant 4n}.$$

Likewise,

$$\mathsf{Q}(0, R+N) \setminus \bar{\mathsf{Q}}(0,R) = \bigcup_{1\leqslant s\leqslant 4n} \mathsf{F}_s \cap (\mathsf{Q}(0,R+N)\setminus \bar{\mathsf{Q}}(0,R)).$$

A proof of Lemma 4.6 will be reached after the following more tractable result.

***Lemma 4.7.*** *Let* $0 < \delta < \frac{1}{2}$ *and* $k \in \mathbb{N}$ *be given. Then, one can find positive integers* $N = N(k,\delta)$ *and* $R_0 = R_0(k,\delta)$, *such that, for any* $R \geqslant R_0$, *for each* $1 \leqslant s \leqslant 4n$, *there exists a family* $\{\bar{\mathsf{Q}}(a_i^{[s]}, k+\delta)\}_{i\in\Gamma^{[s]}_{R,k,\delta}}$ *of pairwise disjoint closed hypercubes, with centers* $a_i^{[s]}$ *in* $\mathsf{F}_s \cap (\mathsf{Q}(0,R+N)\setminus\bar{\mathsf{Q}}(0,R))$, *such that the following two properties hold true:*

(1) *For any* $\theta \in \partial\mathsf{Q}(0,1)\cap\mathsf{F}_s$, *for some* $m \in \mathbb{N}$, *the hypercube* $\mathsf{Q}(m\cdot\theta, k)$ *is contained in one of* $\{\bar{\mathsf{Q}}(a_i^{[s]}, k+\delta)\}_{i\in\Gamma^{[s]}_{R,k,\delta}}$.

(2) *The union of disjoint hypercubes*

$$\bigcup_{i\in\Gamma^{[s]}_{R,k,\delta}} \bar{\mathsf{Q}}(a_i^{[s]}, k+\delta)$$

*is polynomially convex.*

*Proof.* By symmetry, we can assume $s = 1$. Take the integer

$$l := \left\lfloor \frac{4k+2}{\delta} + 1 \right\rfloor + 1$$

where $\lfloor \cdot \rfloor$ means rounding down. Select

$$\begin{aligned}
N &:= (2k+1)\cdot l^{2n-1}+1,\\
R_0 &:= (2k+1)\cdot (l^{2n-1}-2),\\
r_j &:= (2k+1)\cdot j + R \quad (j=1,\dots,l^{2n-1}),\\
A &:= \frac{2k+1}{2k+1+R},\\
\delta_0 &:= \frac{\delta}{(2k+1)\cdot l^{2n-1}+R},\\
\mathcal{A} &:= \{(1, s_2A, s_3A,\dots,s_{2n}A) : (s_2,\dots,s_{2n})\in\mathbb{Z}^{2n-1}\},\\
\mathcal{B} &:= \Big\{(0, t_2\delta_0, t_3\delta_0,\dots,t_{2n}\delta_0) : (t_2,\dots,t_{2n})\in\mathbb{Z}^{2n-1}_{\geqslant 0};\\
&\qquad\qquad\qquad\qquad t_i\leqslant l-1,\ \forall\, i=2,3,\dots,2n\Big\}.
\end{aligned}$$

Enumerate $\mathcal{B}$ as $\{b_i\}_{i=1}^{l^{2n-1}}$. Consider the following grid points on $\partial\mathsf{Q}(0,1)\cap\mathsf{F}_1$:

$$\mathcal{A}_j := (b_j+\mathcal{A})\cap\mathsf{F}_1 \quad (j=1,\dots,l^{2n-1}). \tag{4.10}$$

We claim that the family of hypercubes

$$\{\bar{\mathsf{Q}}(a_i^{[1]}, k+\delta)\}_{i\in\Gamma_{R,k,\delta}^{[1]}} := \bigcup_{1\leqslant j\leqslant l^{2n-1}} \{\bar{\mathsf{Q}}(r_j\cdot a, k+\delta)\}_{a\in\mathcal{A}_j}$$

satisfies the desired properties (1) and (2).

First, by the estimate

$$R < r_j = (2k+1)j + R \leqslant (2k+1)l^{2n-1} + R < R+N,$$

we can check that the centers are in the target region (see (4.10))

$$\{a_i^{[1]}\}_{i\in\Gamma_{R,k,\delta}^{[1]}} = \{r_j\cdot a\}_{a\in\mathcal{A}_j, 1\leqslant j\leqslant l^{2n-1}} \subset (\mathsf{Q}(0,R+N)\setminus\bar{\mathsf{Q}}(0,R))\cap\mathsf{F}_1.$$

Next, to show (1), we associate each point

$$a\in\bigcup_{1\leqslant j\leqslant l^{2n-1}}\mathcal{A}_j$$

(see (4.10)) with a smaller hypercube $\bar{\mathsf{Q}}(a,\delta_0)$. Hence, we can check that

$$\bigcup_{1\leqslant j\leqslant l^{2n-1}}\bigcup_{a\in\mathcal{A}_j}\bar{\mathsf{Q}}(a,\delta_0)\supset\partial\mathsf{Q}(0,1)\cap\mathsf{F}_1$$

by the width estimate

$$
\begin{aligned}
l \cdot \delta_0 &> \frac{4k+2}{\delta} \cdot \frac{\delta}{(2k+1)l^{2n-1}+R} \\
&= \frac{4k+2}{(2k+1)l^{2n-1}+R} \\
(*) \qquad &\geqslant \frac{2k+1}{2k+1+R} = A,,
\end{aligned}
$$

where the inequality $(*)$ is due to $R \geqslant R_0 = (2k+1) \cdot (l^{2n-1} - 2)$.

Hence, any $\theta \in \partial \mathsf{Q}(0,1) \cap \mathsf{F}_1$ is contained in some $\bar{\mathsf{Q}}(a, \delta_0)$ for $a \in \mathcal{A}_j$, $1 \leqslant j \leqslant l^{2n-1}$. Since

$$
r_j \cdot \delta_0 = ((2k+1)j + R) \cdot \frac{\delta}{(2k+1) \cdot l^{2n-1} + R} \leqslant \delta,
$$

we obtain that $r_j \cdot \theta \in \bar{\mathsf{Q}}(r_j \cdot a, \delta)$. Whence we have that

$$
\mathsf{Q}(r_j \cdot \theta, k) \subset \bar{\mathsf{Q}}(r_j \cdot a, k + \delta).
$$

Lastly, we show that the hypercubes in the family $\{\bar{\mathsf{Q}}(a_i, k+\delta)\}_{i \in \Gamma^{[1]}_{R,k,\delta}}$ are pairwise disjoint, and satisfy (2). Indeed, for every $j \in \{1, 2, \dots, l^{2n-1}\}$, since

$$
r_j \cdot A = ((2k+1)j + R) \cdot \frac{2k+1}{2k+1+R} \geqslant 2k+1 > 2(k+\delta),
$$

the hypercubes $\{\bar{\mathsf{Q}}(r_j \cdot a, k+\delta)\}_{a \in \mathcal{A}_j}$ are pairwise disjoint. Noting that centers $a$ are placed on the grid $\mathcal{A}_j$, by using Lemma 4.5 repeatedly, we see that $\bigcup_{a \in \mathcal{A}_j} \bar{\mathsf{Q}}(r_j \cdot a, k+\delta)$ is polynomially convex. Moreover, for any two distinct $j, j'$ in $\{1, 2, \dots, l^{2n-1}\}$, since

$$
|r_j - r_{j'}| = (2k+1) \cdot |j - j'| \geqslant 2k+1 > 2(k+\delta),
$$

it is clear that $\bigcup_{a \in \mathcal{A}_j} \bar{\mathsf{Q}}(r_j \cdot a, k+\delta)$ and $\bigcup_{a \in \mathcal{A}_{j'}} \bar{\mathsf{Q}}(r_{j'} \cdot a, k+\delta)$ are disjoint. Hence, by using Lemma 4.5 repeatedly, we see that

$$
\bigcup_{1 \leqslant j \leqslant l^{2n-1}} \bigcup_{a \in \mathcal{A}_j} \mathsf{Q}(r_j \cdot a, k+\delta)
$$

is polynomially convex. ∎

*Proof of Lemma 4.6.* Borrow two positive integers $N' = N'(k, \delta)$ and $R_0 = R_0(k, \delta)$ from Lemma 4.7. Select $N_0 = (4n+1) \cdot (N' + 2k + 2)$. Now, we claim that, for any $R \geqslant R_0$, $N \geqslant N_0$, there are pairwise disjoint closed hypercubes

$\{\bar{\mathsf{Q}}(a_i, k+\delta)\}_{i\in\Gamma_{R,k,\delta}}$ contained in the domain $\mathsf{Q}(0, R+N)\setminus\bar{\mathsf{Q}}(0,R)$, which satisfy (1), (2).

Take

$$R_m = R + (N' + 2k + 2)\cdot m \quad (m = 1, 2, \ldots, 4n+1).$$

By Lemma 4.7, for each $1 \leqslant m \leqslant 4n$, since

$$(R_{m+1} - k - 1) - (R_m + k + 1) = N',$$

there is a family of pairwise disjoint closed hypercubes

$$\{\bar{\mathsf{Q}}(a_i^{[m]}, k+\delta)\}_{i\in\Gamma^{[m]}_{R_m+k+1,k,\delta}},$$

with centers in $\mathsf{F}_m \cap \big(\mathsf{Q}(0, R_{m+1} - k - 1)\setminus\bar{\mathsf{Q}}(0, R_m + k + 1)\big)$, such that

(a) For any $\theta \in \partial\mathsf{Q}(0,1)\cap\mathsf{F}_m$, there exists some $l\in\mathbb{N}$ and some $i\in\Gamma^{[m]}_{R_m+k+1,k,\delta}$ with

$$\mathsf{Q}(l\cdot\theta, k) \subset \bar{\mathsf{Q}}(a_i^{[m]}, k+\delta).$$

(b) The union of $\{\bar{\mathsf{Q}}(a_i^{[m]}, k+\delta)\}_{i\in\Gamma^{[m]}_{R_m+k+1,k,\delta}}$ is polynomially convex.

Our desired family of hypercubes is nothing but

$$\{\bar{\mathsf{Q}}(a_i, k+\delta)\}_{i\in\Gamma_{R,k,\delta}} := \bigcup_{1\leqslant m\leqslant 4n}\{\bar{\mathsf{Q}}(a_i^{[m]}, k+\delta)\}_{i\in\Gamma^{[m]}_{R_m+k+1,k,\delta}}.$$

Obviously, these hypercubes are pairwise disjoint, since for each $1\leqslant m\leqslant 4n$, the family of hypercubes

$$\{\bar{\mathsf{Q}}(a_i^{[m]}, k+\delta)\}_{i\in\Gamma^{[m]}_{R_m+k+1,k,\delta}} \subset \mathsf{Q}(0, R_{m+1})\setminus\bar{\mathsf{Q}}(0, R_m)$$

are pairwise disjoint, and the domains $\{\mathsf{Q}(0,R_{m+1})\setminus\bar{\mathsf{Q}}(0,R_m)\}_{1\leqslant m\leqslant 4n}$ are also pairwise disjoint.

We can check that all the hypercubes $\{\bar{\mathsf{Q}}(a_i, k+\delta)\}_{i\in\Gamma_{R,k,\delta}}$ are lying in the domain

$$\mathsf{Q}(0, R_{4n+1})\setminus\bar{\mathsf{Q}}(0,R_1) \subset \mathsf{Q}(0, R+N)\setminus\bar{\mathsf{Q}}(0,R).$$

It is clear that (1) can be deduced from (a), since

$$\partial\mathsf{Q}(0,1) = \bigcup_{1\leqslant i\leqslant 4n}\partial\mathsf{Q}(0,1)\cap\mathsf{F}_i.$$

Now we check (2).

For any $L < R$, the coordinate hyperplane $\{x \in \mathbb{R}^{2n} : x_1 = L + 1\}$ separates $\bar{\mathsf{Q}}(0, L)$ and $\bigcup_{i\in\Gamma^{[1]}_{R_1+k+1,k,\delta}} \bar{\mathsf{Q}}(a_i^{[1]}, k+\delta)$. By (b) and Lemma 4.5, we see that the union

$$\bigcup_{i\in\Gamma^{[1]}_{R_1+k+1,k,\delta}} \bar{\mathsf{Q}}(a_i^{[1]}, k+\delta) \cup \bar{\mathsf{Q}}(0, L)$$

is polynomially convex. Subsequently, for all $j = 1, 2, \ldots, 2n$, since

$$\bigcup_{1\leqslant s\leqslant 2j-2}\ \bigcup_{i\in\Gamma^{[s]}_{R_s+k+1,k,\delta}} \bar{\mathsf{Q}}(a_i^{[s]}, k+\delta) \cup \bar{\mathsf{Q}}(0, L),$$
$$\bigcup_{i\in\Gamma^{[2j-1]}_{R_{2j-1}+k+1,k,\delta}} \bar{\mathsf{Q}}(a_i^{[2j-1]}, k+\delta)$$

can be separated by $\{x \in \mathbb{R}^{2n} : x_j = R_{2j-1}\}$, while

$$\bigcup_{1\leqslant s\leqslant 2j-1}\ \bigcup_{i\in\Gamma^{[s]}_{R_s+k+1,k,\delta}} \bar{\mathsf{Q}}(a_i^{[s]}, k+\delta) \cup \bar{\mathsf{Q}}(0, L),$$
$$\bigcup_{i\in\Gamma^{[2j]}_{R_{2j}+k+1,k,\delta}} \bar{\mathsf{Q}}(a_i^{[2j]}, k+\delta)$$

can be separated by $\{x \in \mathbb{R}^{2n} : x_j = -R_{2j}\}$, using (b) and Lemma 4.5 repeatedly, eventually, we get the polynomial convexity of

$$\bigcup_{1\leqslant s\leqslant 4n}\ \bigcup_{i\in\Gamma^{[s]}_{R_s+k+1,k,\delta}} \bar{\mathsf{Q}}(a_i^{[s]}, k+\delta) \cup \bar{\mathsf{Q}}(0, L). \qquad \square$$

***4.4. Proof of Theorem 4.2.*** First of all, we need the following result.

***Lemma 4.8 ([GEPM11]–Lemma 9.5).*** *There exist pairwise disjoint subsets $A(l, \nu)$ of $\mathbb{N}$ for all $l, \nu \in \mathbb{N}$, having positive lower density, such that, for any $n$ in $A(l, \nu)$ and another di erent $m \neq n$ in some $A(k, \mu)$, one has $n \geqslant \nu$ and $|n - m| \geqslant \nu + \mu$.*

Our goal is to show that, for any given holomorphic map $g : \mathbb{C}^n \to Y$ and $R \in \mathbb{Z}_+$, there exists a holomorphic map $f \in \bigcap_{\theta\in\partial\mathsf{Q}(0,1)} \mathsf{FHC}(\mathsf{T}_\theta)$ such that

$$\sup_{z\in\bar{\mathsf{Q}}(0,R)} d_Y(f(z), g(z)) < \frac{1}{R}. \tag{4.11}$$

***Remark 4.9.*** It is well known that, for any two smooth connected manifolds $X'$ and $Y'$, the smooth mapping space $C^\infty(X', Y')$ equipped with the compact-open topology is separable. In particular, $C^\infty(\mathbb{C}^n, Y)$ is separable, so is the closed subset $\mathsf{Hol}(\mathbb{C}^n, Y) \subset C^\infty(\mathbb{C}^n, Y)$.

*Proof of Lemma 4.8.* Take a countable dense subset $\{\psi_i\}_{i\geqslant 1}$ of $\mathsf{Hol}(\mathbb{C}^n, Y)$. For each $j, l \in \mathbb{N}$, by the uniform continuity of $\psi_j$ on the compact closed hypercube $\bar{\mathsf{Q}}(0, l+1)$, we can find a su ciently small positive number $\delta^{[l,j]} \ll 1$ such that

$$
(4.12)\qquad d_Y(\psi_j(z_1), \psi_j(z_2)) < \frac{1}{2l} \qquad (\forall\, z_1, z_2 \in \bar{\mathsf{Q}}(0, l+1),\ \text{with } z_2 \in \bar{\mathsf{Q}}(z_1, \delta^{[l,j]})).
$$

Applying Lemma 4.6 upon $\delta^{[l,j]} \ll 1$ and $l \in \mathbb{N}$, we obtain positive integers

$$
N_0^{[l,j]} = N_0^{[l,j]}(l, \delta^{[l,j]}),\ R_0^{[l,j]} = R_0^{[l,j]}(l, \delta^{[l,j]}) \quad (\forall\, l, j \in \mathbb{N}).
$$

Since there is an enumeration on the index set $\mathbb{N} \times \mathbb{N}$ (see Lemma 3.8), we can subsequently choose positive integers $N^{[l,j]} \geqslant N_0^{[l,j]}$ for each pair $(l, j)$ along the enumeration, such that $(l, 2N^{[l,j]}) \neq (l', 2N^{[l',j']})$ if $(l, j) \neq (l', j')$. Hence, Lemma 4.8 yields a family $\{A(j, 2N^{[l,j]})\}_{j,l\in\mathbb{N}}$ of pairwise disjoint subsets of $\mathbb{N}$ having positive lower density. Set

$$
B(j, 2N^{[l,j]}) := \{p \in A(j, 2N^{[l,j]}) : p \geqslant R_0^{[l,j]} + N^{[l,j]} + R\},
$$

which also has positive lower density.

Write all elements of $\bigcup_{l,j\in\mathbb{N}} B(j, 2N^{[l,j]})$ in the increasing order as

$$
(4.13)\qquad \{n_k\}_{k=1}^{+\infty}.
$$

Then, for each $n_k \in B(j, 2N^{[l,j]})$, we define $j_k = j$, $l_k = l$. Assign each $n_k$ a domain

$$
\mathsf{A}_k := \mathsf{Q}(0, n_k + N^{[l_k,j_k]}) \setminus \bar{\mathsf{Q}}(0, n_k - N^{[l_k,j_k]}) \quad (\forall\, k \geqslant 1).
$$

By Lemma 4.8, we have

$$
(4.14)\qquad \begin{aligned} n_k + N^{[l_k,j_k]} &\leqslant n_{k+1} - (2N^{[l_{k+1},j_{k+1}]} + 2N^{[l_k,j_k]}) + N^{[l_k,j_k]} \\ &< n_{k+1} - N^{[l_{k+1},j_{k+1}]} \quad (\forall\, k \geqslant 1). \end{aligned}
$$

Hence, $\mathsf{A}_k$ are pairwise disjoint for all $k \geqslant 1$.

Applying Lemma 4.6, in each $\mathsf{A}_k$, we can insert pairwise disjoint closed hypercubes

$$
\begin{aligned} \mathsf{C}_k &= \{\bar{\mathsf{Q}}(a_i^{[k]}, l_k + \delta^{[l_k,j_k]})\}_{i\in\Lambda_k} \\ &:= \{\bar{\mathsf{Q}}(a_i, l_k + \delta^{[l_k,j_k]})\}_{i\in\Gamma_{n_k - N^{[l_k,j_k]}, l_k, \delta^{[l_k,j_k]}}}, \end{aligned}
$$

such that following two properties hold:

(i) For any $\theta \in \partial \mathsf{Q}(0,1)$, for some $m \in \mathbb{N}$, the hypercube $\mathsf{Q}(m \cdot \theta, l)$ is contained in one of $\{\bar{\mathsf{Q}}(a_i^{[k]}, l_k + \delta^{[l_k,j_k]})\}_{i\in\Lambda_k}$.

(ii) Set $\mathsf{B}_k := \bigcup_{i\in\Lambda_k} \bar{\mathsf{Q}}(a_i^{[k]}, l_k + \delta^{[l_k,j_k]})$. Then, for $k=1$, $\mathsf{B}_1 \cup \bar{\mathsf{Q}}(0,R)$ is polynomially convex; for every $k \geqslant 2$, by (4.14), the union

$$\mathsf{B}_k \cup \bar{\mathsf{Q}}(0, n_{k-1} + N^{[l_{k-1},j_{k-1}]})$$

is also polynomially convex.

We can define a holomorphic map $g_k$ on a neighborhood of each $\mathsf{B}_k$ by translation:

(4.15)
$$g_k(z) := \psi_{j_k}(z - a_i^{[k]}) \quad (\forall\, k \geqslant 1,\ \forall\, i \in \Lambda_k,\ \forall\, z \in \bar{\mathsf{Q}}(a_i^{[k]}, k + \delta^{[l_k,j_k]}).$$

Since $Y$ is Oka, we can inductively use the BOPA property (see Remark 4.3) to construct a sequence of entire maps $\{f_n\}_{n\geqslant 0}$ as follows. Fix a sequence of positive numbers $\{\varepsilon_k\}_{k\geqslant 1}$ shrinking rapidly, to be specified later. Starting with $f_0 := g$, by (ii) and the BOPA property, we can find a next holomorphic map $f_1 : \mathbb{C}^n \to Y$ such that

$$\sup_{z\in\bar{\mathsf{Q}}(0,R)} d_Y(f_1(z), f_0(z)) < \varepsilon_1,$$
$$\sup_{z\in\mathsf{B}_1} d_Y(f_1(z), g_1(z)) < \varepsilon_1.$$

When $f_0, f_1, \ldots, f_k$ have been selected for $k \geqslant 1$, again we use (ii) and the BOPA property to obtain a subsequent $f_{k+1}$ with

$$\sup_{z\in\bar{\mathsf{Q}}(0,n_k+N^{[l_k,j_k]})} d_Y(f_{k+1}(z), f_k(z)) < \varepsilon_{k+1},$$
$$\sup_{z\in\mathsf{B}_{k+1}} d_Y(f_{k+1}(z), g_{k+1}(z)) < \varepsilon_{k+1}.$$

It is clear that the limit $f : \mathbb{C}^n \to Y$ of $\{f_k\}_{k\geqslant 1}$ is a well-defined holomorphic map satisfying

(4.16a)
$$\sup_{z\in\bar{\mathsf{Q}}(0,R)} d_Y(f(z), g(z)) < \sum_{i=1}^{+\infty} \varepsilon_i,$$

(4.16b)
$$\sup_{z\in\mathsf{B}_k} d_Y(f(z), g_k(z)) < \sum_{i=k}^{+\infty} \varepsilon_i.$$

Now we can subsequently choose $\varepsilon_k$ for $k = 1, 2, 3, \ldots$ shrinking rapidly, such that

(4.17)
$$\sum_{i=1}^{+\infty} \varepsilon_i < \frac{1}{R} \quad \text{and} \quad \sum_{i=k}^{+\infty} \varepsilon_i < \frac{1}{2l_k}.$$

Therefore, the obtained $f$ satisfies (4.11). Moreover, we claim that it approximates each $\psi_j$ frequently under the iterations of the translation operator $\mathsf{T}_\theta$ for any $\theta \in \partial \mathsf{Q}(0,1)$.

Indeed, for each $\psi_j$, and for any error bound $\varepsilon > 0$, by taking a large $l \in \mathbb{N}$ with $1/l < \varepsilon$, we obtain a set $B(j, 2N^{[l,j]})$ with positive lower density. Recalling (4.13), for every $n_t \in B(j, 2N^{[l,j]})$ with $t \geqslant 1$, by (i), we can find an integer $m_t$ and one hypercube $\bar{\mathsf{Q}}(a, l+\delta^{[l,j]})$ in the family $\mathsf{C}_t$ such that

$$\mathsf{Q}(m_t \cdot \theta, l) \subset \bar{\mathsf{Q}}(a, l+\delta^{[l,j]}) \subset \mathsf{Q}(0, n_t + N^{[l,j]}) \setminus \bar{\mathsf{Q}}(0, n_t - N^{[l,j]}). \tag{4.18}$$

By our previous construction, it is clear that $(l_t, j_t) = (l, j)$. Therefore, we can estimate

$$\begin{aligned}
&\sup_{z\in\bar{\mathsf{Q}}(0,l)} d_Y(\mathsf{T}_{m_t\cdot\theta} f(z), \psi_j(z)) \\
&\qquad = \sup_{z\in\bar{\mathsf{Q}}(m_t\cdot\theta,l)} d_Y(f(z), \psi_j(z - m_t\cdot\theta)) \\
&\qquad \leqslant \sup_{z\in\bar{\mathsf{Q}}(a,l+\delta^{[l,j]})} d_Y(f(z), \psi_j(z-a)) \\
&\qquad\qquad + \sup_{z\in\bar{\mathsf{Q}}(a,l+\delta^{[l,j]})} d_Y(\psi_j(z - m_t\cdot\theta), \psi_j(z-a)) && \text{by (4.18)} \\
&\qquad \leqslant \sup_{z\in\bar{\mathsf{Q}}(a,l+\delta^{[l,j]})} d_Y(f(z), \psi_j(z-a)) + \frac{1}{2l} && \text{by (4.12)} \\
&\qquad \leqslant \sum_{i=t}^{+\infty} \varepsilon_i + \frac{1}{2l}\text{by} && \text{by (4.15), (4.16)} \\
&\qquad < \frac{1}{2l} + \frac{1}{2l} = \frac{1}{l} < \varepsilon && \text{by (4.17).}
\end{aligned}$$

Lastly, (4.18) implies that $|m_t - n_t| \leqslant N^{[l,j]}$. By Lemma 4.8, any two distinct integers $n_t$, $n_{t'}$ in $B(j, 2N^{[l,j]})$ satisfy $|n_{t'} - n_t| \geqslant 4N^{[l,j]}$. Therefore,

$$|m_{t'} - m_t| = |(n_{t'} - n_t) - (n_{t'} - m_{t'}) + (n_t - m_t)| \geqslant 4N^{[l,j]} - N^{[l,j]} - N^{[l,j]} > 0.$$

Hence, the corresponding sequence of integers $\{m_t\}_{n_t \in B(j, 2N^{[l,j]})}$ has positive lower density, inherited from that of $B(j, 2N^{[l,j]})$. ◻

## Appendix A. An Alternative Proof of Theorem 1.13

The proof is inspired by that of Theorem 1.7 [CS04] in which $n = 1$, $Y = \mathbb{C}$.

By Lemma 4.1 (1), we only need to prove the following result.

***Theorem A.1.*** *Let $Y$ be a connected Oka manifold, and $n \geqslant 1$ an integer. Then, in $\mathsf{Hol}(\mathbb{C}^n, Y)$ with the compact-open topology, $\bigcap_{a\in\partial\mathsf{Q}(0,1)} \mathsf{HC}(\mathsf{T}_a)$ contains a $G_\delta$-dense subset.*

*Proof.* Take a dense subset $\{\psi_j\}_{j\geqslant 1}$ of $\mathsf{Hol}(\mathbb{C}^n, Y)$ (see Remark 4.9). For any $(a,b,k,j) \in \mathbb{N}^4$, set

$$E(a,b,k,j) = \Big\{ f \in \mathsf{Hol}(\mathbb{C}^n, Y) : \forall\, \theta \in \partial\mathsf{Q}(0,1),\ \exists \text{ integer } s = s(\theta) \leqslant a \\ \text{such that } \sup_{z\in\bar{\mathsf{Q}}(0,k)} d_Y(\mathsf{T}_{s\cdot\theta} f(z), \psi_j(z)) < \frac{1}{b} \Big\}.$$

By Definition 1.3, $\bigcap_{\theta\in\partial\mathsf{Q}(0,1)} \mathsf{HC}(\mathsf{T}_\theta) \supseteq \bigcap_{b,k,j\geqslant 1}(\bigcup_{a\geqslant 1} E(a,b,k,j))$. Hence, we only need to show that $\bigcup_{a\geqslant 1} E(a,b,k,j)$ is a dense open set.

First, every $E(a,b,k,j)$ is open. To see this, for any $f \in E(a,b,k,j)$, we have

$$\sup_{\theta\in\partial\mathsf{Q}(0,1)} \min_{1\leqslant s\leqslant a} \{ \sup_{z\in\bar{\mathsf{Q}}(s\cdot\theta,k)} d_Y(f(z), \mathsf{T}_{-s\cdot\theta}\psi_j(z))\} < \frac{1}{b}.$$

Hence, we can take a positive $\delta$ with

$$\delta < \frac{1}{b} - \sup_{\theta\in\partial\mathsf{Q}(0,1)} \min_{1\leqslant s\leqslant a} \{ \sup_{z\in\bar{\mathsf{Q}}(s\cdot\theta,k)} d_Y(f(z), \mathsf{T}_{-s\cdot\theta}\psi_j(z))\},$$

and obtain an open neighbourhood of $f$

$$N_{f,\delta,\bar{\mathsf{Q}}(0,5a\cdot k)} := \{h \in \mathsf{Hol}(\mathbb{C}^n, Y) : \sup_{z\in\bar{\mathsf{Q}}(0,5a\cdot k)} d_Y(h(z), f(z)) < \delta\},$$

which is clearly a subset of $E(a,b,k,j)$.

Next, we claim that each $\bigcup_{a\geqslant 1} E(a,b,k,j)$ is dense. We only need to show that, for any given holomorphic map $g : \mathbb{C}^n \to Y$ and $R > 0$, there exists some $F \in \bigcup_{a\geqslant 1} E(a,b,k,j)$ such that

$$\sup_{z\in\bar{\mathsf{Q}}(0,R)} d_Y(F(z), g(z)) < \frac{1}{R}. \tag{A.1}$$

By the uniform continuity of $\psi_j$ on compact sets, we can find a sufficiently small positive $\delta \ll 1$ such that

$$d_Y(\psi_j(z_1), \psi_j(z_2)) < \frac{1}{2b} \\ (\forall\, z_1, z_2 \in \bar{\mathsf{Q}}(0, k+\delta),\ \text{with } z_2 \in \bar{\mathsf{Q}}(z_1,\delta)). \tag{A.2}$$

Applying Lemma 4.6 for the $\delta$ and $k \in \mathbb{N}$, we obtain a sufficiently large number $r > \lceil R \rceil + k$ satisfying that there are finitely many pairwise disjoint closed hypercubes $\{\bar{\mathsf{Q}}(a_i, k+\delta)\}_{i\in\Gamma_{r,k,\delta}}$ outside $\mathsf{Q}(0,r)$ such that for any $\theta \in \partial\mathsf{Q}(0,1)$, there are some $m \in \mathbb{N}$ and $l \in \Gamma_{r,k,\delta}$ with

$$\mathsf{Q}(m\cdot\theta, k) \subset \bar{\mathsf{Q}}(a_l, k+\delta). \tag{A.3}$$

On some small neighborhood of the union of the disjoint hypercubes

$$\bigcup_{i\in\Gamma_{r,k,\delta}} \bar{\mathsf{Q}}(a_i, k+\delta) \cup \bar{\mathsf{Q}}(0,R),$$

which is polynomially convex by Lemma 4.6, we define

$$G(z) = \begin{cases} g(z), & \text{for } z \text{ near } \bar{\mathsf{Q}}(0,R), \\ \psi_j(z-a_i), & \text{for } z \text{ near } \bar{\mathsf{Q}}(a_i, k+\delta),\ \forall i \in \Gamma_{r,k,\delta}. \end{cases} \tag{A.4}$$

By the BOPA property of the Oka manifold $Y$ ([For17, p. 258] and Remark 4.3), we can find some holomorphic map $F : \mathbb{C}^n \to Y$ approximating $G$ within tiny deviation

$$d_Y(F(z), G(z)) < \frac{\min\left\{\frac{1}{b}, \frac{1}{R}\right\}}{2} \quad \Big(\forall\, z \in \bigcup_{i\in\Gamma_{r,k,\delta}} \bar{\mathsf{Q}}(r_d\cdot\theta_{d,l}, k+\delta) \cup \bar{\mathsf{Q}}(0,R)\Big). \tag{A.5}$$

Then, (A.1) is automatically satisfied, and $F \in \bigcup_{a\geqslant 1} E(a,b,k,j)$, because

$$\begin{aligned}
&\sup_{z\in\bar{\mathsf{Q}}(0,k)} d_Y(\mathsf{T}_{m\cdot\theta}F(z), \psi_j(z)) \\
&\quad= \sup_{z\in\bar{\mathsf{Q}}(m\cdot\theta,k)} d_Y(F(z), \psi_j(z-m\cdot\theta)) \\
&\quad\leqslant \sup_{z\in\bar{\mathsf{Q}}(a_l,k+\delta)} d_Y(F(z), \psi_j(z-a_l)) \\
&\qquad\quad + \sup_{z\in\bar{\mathsf{Q}}(a_l,k+\delta)} d_Y(\psi_j(z-m\cdot\theta), \psi_j(z-a_l)) && \text{(by (A.3))} \\
&\quad\leqslant \sup_{z\in\bar{\mathsf{Q}}(a_l,k+\delta)} d_Y(F(z), \psi_j(z-a_l)) + \frac{1}{2b} && \text{(by (A.2))} \\
&\quad= \sup_{z\in\bar{\mathsf{Q}}(a_l,k+\delta)} d_Y(F(z), G(z)) + \frac{1}{2b} && \text{(by (A.4))} \\
&\quad< \frac{1}{b} && \text{(by (A.5)).} \quad \square
\end{aligned}$$

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