Mathematische Annalen (2024) 389:3349–3378
https://doi.org/10.1007/s00208-023-02719-2

Mathematische Annalen

Universal holomorphic maps with slow growth I: an algorithm

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Received: 27 April 2023 / Revised: 12 August 2023 / Accepted: 2 September 2023 /
Published online: 25 September 2023

Abstract

We design an *Algorithm* to fabricate universal holomorphic maps between any two complex Euclidean spaces, within preassigned transcendental growth rate. As by-products, universal holomorphic maps from $\mathbb{C}^n$ to $\mathbb{CP}^m$ $(n \leq m)$ and to complex tori having slow growth are obtained. We take inspiration from Oka manifolds theory, Nevanlinna theory, and hypercyclic operators theory.

Mathematics Subject Classification 32A22 · 30D35 · 47A16 · 41A20 · 32H30

1 Introduction

About one century ago, in the space $\mathcal{H}(\mathbb{C})$ of entire holomorphic functions, Birkhoff [3] constructed surprising examples $F : \mathbb{C} \to \mathbb{C}$ with the *universal* property that every $H \in \mathcal{H}(\mathbb{C})$ can be reached as a limit by translations of F, precisely, for certain sequence $\{a_{H,i}\}_{i \geq 1}$ of complex numbers there holds

$$\lim_{i \to \infty} F(z + a_{H,i}) = H(z) \quad (\forall z \in \mathbb{C}). \tag{1.1}$$

Dedicated to the memory of Nessim Sibony.

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The original construction often appears in exercise of complex analysis (e.g. [32, pp. 69–70, ex. 5]).

For source spaces with certain symmetry, there might be also "universal" holomorphic maps into a certain class of complex manifolds. In hindsight, the existence of such maps is intimately related to Oka theory [14, 15], which originated (1939) in Kiyoshi Oka's celebrated articles on Cousin Problems I and II [28]. Major developments have been made by Hans Grauert [18], Mikhail Gromov [19] and Franc Forstnerič [12, 13] (cf. [15] for detailed account). The simplest, among a dozen, characterization of Oka manifolds is given by Runge's approximation property [12].

Definition A complex manifold Y is an Oka manifold if every holomorphic map from a neighbourhood of a compact convex set K in a Euclidean space $\mathbb{C}^n$ (for any $n \in \mathbb{Z}_+$) to Y is a uniform limit on K of entire maps $\mathbb{C}^n \to Y$.

Basic examples of Oka manifolds include $\mathbb{C}^m$, $\mathbb{CP}^m$, and complex tori $\mathbb{T}^m$. A result of Kusakabe [24, Theorem 1.4] shows that, given an Oka manifold Y, for any source space X with mild symmetry, there are plenty of "universal" holomorphic maps from X to Y.

One philosophy in algebraic geometry is that key information of a variety can be read off from moduli spaces of certain curves or subvarieties it contains. Similarly, for seeking quantitative invariants of a given Oka manifold Y, we shall likewise study the spaces $\mathsf{Hol}(X, Y)$ of holomorphic maps from various open source spaces X to Y, and certain subspaces $\mathscr{U}(X, Y)$ consist of "universal" holomorphic maps are particularly interesting.

There is a natural *Nevanlinna functional* on the space $\mathsf{Hol}(X, Y)$, which associates an element $F : X \to Y$ with its (certain) Nevanlinna characteristic function $T_F(\bullet) : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 0}$, capturing the asymptotic growth rate of F. An interesting problem is thus studying "possible minimum" of the Nevanlinna functional on subspaces $\mathscr{U}(X, Y) \subset \mathsf{Hol}(X, Y)$ for Oka manifolds Y and various source spaces X. One potential application is about classification of Oka manifolds.

We start with the fundamental case where $X = \mathbb{C}^n$ and $Y = \mathbb{C}^m, \mathbb{CP}^m$, or $\mathbb{T}^m$, having certain linear structure. One motivation is a challenge [10, Problem 9.1] raised by Dinh and Sibony, about determining minimal possible growth $T_f(r)$ of universal entire holomorphic/meromorphic functions $f : \mathbb{C} \to \mathbb{CP}^1$, in terms of the Nevanlinna–Shimizu–Ahlfors characteristic function

$$T_f(r) := \int_1^r \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}_t} f^*\omega \quad (\forall\, r \geq 1),$$

where ω is the Fubini-Study form on $\mathbb{CP}^1$ giving unit area, and where $\mathbb{D}_t := \{z \in \mathbb{C} : |z| < t\}$.

An easy observation is that, a universal holomorphic/meromorphic function f must have infinite area

$$\int_{\mathbb{D}_t} f^*\omega \to \infty, \quad \text{as } t \to \infty,$$

for otherwise by complex analysis [27, p. 10, Theorem 1.1.26], f can be extended holomorphically across the infinity $\{\infty\} = \mathbb{CP}^1 \backslash \mathbb{C}$ and hence $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ is a rational function, contradicting the presumed universality.

Thus the growth T_f of a universal holomorphic/meromorphic function f must be sufficiently fast

$$\liminf_{r\to+\infty} T_f(r)/\log r \geq \lim_{r\to+\infty} \frac{1}{\log r} \int_{r_0}^{r} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}_{r_0}} f^*\omega = \int_{\mathbb{D}_{r_0}} f^*\omega,$$

where $\int_{\mathbb{D}_{r_0}} f^*\omega$ tends to infinity as $r_0 \to +\infty$.

On the other hand, given any positive continuous nondecreasing function $\psi : \mathbb{R}_{\geq 1} \to \mathbb{R}_+$ tending to infinity $\lim_{r\to+\infty} \psi(r) = +\infty$, it is shown in [6] that

Theorem 1.1 *There exists some universal meromorphic function $f : \mathbb{C} \to \mathbb{CP}^1$ with slow growth*

$$T_f(r) \leq \psi(r) \cdot \log r \quad (\forall\, r \geq 1). \tag{1.2}$$

This result is an optimal answer to the meromorphic part of [10, Problem 9.1]. Indeed, given any universal meromorphic function f, one can construct an auxiliary positive continuous nondecreasing function

$$\hat{\psi}(t) = \inf_{r>t} \{T_f(r)/\log r\} \;:\; \mathbb{R}_{\geq 1} \longrightarrow \mathbb{R}_+$$

such that an opposite growth rate estimate holds automatically

$$T_f(r) \geq \hat{\psi}(r) \cdot \log r \quad (\forall\, r \geq 1),$$

where $\hat{\psi}(r)$ tends to infinity as $r \to \infty$.

However, for the holomorphic part of [10, Problem 9.1], aiming at the same slow growth rate (1.2), the authors had encountered much more difficulties. Generally, for constructing holomorphic functions, the most successful method shall be about solving $\overline{\partial}$-equations, by either (1) explicit general Cauchy integral formula [21, p. 3, Theorem 1.2.1], or by (2) Hörmander's L^2 estimates [21, p. 94, Theorem 4.4.2]. Both machinery, together with several plausible ideas, failed for the authors. As a matter of fact, there was also psychological obstacle about wondering whether such universal entire functions having slow growth actually exist, and our faith in *Oka principle* [15, pp. 368–369] was in the test.

Nevertheless, a surprise came to the authors, shortly after they successfully designed an *Algorithm* producing the desired universal entire functions, that such result was already sketched in [9].

Unlike the ignorance of [9] in the community of complex analysis, in hypercyclic operators theory, Duĭos Ruis' result is widely known (see [17, pp. 109–110, Theorem 4.23]). Here we rephrase

Theorem 1.2 (Duĭos Ruis) *Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be a continuous function growing faster than any polynomial*

$$\lim_{r\to\infty} \phi(r)/r^N = \infty \quad (\forall\, N \geq 1).$$

Let a be a nonzero complex number. Then there exists an entire function f hypercyclic for T_a with slow growth

$$|f(z)| \leq \phi(|z|) \quad (\forall\, z \in \mathbb{C}). \tag{1.3}$$

Let us explain the notation and terminology. The space $\mathcal{H}(\mathbb{C})$ of entire holomorphic functions has a natural topology given by uniform convergence on compact sets. For a nonzero complex number a, the translation operator $\mathsf{T}_a : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ is defined by $f(\bullet) \mapsto f(\bullet + a)$. An element $f \in \mathcal{H}(\mathbb{C})$ is said to be *hypercyclic* for T_a if the orbit $\{f, \mathsf{T}_a^{(1)}(f), \mathsf{T}_a^{(2)}(f), \dots\}$ is dense in $\mathcal{H}(\mathbb{C})$.

In fact, the original statement in [9] only concerns slow growth real analytic functions ϕ having positive coefficients shrinking to zero rapidly

$$\phi(r) = \sum_{i=0}^{\infty} A_i\, r^i, \quad A_i > 0 \quad (\forall\, i \geq 0), \quad \lim_{i\to\infty} A_{i+1}/A_i = 0. \tag{1.4}$$

The two statements are indeed equivalent, see Lemma 2.1 below. Hence the holomorphic part of [10, Problem 9.1] has a clear answer of the shape (1.2) after Duĭos Ruis' Theorem 1.2, by using H. Cartan's equivalent version of Nevanlinna's characteristic function

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, \mathrm{d}\,\theta + O(1), \tag{1.5}$$

where $O(1)$ stands for some uniformly bounded term [27, p. 9].

In retrospect, outside complex analysis, Birkhoff's example of universal entire functions was phenomenal. It foreshadowed a new branch of functional analysis/dynamical systems about linear chaos [4, 16, 17], according to which an infinite dimensional linear system may exhibit nonlinear and unpredictable patterns. The typical example is

Theorem 1.3 (Birkhoff [3]) *For $a \in \mathbb{C} \setminus \{0\}$, some $f \in \mathcal{H}(\mathbb{C})$ is hypercyclic for T_a.*

Question 1.4 Within transcendental growth rate ϕ of the shape (1.4), can we construct universal holomorphic functions $F \in \mathcal{H}(\mathbb{C}^n)$ in several variables which are hypercyclic for a given T_a where $a \in \mathbb{C}^n \setminus \{\mathbf{0}\}$?

Note that the approach of Duĭos Ruis [9] only works for the source space $\mathbb{C}$, see the discussion in Sect. 2.1. Our initial goal is thus to answer Question 1.4 *in positive* by adapting our *Algorithm*. It turns out that we can prove much more.

Here is a reminiscent of [8, Theorem 8].

Lemma 1.5 *Let X be an Oka manifold. In the space $\mathsf{Hol}(\mathbb{C}^n, X)$, equipped with compact-open topology, of holomorphic maps from $\mathbb{C}^n$ to X, if f is a hypercyclic element for the translation operator T_a where $a \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, then f is hypercyclic simultaneously for all T_b where $b \in \mathbb{R}_+ \cdot a$.*

Hence we shall consider the *hypercyclic directions I* in the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ such that F is hypercyclic for T_a, where $a \in I$.

Convention. $|| \bullet ||$ denotes the standard Euclidean norm on any $\mathbb{C}^k$, which shall be clear according to the context. $\mathbb{B}(a, r)$ stands for the ball centered at a having radius r with respect to the norm $|| \bullet ||$. When a is the origin, we write $\mathbb{B}(r)$ for shortness of $\mathbb{B}(\mathbf{0}, r)$. We abuse $O(1)$ for uniformly bounded terms, which will vary. The differential operator d^c stands for $\frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$.

Theorem A *Given any transcendental growth function ϕ of the shape (1.4), for countable directions $\{\theta_i\}_{i\geq 1}$ in the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$, there exist universal holomorphic maps $F : \mathbb{C}^n \to \mathbb{C}^m$ with slow growth*

$$||F(z)|| \leq \phi(||z||) \quad (\forall z \in \mathbb{C}^n), \tag{1.6}$$

such that the hypercyclic directions of F include $\{\theta_i\}_{i\geq 1}$. Namely, F are hypercyclic for all T_a, where a are in some ray $\mathbb{R}_+ \cdot \theta_i$.

This is the main result of this paper, which answers Question 1.4. The proof is by the aforementioned constructive *Algorithm*, whose underling idea goes back to an insight of Sibony, see Sect. 2.2.

One application of Theorem A is generalizing Theorem 1.1 to higher dimensional universal holomorphic maps $f : \mathbb{C}^n \to \mathbb{CP}^m$ ($m \geq n \geq 1$), keeping in mind that the method of [6] only works for $f : \mathbb{C} \to \mathbb{CP}^m$ (see Sect. 2.3).

Recall that a Nevanlinna characteristic function of $f : \mathbb{C}^n \to \mathbb{CP}^m$ is given by

$$T_f(r) := \int_1^r \frac{\mathrm{d}\,t}{t^{2n-1}} \int_{\mathbb{B}(t)} f^*\omega_{\mathsf{FS}} \wedge \alpha^{n-1},$$

where ω_{FS} is the Fubini-Study form on $\mathbb{CP}^m$ and where $\alpha := \mathrm{d}\,\mathrm{d}^c||z||^2$ (cf. [27, 29]).

Theorem B *Let $n \leq m$ be two positive integers. Let $\psi : \mathbb{R}_{\geq 1} \to \mathbb{R}_+$ be a continuous nondecreasing function tending to infinity. Then for given directions $\{\theta_i\}_{i\geq 1}$ in the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$, there exist universal entire holomorphic maps $f : \mathbb{C}^n \to \mathbb{CP}^m$ with slow growth*

$$T_f(r) \leq \psi(r) \cdot \log r \quad (\forall r \geq 1), \tag{1.7}$$

such that f are hypercyclic for all nontrivial translations T_a, where a are in some ray $\mathbb{R}_+ \cdot \theta_i$.

Another application of Theorem A is determining minimal growth of universal holomorphic maps from $\mathbb{C}^n$ to a complex m-tori $\mathbb{T}^m$, i.e., $\mathbb{T}^m = \mathbb{C}^m/\Gamma$ for some lattice Γ of full rank $2m$.

Theorem C *Given any continuous nondecreasing function $\psi : \mathbb{R}_{\geq 1} \to \mathbb{R}_+$ tending to infinity, for countably many directions $\{\theta_i\}_{i\geq 1}$ in the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$, there exist universal holomorphic maps $F : \mathbb{C}^n \to \mathbb{T}^m$ with slow growth*

$$T_F(r) \leq r^{\psi(r)} \quad (\forall\, r \geq 1), \tag{1.8}$$

such that the hypercyclic directions of F contains $\{\theta_i\}_{i\geq 1}$.

Without loss of generality, in (1.8) we take

$$T_F(r) := \int_1^r \frac{\mathrm{d}\, t}{t^{2n-1}} \int_{\mathbb{B}(t)} F^* \omega_{\mathsf{std}} \wedge \alpha^{n-1},$$

where $\omega_{\mathsf{std}} := \mathrm{d}\,\mathrm{d}^c ||z||^2$ is a positive $(1, 1)$ form induced from the Euclidean space $\mathbb{C}^m$.

Recall a surprising result [2] that some holomorphic functions $f \in \mathcal{H}(\mathbb{C}^n)$ in several variables are simultaneously hypercyclic for all nontrivial translations T_a where $a \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. It is thus natural to ask

Question 1.6 Can we strengthen Theorem A by moreover requiring the hypercyclic directions of F to be the whole unit sphere $\mathbb{S}^{2n-1}$?

Keeping in mind the slogan of modern Oka principle (see Sect. 5), for the above question, there seems no obvious topological obstruction at first glance. However, the answer is *No*, by deep reason in Nevanlinna theory. Here for simplicity we only analyze the basic case $F : \mathbb{C} \to \mathbb{C}$.

Theorem D *There exist some transcendental growth function ϕ of the shape (1.4), such that for any entire holomorphic function $F \in \mathcal{H}(\mathbb{C})$ with slow growth*

$$|F(z)| \leq \phi(|z|) \quad (\forall\, z \in \mathbb{C}), \tag{1.9}$$

the hypercyclic directions $I \subset \mathbb{S}^1$ of F must have Hausdorff dimension zero.

Since the hypercyclic directions I of a universal holomorphic function F shall be of small size and could contain countable points, it is natural to ask if some I of F can contain uncountable directions. This time the answer is *yes*. For shortness we again only consider the case $\mathcal{H}(\mathbb{C})$.

Theorem E *Given any transcendental growth function ϕ of the shape (1.4), there exists some universal entire holomorphic function $F \in \mathcal{H}(\mathbb{C})$ with slow growth*

$$|F(z)| \leq \phi(|z|) \quad (\forall\, z \in \mathbb{C}),$$

such that the hypercyclic directions $I \subset \mathbb{S}^1$ of F is uncountable.

Thus one shall be prudent when using Oka principle, in case there might be certain delicate subtlety depending on the nature of the question.

Here is the structure of this paper. In Sect. 2 we prove a comparison Lemma 2.1 to show that the class of slow growth functions of shape (1.4) in our theorems is optimal, and we explain our central idea of "patching" together holomorphic discs inspired by Question 2.2 of Sibony. A relevant speculation of Brunella is also recalled. In Sect. 3 we present an *Algorithm*, which is the engine in our approach for constructing universal entire holomorphic maps with slow growth. In Sect. 4 we prove Lemma 1.5 and all our Theorems A, B, C, D, E. Lastly, in Sect. 5, we highlight key questions for future research.

2 Background

2.1 About the construction of Duos Ruis

The original article [9] of the Theorem 1.2 only concerns a special class of analytic functions ϕ, by the following reason.

Lemma 2.1 *Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be a continuous function growing faster than any polynomials*

$$\lim_{r\to\infty} \phi(r)/r^N = \infty \quad (\forall\, N \geq 1).$$

Then the function $\phi(r)$ dominates some analytic function $\sum_{i=0}^{\infty} A_i\, r^i$ with positive coefficients A_i shrinking to zero rapidly

$$\lim_{i\to\infty} A_{i+1}/A_i = 0.$$

Proof Set A_0 to be half of the minimum value of the function $\phi(r)$ on the interval $[0, \infty[$. It is clear that $A_0 > 0$. For $i = 0, 1, 2, \ldots$, inductively we set A_{i+1} to be the smaller number between $\frac{A_i}{i+1}$ and half of the minimum value of the continuous positive function $\big(\phi(r) - \sum_{j=0}^{i} A_j r^j\big)/r^{i+1}$ on the interval $]0, \infty[$. It is clear that $A_{i+1} > 0$, and that $A_{i+1}/A_i \leq \frac{1}{i+1}$, $\phi(r) > \sum_{j=0}^{i+1} A_j\, r^j$ for all $r \geq 0$. By taking $i \to \infty$ we conclude $\phi(r) \geq \sum_{j=0}^{\infty} A_j\, r^j$ for all $r \geq 0$. □

Theorem 1.2 is claimed in the article [9], with no explanations but a few formulas to indicate the construction, in which the fundamental theorem of algebra plays an important role:

$$\text{every polynomial in } \mathbb{C}[z] \text{ factors as products of one forms.} \tag{2.1}$$

Duĭos Ruis' universal entire functions with slow growth are constructed by infinite products of certain well-chosen one forms. Therefore, this method is strictly one dimensional, as (2.1) fails for $\mathbb{C}[z_1, \ldots, z_n]$ $(n \geq 2)$.

2.2 An insight of Sibony

After receiving an early manuscript of [22], Sibony asked

Question 2.2 [31] Show that for $X = \mathbb{CP}^n$, or complex tori, all Ahlfors currents can be obtained by single entire curve $f : \mathbb{C} \to X$.

Sibony suggested a hint [20]. We regret that we had not discussed this problem with him before it was too late.

If only concentric holomorphic discs $\{f(\mathbb{D}(r))\}_{r>0}$ are allowed to use (as [22]), then we do not know the answer. If one can use any holomorphic discs $\{f(\mathbb{D}(a, r))\}_{a\in\mathbb{C}, r>0}$, then we can follow Sibony's insight by Oka theory. In fact, such phenomenon holds for compact complex manifolds Y satisfying a weak Runge's approximation property that *every holomorphic map from a neighborhood of two disjoint closed discs* $D_1, D_2 \subset D$ *into* Y *can be "extended" to a holomorphic map from a neighborhood of a larger closed disc* $D \subset \mathbb{C}$ to Y *within given small positive error bound with respect to a complete distance* d_Y on Y. Such manifolds Y include all Oka-1 manifolds, in particular all rationally connected manifolds [1]. Modulo certain technical details, such entire curve can be constructed by "patching" together some countable holomorphic discs. The same method can also construct universal holomorphic maps into Oka manifolds (a result due to Kusakabe [24, Theorem 1.4]). For shortness, we illustrate the method by the following

Proof of Birkhoff's Theorem 1.3 using Runge's approximation theorem. Fix a countable and dense subfield of $\mathbb{C}$, say complex rational numbers

$$\mathbb{Q}_c := \{z = x + y\sqrt{-1} : x, y \in \mathbb{Q}\}.$$

Since $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable, there is some bijection

$$\varphi = (\varphi_1, \varphi_2) : \mathbb{Z}_+ \xrightarrow{\sim} \mathbb{Z}_+ \times \mathbb{Z}_+.$$

Fix small positive error bounds $\{\epsilon_i\}_{i\geq 1}$, say $\epsilon_i = 1/2^i$, such that

$$\lim_{j\to\infty} \sum_{i\geq j} \epsilon_i = 0. \tag{2.2}$$

We arrange all elements in the countable set $\mathbb{Q}_c[z] \times \mathbb{Q}_+$, in some subsequent order $\{(f_i, r_i)\}_{i\geq 1}$.

The punchline, which will be understood later, is to make another sequence $\{(g_j, \hat{r}_j)\}_{j\geq 1}$ so that each (f_i, r_i) repeats infinitely many times in $(g_j, \hat{r}_j)_{j\geq 1}$, e.g.

$$g_j := f_{\varphi_1(j)}, \quad \hat{r}_j := r_{\varphi_1(j)}. \tag{2.3}$$

Now we construct a desired entire function f by taking the limits of some convergent holomorphic functions

$$F_i : \mathbb{D}_{R_i} \to \mathbb{C}, \quad R_i \nearrow \infty \quad (i = 1, 2, 3, \dots).$$

In *Step* 1, we set three key data $F_1 := g_1$, $R_1 := \hat{r}_1$, $c_1 := 0$.

Subsequently, in *Step* $i+1$ for $i = 1, 2, 3, \ldots$, we choose a center $c_{i+1} \in \mathbb{Z}_+ \cdot a$ far from the origin, so that the two discs $\mathbb{D}_{R_i}$ and $\mathbb{D}(c_{i+1}, \hat{r}_{i+1})$ keep positive distance, i.e., $|c_{i+1}| > R_i + \hat{r}_{i+1}$. We define a polynomial $\hat{g}_{i+1}(\bullet) := g_{i+1}(\bullet - c_{i+1})$ on $\mathbb{D}(c_{i+1}, \hat{r}_{i+1})$ by translation of g_{i+1}. Now we take a large disc $\mathbb{D}_{R_{i+1}}$ having radius $R_{i+1} > |c_{i+1}| + \hat{r}_{i+1}$, so that it contains $\mathbb{D}_{R_i}$ and $\mathbb{D}(c_{i+1}, \hat{r}_{i+1})$ as relatively compact subsets. By Runge's approximation theorem, we can "extend" F_i and $\hat{g}_{i+1}$ to some polynomial $F_{i+1} : \mathbb{D}_{R_{i+1}} \to \mathbb{C}$ within admissible errors ($d_{\mathbb{C}}$ stands for the usual Euclidean distance on $\mathbb{C}$)

$$\sup_{z \in \mathbb{D}_{R_i}} d_{\mathbb{C}}(F_{i+1}(z), F_i(z)) \leq \epsilon_{i+1}, \qquad \sup_{z \in \mathbb{D}(c_{i+1}, \hat{r}_{i+1})} d_{\mathbb{C}}(F_{i+1}(z), \hat{g}_{i+1}(z)) \leq \epsilon_{i+1}. \tag{2.4}$$

By (2.2), it is clear that $\{F_i\}_{i \geq 1}$ converges to an entire curve $f : \mathbb{C} \to \mathbb{C}$. Moreover, by using (2.4) repeatedly, we see that f approximates each $\hat{g}_{i+1}$ effectively

$$\begin{aligned}
\sup_{z \in \mathbb{D}(c_{i+1}, \hat{r}_{i+1})} d_{\mathbb{C}}(f(z), \hat{g}_{i+1}(z)) &\leq \sup_{z \in \mathbb{D}(c_{i+1}, \hat{r}_{i+1})} d_{\mathbb{C}}(F_{i+1}(z), \hat{g}_{i+1}(z)) \\
&\quad + \sum_{j \geq i+1} \sup_{z \in \mathbb{D}_{R_j}} d_{\mathbb{C}}(F_{j+1}(z), F_j(z)) \\
&\leq \epsilon_{i+1} + \sum_{j \geq i+2} \epsilon_j \ \to 0 \quad \text{(as } i \to \infty\text{)} \quad \text{[by 2.2]}.
\end{aligned}$$

In other words

$$\sup_{z \in \mathbb{D}_{\hat{r}_{i+1}}} d_{\mathbb{C}}(f(c_{i+1} + z), g_{i+1}(z)) \to 0 \quad \text{(as } i \to \infty\text{)}.$$

This finishes the proof, since $\mathbb{Q}_c[z]$ is dense in $\mathcal{H}(\mathbb{C})$ under the compact-open topology. □

The same idea also plays a key rôle in the proof of Theorem 1.1, as we shall now explain.

2.3 Thinking process of [6]

Without knowing Duĭos Ruis' Theorem 1.2, the estimate (1.2) was anticipated by Dinh Tuan Huynh as an optimal answer to [10, Problem 9.1], based on evidence given by [25], in which some meromorphic functions g with growth $T_g(r) \leq \psi(r)(\log r)^2$ were constructed so that any entire function can be approximated by translations of g.

In the preceding proof, one sees clearly the idea of "patching together" distinct holomorphic discs, up to small errors, by Runge's approximation property of the ambient manifold X. For the meromorphic part of [10, Problem 9.1], where $X = \mathbb{CP}^1$, such idea works quite effectively as follows.

Fix a countable and dense subfield of $\mathbb{C}$, say

$$\mathbb{Q}_c := \{z = x + y\sqrt{-1} : x, y \in \mathbb{Q}\}.$$

First, we can find countable rational functions

$$f_i : \mathbb{D}_{r_i} \to \mathbb{CP}^1 \quad (i = 1, 2, 3, \ldots;\ 0 < r_i < \infty)$$

such that any meromophic function can be reached as a limit by elements in $(f_i)_{i\geq 1}$. Indeed, the Cartesian product $\mathbb{Q}_c(z) \times \mathbb{Q}_+$ of all rational functions with $\mathbb{Q}_c$ coefficients and all positive rational radii, is countable and sufficient.

Next, we play the same trick as (2.3) to gather a sequence of rational functions

$$g_j : \mathbb{D}_{\ell_j} \to X \quad (j \geq 1)$$

by elements of $(f_i)_{i\geq 1}$, such that each f_i for $i = 1, 2, 3, \ldots$ repeats infinitely many times in $(g_j)_{j\geq 1}$.

If all the rational functions f_i decay to zero near the infinity point, i.e.,

$$f_i = p_i/q_i \text{ for some polynomials } p_i, q_i \in \mathbb{Q}_c[z] \text{ with } \deg p_i < \deg q_i \quad (i = 1, 2, 3, \ldots), \tag{2.5}$$

by the argument in the preceding proof, we can construct a universal meromorphic function

$$F(z) := \sum_{j=1}^{\infty} g_j(z - c_j), \tag{2.6}$$

where $c_j \in \mathbb{C}$ are subsequently well chosen, having absolute values increasing sufficiently fast

$$1 \ll |c_1| \ll |c_2| \ll |c_3| \ll \cdots .$$

Hence the remaining task for constructing universal meromorphic function is to make sure that (2.5) holds. This can be achieved by approximating and replacing each f_i on $\mathbb{D}_{r_i}$ by countably many rational functions $\{h_{i,j}\}_{j\geq 1}$ vanishing at the infinity. Indeed, if f_i is a polynomial, then the same as [25] we can use Runge's approximation theorem [30] to find rational functions of the shape

$$\hat{h}_{i,j} = \sum_{k=1}^{N_{i,j}} \frac{C_{i,j,k}}{z - \xi_{i,j,k}} \quad (j = 1, 2, 3, \ldots;\ N_{i,j} \in \mathbb{Z}_+;\ C_{i,j,k} \in \mathbb{C};\ \xi_{i,j,k} \in \partial\mathbb{D}_{r_i+1})$$

to approximate f_i on $\mathbb{D}_{r_i}$ uniformly. Therefore, since $\mathbb{Q}_c \subset \mathbb{C}$ is dense, f_i can be approximated by

$$\text{rational functions with } \mathbb{Q}_c \text{ coefficients of the shape (2.5) (countable!)} \tag{2.7}$$

uniformly on $\mathbb{D}_{r_i}$.

Yet how to deal with a general rational function f_i? A trick comes naturally:

$$\text{decompose } f_i = f_{i,\,\text{pole}} + f_{i,\,\text{holo}}$$

as the sum of the principle pole part $f_{i,\,\text{pole}}$ and the remaining polynomial part $f_{i,\,\text{holo}}$. The first part $f_{i,\,\text{pole}}$ automatically enjoys the desired decay near $\infty \in \mathbb{CP}^1$, and obviously can be approximated by (2.7). The later part can be handled likewise by using Runge's approximation theorem. Hence (2.5) can be achieved in practice.

Lastly, by Nevanlinna theory (cf. [27, 29]), the Shimizu-Ahlfors geometric version of $T_F(r)$ is equivalent to Nevanlinna's characteristic function [26]

$$T_F(r) = N_F(r, \infty) + m_F(r, \infty) + O(1),$$

where $N_F(r, \infty)$ is the counting function (with logarithmic weight) about poles of F, and where the proximity function $m_F(r, \infty)$ measures the closeness of $F(\partial\mathbb{D}_r)$ to $\infty \in \mathbb{CP}^1$. Since every $g_i(z) \approx 0$ for $|z| \gg 1$, we have $m_F(r, \infty) = 0$ for most r. Thus $T_F(r) \approx N_F(r, \infty)$ can be made to grow as slowly as possible, by throwing subsequently all the centers

$$1 \ll |c_1| \ll |c_2| \ll |c_3| \ll \cdots$$

far away from the origin.

Alternatively, Zhangchi Chen found an elegant trick to achieve (2.5):

$$\text{replacing } f_i \text{ by countably many rational functions}$$
$$f_i \cdot \Big(\frac{M_{i,j}}{z + M_{i,j}}\Big)^{\deg f_i + 1} \qquad (j = 1, 2, 3, \ldots;\ M_{i,j} \in \mathbb{Q}_c)$$

where $r_i \ll |M_{i,1}| \ll |M_{i,2}| \ll |M_{i,3}| \ll \cdots \nearrow \infty$. Thus one avoids using Runge's theorem, since

$$\Big(\frac{M_{i,j}}{z + M_{i,j}}\Big)^{\deg f_i + 1} \to 1 \quad (\text{as } j \to \infty)$$

uniformly for all $z \in \mathbb{D}_{r_i}$.

The same method works for $\mathbb{C}$ to $\mathbb{CP}^m$ (any $m \geq 1$) as well, see [6] for details. However, for constructing universal holomorphic maps from $\mathbb{C}^n$ to $\mathbb{CP}^m$ ($m \geq n \geq 2$) with slow growth, the approach of [6] does not work, because poles of meromorphic functions in several variables are no longer discrete points.

2.4 A question of Brunella

In [5, p. 200] Brunella asked: if for any entire curve $f : \mathbb{C} \to X$ into a compact complex manifold X, one can "cleverly" choose some radii sequence $\{r_i\}_{i \geq 1}$ such that $\{f(\mathbb{D}_{r_i})\}_{i \geq 1}$ produces some Nevanlinna/Ahlfors current with either trivial singular part or trivial diffuse part.

Note that for the exotic entire curves in [22], for generic (in probability sense) choices of $\{r_i\}_{i\geq 1} \nearrow \infty$, Brunella's anticipation is correct. One may wonder what shall be a canonical Nevanlinna/Ahlfors current associates with an entire curve $f : \mathbb{C} \to X$. This problem is intimately caused by *flexibility v.s. rigidity* in complex analysis, for which Oka and Kobayashi shall take some responsibility.

If counterexamples $f : \mathbb{C} \to X$ to Brunella's question ever exist, one shall search X among compact Oka manifolds.

3 Algorithm

3.1 One variable case

We use the idea from Sect. 2.2 to provide a constructive proof, different to [9] and [7, p. 1433], of Duĭos Ruis' Theorem 1.2.

We enumerate all elements in the countable set $\mathbb{Q}_c[z] \times \mathbb{Q}_+$ in some subsequent order $\{(f_i, r_i)\}_{i\geq 1}$. Fix an auxiliary bijection

$$\varphi = (\varphi_1, \varphi_2) : \mathbb{Z}_+ \xrightarrow{\sim} \mathbb{Z}_+ \times \mathbb{Z}_+. \tag{3.1}$$

We make another sequence $\{(g_j, \hat{r}_j)\}_{j\geq 1}$ so that

$$\text{each } (f_i, r_i) \text{ repeats infinitely many times in } (g_j, \hat{r}_j)_{j\geq 1}. \tag{3.2}$$

For instance, we can take

$$g_j := f_{\varphi_1(j)}, \quad \hat{r}_j := r_{\varphi_1(j)}. \tag{3.3}$$

Such trick was also visible in [22, p. 321] for some exotic construction.

Given analytic growth function ϕ of the shape (1.4), our *Algorithm* will construct universal entire functions $f := S_0 + S_1 + S_2 + S_3 + \cdots$ inductively, where in each *Step* k for $k = 0, 1, 2, \ldots$ we add one piece of jigsaw puzzles

$$S_k := \sum_{j=m_k}^{M_k} a_j\, z^j \quad \text{having small coefficients } |a_j| \leq A_j \tag{3.4}$$

where $0 = m_0 = M_0 < m_1 \leq M_1 < m_2 \leq M_2 < m_3 \leq M_3 < \cdots$. Thus the desired estimate (1.3) holds immediately

$$\begin{aligned} |f(z)| &\leq \sum_{k=0}^{\infty} |S_k(z)| \leq \sum_{k=0}^{\infty} \sum_{j=m_k}^{M_k} |a_j|\,|z|^j \\ &\leq \sum_{k=0}^{\infty} \sum_{j=m_k}^{M_k} |A_j|\,|z|^j \leq \sum_{j=0}^{\infty} |A_j|\,|z|^j = \phi(|z|). \end{aligned}$$

Fix small positive error bounds $\{\epsilon_i\}_{i\geq 1}$, say $\epsilon_i := 2^{-i}$, so that

$$\lim_{j\to\infty} \sum_{i\geq j} \epsilon_i \;=\; 0. \tag{3.5}$$

Set the initial three key data

$$S_0 = \mathbf{0} \in \mathbb{C}[z], \quad R_0 = 2023 \in \mathbb{R}_+, \quad c_0 = 0 \in \mathbb{C}.$$

In each Step $k = 1, 2, 3, \ldots$, we pick a center $c_k \in \mathbb{C}$ far away from the disc $\mathbb{D}_{R_{k-1}}$ and then cook up some S_k satisfying:

(1) on the disc $\mathbb{D}_{R_{k-1}}$ the polynomial S_k is negligible

$$|S_k(z)| \leq \epsilon_k \quad (\forall\, z \in \mathbb{D}_{R_{k-1}}); \tag{3.6}$$

(2) the partial sum $G_k := (S_0 + S_1 + \cdots + S_{k-1}) + S_k$ approximates $\mathsf{T}_{-c_k}\, g_k$ on $\mathbb{D}(c_k, \hat{r}_k)$ within small error ϵ_k, i.e.,

$$|G_k(c_k + z) - g_k(z)| \leq \epsilon_k \quad (\forall\, z \in \mathbb{D}_{\hat{r}_k}). \tag{3.7}$$

We end Step k by choosing a large radius $R_k > |c_k| + \hat{r}_k$, and then move on to Step $k+1$.

If the above algorithm runs successfully in all Steps $k = 1, 2, 3, \ldots$, then f is universal by virtue of (3.7), (3.2), (3.5). We now fulfill the details.

The restriction (3.6) can easily be guaranteed by requiring all coefficients of S_k to be small enough, say

$$|a_j| \;\leq\; \min\left\{A_j,\, \epsilon_k \cdot \left[(M_k - m_k + 1)\cdot R_{k-1}^{j}\right]^{-1}\right\} \quad (m_k \leq j \leq M_k). \tag{3.8}$$

However, the proximity goal (3.7) seems doubtful at first glance. The existence of such S_k is indicated by Runge's theorem on polynomial approximations. Indeed, if ignoring the boundedness restriction (3.4) on the coefficients of S_k, such S_k certainly exists by Runge's theorem. By varying c_k far away from the origin continuously, we may expect that certain equivariant $S_k \in \mathbb{C}(c_k)[z]$ exists, and the coefficients are decaying to zero as $c_k \to \infty$. Moreover, we may expect that the degree of S_k with respect to variable z is exactly $\max\{\deg G_{k-1}, \deg g_k\}$.

Now let us clarify the above imagination by explicit computation. The idea is, regarding c_k as a new variable, to construct polynomial S_k with coefficients in the function field $\mathbb{C}(c_k)$ of the shape

$$a_j \;\in\; \frac{1}{c_k}\cdot\mathbb{C}\left[\frac{1}{c_k}\right] \quad (m_k \leq j \leq M_k), \tag{3.9}$$

so that (3.6) automatically holds by later specializing $|c_k| \gg 1$. Similarly, we can fix (3.7) by showing that

$$G_k(c_k + z) - g_k(z) \text{ is a polynomial of variable } z, \text{ having coefficients in } \frac{1}{c_k} \cdot \mathbb{C}\Big[\frac{1}{c_k}\Big]. \tag{3.10}$$

Firstly, by gathering binomial expansions, we rewrite

$$(S_0 + S_1 + \cdots + S_{k-1})(c_k + z) - g_k(z) = \sum_{i=0}^{\ell_k} \hat{a}_{k,i}\, z^i, \tag{3.11}$$

where $\ell_k := \max\{M_{k-1}, \deg g_k\}$, and where the coefficients

$$\hat{a}_{k,i} = \sum_{i \le j \le M_{k-1}} a_j \binom{j}{i} c_k^{j-i} - \frac{1}{i!} g_k^{(i)}(0) \quad \in \mathbb{C}[c_k] \quad (i = 0, 1, 2, \ldots, \ell_k). \tag{3.12}$$

Next, we collect (we will take $m_k > \ell_k$)

$$S_k(z + c_k) = \sum_{j=m_k}^{M_k} a_j\, (z + c_k)^j = \sum_{i=0}^{\ell_k} \Big[\sum_{j=m_k}^{M_k} a_j \binom{j}{i} c_k^{j-i} \Big] \cdot z^i + (\mathsf{h.o.t.}) \tag{3.13}$$

where $(\mathsf{h.o.t})$ are the remaining higher order terms with respect to variable z.

Declare $m_k := \ell_k + 1$ and $M_k := 2\ell_k + 1$ for solving the system of linear equations

$$\sum_{j=m_k}^{M_k} a_j \binom{j}{i} c_k^{j-i} = -\hat{a}_{k,i} \quad (i = 0, 1, 2, \ldots, \ell_k) \tag{3.14}$$

with the just number $\ell_k + 1$ of unknown variables $\{a_j\}_{m_k \le j \le M_k}$.

Lemma 3.1 *The determinant of the above $(\ell_k + 1) \times (\ell_k + 1)$ coefficient matrix has neat expression*

$$\det \Big[\binom{j}{i} c_k^{j-i} \Big]_{\substack{0 \le i \le \ell_k \\ m_k \le j \le M_k}} = c_k^{(\ell_k+1)^2}. \tag{3.15}$$

That the determinant is of the shape $O(1) \cdot c_k^{(\ell_k+1)^2}$ for some nonzero integer $O(1)$ is natural, and is sufficient for our purpose.

Proof Observe that

$$\Big[\tbinom{j}{i}c_k^{j-i}\Big]_{\substack{0\le i\le \ell_k\\ m_k\le j\le M_k}}=\begin{pmatrix} c_k^{-0} & 0 & \cdots & 0 & 0\\ 0 & c_k^{-1} & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & c_k^{-(\ell_k-1)} & 0\\ 0 & 0 & \cdots & 0 & c_k^{-\ell_k}\end{pmatrix}\cdot\Big[\tbinom{j}{i}\Big]_{\substack{0\le i\le \ell_k\\ m_k\le j\le M_k}}\cdot\begin{pmatrix} c_k^{m_k} & 0 & \cdots & 0 & 0\\ 0 & c_k^{m_k+1} & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & c_k^{M_k-1} & 0\\ 0 & 0 & \cdots & 0 & c_k^{M_k}\end{pmatrix}.$$

The determinants of the first and the last diagonal matrices contribute $c_k^{m_k^2}$. Now we compute the middle

$$\det\left[\binom{j}{i}\right]_{\substack{0\le i\le \ell_k\\ m_k\le j\le M_k}}=\det\left[\binom{j}{i}\right]_{\substack{0\le i\le \ell_k\\ \ell_k+1\le j\le 2\ell_k+1}}$$

$$=\det\begin{pmatrix}\binom{\ell_k+1}{0} & \binom{\ell_k+2}{0} & \cdots & \binom{2\ell_k}{0} & \binom{2\ell_k+1}{0}\\ \binom{\ell_k+1}{1} & \binom{\ell_k+2}{1} & \cdots & \binom{2\ell_k}{1} & \binom{2\ell_k+1}{1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \binom{\ell_k+1}{\ell_k-1} & \binom{\ell_k+2}{\ell_k-1} & \cdots & \binom{2\ell_k}{\ell_k-1} & \binom{2\ell_k+1}{\ell_k-1}\\ \binom{\ell_k+1}{\ell_k} & \binom{\ell_k+2}{\ell_k} & \cdots & \binom{2\ell_k}{\ell_k} & \binom{2\ell_k+1}{\ell_k}\end{pmatrix}.$$

The trick is to use the identity $\binom{j}{i}-\binom{j-1}{i}=\binom{j-1}{i-1}$ for $j\ge i\ge 1$. Subtracting the s-th column from the $(s+1)$-th column subsequently for $s=\ell_k-1,\ell_k-2,\dots,1$, we receive

$$\det\left[\binom{j}{i}\right]_{\substack{0\le i\le \ell_k\\ \ell_k+1\le j\le 2\ell_k+1}}=\det\begin{pmatrix}1 & 0 & \cdots & 0 & 0\\ \binom{\ell_k+1}{0} & \binom{\ell_k+1}{0} & \cdots & \binom{2\ell_k-1}{0} & \binom{2\ell_k}{0}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \binom{\ell_k}{\ell_k-2} & \binom{\ell_k+1}{\ell_k-2} & \cdots & \binom{2\ell_k-1}{\ell_k-2} & \binom{2\ell_k}{\ell_k-2}\\ \binom{\ell_k}{\ell_k-1} & \binom{\ell_k+1}{\ell_k-1} & \cdots & \binom{2\ell_k-1}{\ell_k-1} & \binom{2\ell_k}{\ell_k-1}\end{pmatrix}$$

$$=\det\left[\binom{j}{i}\right]_{\substack{0\le i\le \ell_k-1\\ \ell_k+1\le j\le 2\ell_k}}.$$

This is the pattern for induction process. Eventually the outcome is $\det[\binom{\ell_k+1}{0}]=1$. $\square$

Observation 3.2 *In* (3.11), (3.12), (3.14) *the polynomials* $\hat{a}_{k,i}\in\mathbb{C}[c_k]$ *have degrees* $\le \ell_k-i$ *for* $0\le i\le \ell_k$*. Hence by Cramer's rule, the solutions* a_j *of* (3.14) *have shape*

$$a_j\ \in\ \frac{1}{c_k^{j-m_k+1}}\cdot\mathbb{C}\left[\frac{1}{c_k}\right]\quad (m_k\le j\le M_k).$$

Therefore, coefficients of (h.o.t.) *in* (3.13) *are in* $\frac{1}{c_k}\cdot\mathbb{C}[\frac{1}{c_k}]$, *whence* (3.10) *is satisfied.*

Proof By (3.12), $\hat{a}_{k,i}$ is a polynomial of variable c_k with degree $\leq M_{k-1} - i$. Note that $\binom{j}{i} c_k^{j-i}$ is of degree $j - i$. Observe that each term in the Laplace expansion of the determinant (3.15) is of the same degree with respect to variable c_k. By Cramer's rule, we see that the pole order of a_j with respect to variable c_k is at least $(j - i) - (M_{k-1} - i) = j - M_{k-1} \geq j - \ell_k = j - (m_k - 1)$. □

Summarizing, for every Step $k = 1, 2, 3, \ldots$, by solving the system of linear equations (3.14), and by specializing sufficiently large $|c_k| \gg 1$, we can obtain a desired S_k to move on in the Algorithm. In the limit we receive some universal entire functions $f = \sum_{i=0}^{\infty} S_i$ within admissible slow growth ϕ.

Remark 3.3 The trick of choosing c_k far away from the origin is not only sufficient but also necessary, by reason of Nevanlinna theory (see (4.25) below), since g_k might contain too many zeros in $\mathbb{D}_{\hat{r}_k}$.

By improving our algorithm, we can make f to be hypercyclic simultaneously for given countable nontrivial translations $\{\mathsf{T}_{a_i}\}_{i\geq 1}$.

Moreover, our Algorithm is flexible for higher dimensional situation, as we now explain.

3.2 General case

Given any transcendental growth function ϕ of the shape (1.4), and given countably many directions $\{\theta_i\}_{i\geq 1}$ in the unit sphere of $\mathbb{C}^n$, we now upgrade our *Algorithm* to prove Theorem A, by constructing universal entire holomorphic maps $F \in \mathsf{Hol}(\mathbb{C}^n, \mathbb{C}^m)$ with slow growth (1.6) such that F are hypercyclic for all nontrivial translations T_{θ_i} for $i \geq 1$.

We start with the countable set

$$\Big(\oplus_{j=1}^{m} \mathbb{Q}_c[z_1, \ldots, z_n] \Big) \times \mathbb{Q}_+, \tag{3.16}$$

in which we enumerate all elements as $\big((f_i^{[j]})_{1\leq j\leq m}, r_i\big)_{i\geq 1}$. Using again the trick (3.2), we make another sequence $\big((g_i^{[j]})_{1\leq j\leq m}, \hat{r}_i\big)_{i\geq 1}$ by declaring

$$g_i^{[j]} := f_{\varphi_1\circ\varphi_1(i)}^{[j]}; \quad \hat{r}_i := r_{\varphi_1\circ\varphi_1(i)} \quad (j = 1, 2, \ldots, m;\ i \geq 1). \tag{3.17}$$

Our *Algorithm* produces universal holomorphic maps

$$F = (F^{[j]})_{1\leq j\leq m} : \mathbb{C}^n \longrightarrow \mathbb{C}^m \tag{3.18}$$

by induction, where each coordinate component

$$F^{[j]} := S_0^{[j]} + S_1^{[j]} + S_2^{[j]} + S_3^{[j]} + \cdots \quad (j = 1, 2, \ldots, m). \tag{3.19}$$

In *Step* k for $k = 0, 1, 2, \ldots$ we fabricate

$$S_k^{[j]} := \sum_{j=m_k}^{M_k} \sum_{|I|=j} a_I^{[j]}\, Z^I \quad {\scriptstyle (j=1,2,\ldots,m)}. \tag{3.20}$$

with sufficiently small coefficients $a_I^{[j]} \in \mathbb{C}$, so that the desired estimate (1.6) holds automatically. Here we use standard notation of multi-indices $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$ with norm $|I| = i_1 + \cdots + i_n$, and $Z^I = z_1^{i_1} \cdots z_n^{i_n}$, and likewise we require $0 = m_0 = M_0 < m_1 \leq M_1 < m_2 \leq M_2 < \cdots$.

Fix small positive error bounds $\{\epsilon_i\}_{i\geq 1}$ as (3.5). Set the initial key data

$$S_0^{[1]} = \cdots = S_0^{[m]} = \mathbf{0} \in \mathbb{C}[z_1, \ldots, z_n], \quad R_0 = 2023 \in \mathbb{R}_+, \quad c_0 = \mathbf{0} \in \mathbb{C}^n.$$

In each Step $k = 1, 2, 3, \ldots$, we first pick some center

$$c_k \in \mathbb{Z}_+ \cdot \theta_{\varphi_2 \circ \varphi_1(k)} \tag{3.21}$$

far away from $\mathbb{B}_{R_{k-1}} := \{||z|| \leq R_{k-1}\}$ in $\mathbb{C}^n$. Next, we fabricate some $S_k^{[j]}$ of the shape (3.20) such that:

(1) on the ball $\mathbb{B}_{R_{k-1}}$ the polynomials $S_k^{[j]}$ are negligible

$$|S_k^{[j]}(z)| \leq \epsilon_k \quad {\scriptstyle (j=1,2,\ldots,m;\ \forall z \in \mathbb{B}_{R_{k-1}})}; \tag{3.22}$$

(2) the partial sums $G_k^{[j]} := (S_0^{[j]} + S_1^{[j]} + \cdots + S_{k-1}^{[j]}) + S_k^{[j]}$ approximates $\mathsf{T}_{-c_k}\, g_k^{[j]}$ on $\mathbb{B}(c_k, \hat{r}_k)$ within small error ϵ_k, i.e.,

$$|G_k^{[j]}(c_k + z) - g_k^{[j]}(z)| \leq \epsilon_k \quad {\scriptstyle (j=1,2,\ldots,m;\ \forall z \in \mathbb{B}_{\hat{r}_k})}. \tag{3.23}$$

We end Step k by setting large radius $R_k > ||c_k|| + \hat{r}_k$, and then move on to Step $k+1$.

As before, the restriction (3.22) can easily be satisfied by requiring all coefficients of $S_k^{[j]}$ to be sufficiently small. The main trouble comes from (3.23), which we now handle by the same method.

Complete the unit vector $v_1 := \theta_{\varphi_2 \circ \varphi_1(k)}$ to an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{C}^n$. Define a new linear coordinate system $(w_1, \ldots, w_n)$ for $\mathbb{C}^n$ in which v_ℓ reads as $(0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 appears in the ℓ-th position, for $\ell = 1, \ldots, n$. Rewrite $c_k = n_k \cdot v_1$ where $n_k \in \mathbb{Z}_+$. The idea is, regarding n_k as a new variable, up to changing coordinates linearly between $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$, to construct polynomial

$$S_k^{[j]} = \sum_{\ell=m_k}^{M_k} a_\ell^{[j]}\, w_1^\ell \quad {\scriptstyle (j=1,2,\ldots,m)} \tag{3.24}$$

with respect to variable w_1 having coefficients

$$a_\ell^{[j]} \in \frac{1}{n_k} \cdot \mathbb{C}[w_2, \dots, w_n]\Big[\frac{1}{n_k}\Big] \quad (j=1,2,\dots,m;\ m_k \leq \ell \leq M_k), \tag{3.25}$$

so that (3.22) holds by specializing $n_k \gg 1$. Similarly, we can fix (3.23) by showing that each polynomial $G_k^{[j]}(c_k+z) - g_k^{[j]}(z)$, after changing variables, is a polynomial of variable w_1

$$\text{having coefficients in } \frac{1}{n_k} \cdot \mathbb{C}[w_2, \dots, w_n]\Big[\frac{1}{n_k}\Big]. \tag{3.26}$$

Firstly, by changing coordinates and by gathering binomial expansions, we rewrite

$$(S_0^{[j]} + S_1^{[j]} + \cdots + S_{k-1}^{[j]})(c_k+z) - g_k^{[j]}(z) = \sum_{i=0}^{\ell_k} \hat{a}_{k,i}^{[j]}\, w_1^i \quad (j=1,2,\dots,m), \tag{3.27}$$

where $\ell_k := \max_{1\leq j\leq m}\{M_{k-1}, \deg g_k^{[j]}\}$, and where the coefficients

$$\hat{a}_{k,i}^{[j]} \in \mathbb{C}[w_2, \dots, w_n][n_k] \quad (j=1,2,\dots,m;\ i=0,1,2,\dots,\ell_k). \tag{3.28}$$

Next, by (3.24) we rewrite

$$\begin{aligned} S_k^{[j]}(z+c_k) &= \sum_{\ell=m_k}^{M_k} a_\ell^{[j]}\,(w_1+n_k)^\ell \\ &= \sum_{i=0}^{\ell_k}\Big[\sum_{\ell=m_k}^{M_k} a_\ell^{[j]}\binom{\ell}{i} n_k^{\ell-i}\Big]\cdot w_1^i + (\mathsf{h.o.t.}) \quad (j=1,2,\dots,m) \end{aligned} \tag{3.29}$$

where $(\mathsf{h.o.t})$ are the remaining higher order terms with respect to variable w_1.

Declare $m_k := \ell_k + 1$, $M_k := 2\ell_k + 1$. By solving the linear equations

$$\sum_{\ell=m_k}^{M_k} a_\ell^{[j]}\binom{\ell}{i} n_k^{\ell-i} = -\,\hat{a}_{k,i}^{[j]} \quad (j=1,2,\dots,m;\ i=0,1,2,\dots,\ell_k), \tag{3.30}$$

we obtain the desired $S_k^{[j]}$ as polynomials of variable w_1 with coefficients in

$$\frac{1}{n_k} \cdot \mathbb{C}[w_2, \dots, w_n]\Big[\frac{1}{n_k}\Big][w_1].$$

By mimicking the argument of Observation 3.2, we can verify (3.26). Lastly, by specializing $n_k \gg 1$ and by changing coordinates, we recover $S_k^{[j]}$ in variables $z_1, \dots, z_n$ satisfying both (3.22) and (3.23).

4 Proofs

4.1 Proof of Lemma 1.5

We follow essentially the strategy of [8, Theorem 8].

Our goal is to show that, for any holomorphic map $g : \overline{\mathbb{B}}(0,r) \to X$ defined in some open neighborhood of the closed ball

$$\overline{\mathbb{B}}(0,r) := \{||z|| \leq r\} \subset \mathbb{C}^n,$$

for any positive error bound $\epsilon > 0$, we can find some positive integer m such that $\mathsf{T}_b^{(m)}(f)$ approximates g on $\overline{\mathbb{B}}(0,r)$ up to error ϵ with respect to a fixed distance d_X on X

$$\max_{z\in\overline{\mathbb{B}}(0,r)} d_X(f(z+m\cdot b), g(z)) < \epsilon. \tag{4.1}$$

Of course we must use the condition that f is hypercyclic for T_a. A natural idea is to start with a sequence of increasing positive integers $\{n_i\}_{i\geq 1}$ such that $\{f(\bullet + n_i \cdot a)\}_{i\geq 1}$ approximate $g(\bullet)$ near $\overline{\mathbb{B}}(0,r)$. If moreover $\{n_i\}_{i\geq 1}$ has a subsequence $\{n_{k_i}\}_{i\geq 1}$ such that

$$n_{k_i} \cdot a \approx 0 \quad \text{in} \quad \mathbb{R}\cdot b/\mathbb{Z}\cdot b, \tag{4.2}$$

then by the continuity of g near $\overline{\mathbb{B}}(0,r)$ we are done. However, due to the lack of control on $\{n_i\}_{i\geq 1}$, such approach does not work.

The trick in the proof of [8, Theorem 8] is that, instead of approximating g by orbit of f under T_a, one constructs certain thoughtful $\hat{g}$ so that an analogue of (4.2) holds automatically.

Take a small positive $\delta \ll 1$ such that g is defined on $\overline{\mathbb{B}}(0, r+\delta)$ and uniformly continuous

$$d_X(g(z_1), g(z_2)) < \epsilon/2023 \quad (\forall z_1, z_2 \in \overline{\mathbb{B}}(0,r+\delta) \text{ with } ||z_1 - z_2|| < \delta). \tag{4.3}$$

Choose sufficiently many residue classes $\{[c_i]\}_{i=1}^k$ in $\mathbb{R}\cdot b/\mathbb{Z}\cdot b$ such that

$$\{[c_i]\}_{i=1}^k \text{ are } \delta\text{ -dense}, \tag{4.4}$$

i.e. any vector $c \in \mathbb{R}\cdot b$ can be approximated by a representative c_i' of some class $[c_i]$ within tiny distance $||c - c_i'|| < \delta$. This can be achieved by requiring $k > ||b||/2\delta$. Next, for $i = 1, 2, \ldots, k$, we subsequently select some representatives $\hat{c}_i \in \mathbb{R}\cdot b$ of $[c_i]$ far away from each other so that the closed balls $\overline{\mathbb{B}}(\hat{c}_i, r+\delta)$ are pairwise disjoint. Therefore $\cup_{i=1}^k \overline{\mathbb{B}}(\hat{c}_i, r+\delta)$ is polynomial convex [23]. One can also replace balls with cubes to avoid using [23].

Now we define a holomorphic function g_2 on a neighborhood of $\cup_{i=1}^k \overline{\mathbb{B}}(\hat{c}_i, r+\delta)$ by translations of g

$$g_2 \restriction_{\overline{\mathbb{B}}(\hat{c}_i, r+\delta)} (\bullet) := g(\bullet - \hat{c}_i) \quad {\scriptstyle (i=1,2,...,k)}. \tag{4.5}$$

Since X is Oka, by the BOPA property ([14, p. 235]), we can "extend" g_2 to a holomorphic map $\hat{g}$ from a neighborhood of large ball $\overline{\mathbb{B}}(0, R)$ with radius $R > \max_{1\leq i\leq k}\{||\hat{c}_i||\} + r + \delta$ to X up to small deviation

$$d_X(\hat{g}(z), g_2(z)) < \epsilon/2023 \quad {\scriptstyle (\forall z \in \cup_{i=1}^k \overline{\mathbb{B}}(\hat{c}_i, r+\delta))}. \tag{4.6}$$

Now we take a new sequence of increasing positive integers $\{n_j\}_{j\geq 1}$ such that $\{\mathsf{T}_a^{(n_j)}(f)\}_{j\geq 1}$ converges to $\hat{g}$ on $\overline{\mathbb{B}}(0, R)$, and that in particular

$$\max_{z\in\overline{\mathbb{B}}(0,R)} d_X(f(n_j \cdot a + z), \hat{g}(z)) < \epsilon/2023 \quad {\scriptstyle (\forall j\geq 1)}. \tag{4.7}$$

For each $j \geq 1$, note that at least one vector in $\{n_j \cdot a + \hat{c}_i\}_{i=1}^k$ is near to $0 + \mathbb{Z} \cdot b$ within distance δ, since the segment between $-n_j \cdot a - \delta \cdot \frac{b}{||b||}$ and $-n_j \cdot a + \delta \cdot \frac{b}{||b||}$ cannot avoid all vectors in $\{\hat{c}_i + \mathbb{Z} \cdot b\}_{1\leq i\leq k}$ by our construction (4.4). Hence we can rewrite

$$n_j \cdot a + \hat{c}_i = m_j \cdot b + \delta' \tag{4.8}$$

for one $i \in \{1, 2, \ldots, k\}$, $m_j \in \mathbb{Z}$ and some residue $||\delta'|| < \delta$.

Summarizing (4.3), (4.5), (4.6), (4.7), for any z in $\overline{\mathbb{B}}(0, r)$, we obtain

$$\begin{aligned}
\mathsf{T}_b^{(m_j)} f(z) &= f(m_j \cdot b + z) \\
\text{[from (4.8)]} \quad &= f(n_j \cdot a + \hat{c}_i - \delta' + z) \\
\text{[by (4.7)]} \quad &\approx \hat{g}(\hat{c}_i - \delta' + z) \\
\text{[use(4.6)]} \quad &\approx g_2(\hat{c}_i - \delta' + z) \\
\text{[see (4.5)]} \quad &= g(-\delta' + z) \\
\text{[check (4.3)]} \quad &\approx g(z).
\end{aligned}$$

Thus the estimate (4.1) is guaranteed by adding up small differences in the above three "$\approx$". □

4.2 Proof of Theorem A

We now show that for every θ_p, by our Algorithm, any obtained F in (3.18) is hypercyclic for T_{θ_p}. Indeed, for any pair $\big(f_i = (f_i^{[j]})_{1\leq j\leq m}, r_i\big)$ in (3.16), for any $\epsilon > 0$, we will show that for some integer $n \geq 1$ there holds

$$||F(z + n \cdot \theta_p) - f_i(z)|| \leq \epsilon \quad {\scriptstyle (\forall z \in \mathbb{B}(r_i))} \tag{4.9}$$

By $\{\epsilon_\ell\}_{\ell\geq 1}$ in (3.5), we can find a large M such that $\sum_{\ell\geq M}\epsilon_\ell<\epsilon$. Recalling (3.17) and (3.21), we now select a step number $k>M$ such that $\varphi_1\circ\varphi_1(k)=i$ and $\varphi_2\circ\varphi_1(k)=p$. It is easy to check that $n=n_k$ satisfies the proximity requirement (4.9) by keeping track of the Algorithm.

Lastly, by Lemma 1.5, F is hypercyclic for all nontrivial translations T_a where $a\in\mathbb{R}_+\cdot\theta_p$. □

4.3 Proof of Theorem B

The idea is to choose a holomorphic function $F:\mathbb{C}^n\to\mathbb{C}^{m+1}$ such that:

◇ 1. F has sufficiently slow growth and is hypercyclic for translation operators $\{\mathsf{T}_{\theta_i}\}_{i\geq 1}$;
◇ 2. $F(\mathbb{C}^n)$ avoids the origin $\mathbf{0}\in\mathbb{C}^{m+1}$.

It is clear that $f:=\pi_m\circ F$ satisfies the required hypercyclicity in Theorem B, where π_m is the canonical projection from $\mathbb{C}^{m+1}\setminus\{\mathbf{0}\}$ to $\mathbb{CP}^m$.

The existence of such F is guaranteed by Theorem A and the following

Observation 4.1 *If $m\geq n$, then for any holomorphic map $h:\mathbb{C}^n\to\mathbb{C}^{m+1}$, by Sard's theorem, the image $h(\mathbb{C}^n)$ has zero Lebesgue measure in $\mathbb{C}^{m+1}$.* □

Later, we will take some universal holomorphic function $\tilde{F}:\mathbb{C}^n\to\mathbb{C}^{m+1}$ with slow growth, which is hypercyclic for all $\{\mathsf{T}_{\theta_i}\}_{i\geq 1}$, and choose certain $v\notin\tilde{F}(\mathbb{C}^n)$, so that $F:=\tilde{F}-v$ satisfies our requirements ◇ 1 and ◇ 2 above. The remaining difficulty is to make sure (1.7).

We need some insight from higher dimensional Nevanlinna theory for constructing such F.

By homogeneity, the angular form $\gamma:=\mathrm{d}^c\log||z||^2\wedge(\mathrm{d}\,\mathrm{d}^c\log||z||^2)^{n-1}$ contributes constant integral

$$\int_{||z||=r}\gamma=1\quad(\forall r>0)\tag{4.10}$$

on each sphere $\{||z||=r\}$. By Jensen's formula [27, Lemma 2.1.33], the characteristic function of $f:=\pi_m\circ F$ can be rephrased as (denote $\alpha:=\mathrm{dd}^c||z||^2$)

$$\begin{aligned}T_f(r)&=\int_1^r\frac{\mathrm{d}\,t}{t^{2n-1}}\int_{\mathbb{B}(t)}\mathrm{d}\,\mathrm{d}^c\log\Big(\sum_{i=0}^m|F_i|^2\Big)\wedge\alpha^{n-1}\\&=\int_{||z||=r}\log\Big(\sum_{j=0}^m|F_j(z)|^2\Big)^{\frac12}\gamma(z)-\int_{||z||=1}\log\Big(\sum_{j=0}^m|F_j(z)|^2\Big)^{\frac12}\gamma(z).\end{aligned}\tag{4.11}$$

Define $\phi:\mathbb{R}_+\to\mathbb{R}_+$ to be constantly 1 on the interval (0, 1) and

$$\phi(r):=r^{\frac{\psi(r)}{2}}\quad(\forall r\geq 1).\tag{4.12}$$

Then ϕ is continuous and nondecreasing. By Theorem A, given $\{\theta_i\}_{i\geq 1} \subset \mathbb{S}^{2n-1}$, there is some universal holomorphic map $\tilde{F} : \mathbb{C}^n \to \mathbb{C}^{m+1}$ with slow growth

$$||\tilde{F}(z)|| \leq \frac{\phi(||z||)}{2^k} \quad {\scriptstyle (\forall\, z \in \mathbb{C}^n)}, \tag{4.13}$$

such that $\tilde{F}$ is hypercyclic for all translations $\{\mathsf{T}_{\theta_i}\}_{i\geq 1}$. Here, the value $k \gg 1$ is to be determined.

By Observation 4.1, we can find a vector $v \in \mathbb{C}^{m+1} \setminus \tilde{F}(\mathbb{C}^n)$ in the open region

$$1 - \frac{1}{2^{k-1}} < ||v|| < 1 - \frac{1}{2^k}. \tag{4.14}$$

Set $F(z) := \tilde{F}(z) - v$. It is clear that F is hypercyclic for all $\{\mathsf{T}_{\theta_i}\}_{i\geq 1}$. By (4.13), (4.14), and by $\phi(r) \geq 1$ for $r \geq 1$, we have

$$||F(z)|| \leq \frac{\phi(||z||)}{2^k} + 1 - \frac{1}{2^k} \leq \phi(||z||) \quad {\scriptstyle (\forall\, |z| \geq 1)}. \tag{4.15}$$

In particular

$$\sup_{||z||=1} ||F(z)|| \leq \phi(1) = 1. \tag{4.16}$$

Applying (4.13) and (4.14) again, we can estimate the lower bound of $||F||$ on the unit sphere

$$\inf_{||z||=1} ||F(z)|| \geq ||v|| - \max_{||z||=1} ||\tilde{F}(z)|| > \left(1 - \frac{1}{2^{k-1}}\right) - \frac{1}{2^k}. \tag{4.17}$$

Thus we can estimate (4.11) by

$$\begin{aligned} T_f(r) &= \int_{||z||=r} \log\left(\sum_{j=0}^m |F_j(z)|^2\right)^{\frac{1}{2}} \gamma(z) - \int_{||z||=1} \log\left(\sum_{j=0}^m |F_j(z)|^2\right)^{\frac{1}{2}} \gamma(z) \\ &\leq \log\phi(r) \cdot \int_{||z||=r} \gamma(z) - \int_{||z||=1} \log||F(z)|| \cdot \gamma(z) \quad [\text{by } ((4.15))] \\ &\leq \frac{1}{2}\psi(r) \cdot \log r - \log\left(1 - \frac{1}{2^{k-1}} - \frac{1}{2^k}\right) \quad [\text{by } (4.10), (4.12), (4.16), (4.17)]. \end{aligned} \tag{4.18}$$

Fix $\epsilon_0 \in (0, 1)$. Choose some large $M_1 > 0$ such that

$$-\log\left(1 - \frac{1}{2^{k-1}} - \frac{1}{2^k}\right) \leq \log\,(1 + \epsilon_0) \cdot \frac{\psi(1)}{2} \quad {\scriptstyle (\forall\, k \geq M_1)}.$$

Plugging the above estimate into (4.18), we get

$$T_f(r) \leq \psi(r) \cdot \log r \quad (\forall\, r \geq 1+\epsilon_0). \tag{4.19}$$

Next, we prove the remaining part of (1.7) for $1 < r < 1+\epsilon_0 < 2$ by an alternative method.

The pullback of Fubini-Study form $f^*\omega_{\mathsf{FS}}$ is

$$\mathrm{d}\,\mathrm{d}^c\log\left(\sum_{i=0}^m ||F_i(z)||^2\right) = \frac{\sqrt{-1}}{2\pi}\left(\frac{\sum_{i,s,l}\frac{\partial F_i}{\partial z_s}dz_s \wedge \frac{\partial \overline{F_i}}{\partial \overline{z_l}}d\overline{z_l}}{||F(z)||^2} - \frac{(\sum_{i,s}\overline{F_i}\frac{\partial F_i}{\partial z_s}dz_s)\wedge(\sum_{j,l}F_j\frac{\partial \overline{F_j}}{\partial \overline{z_l}}d\overline{z_l})}{||F(z)||^4}\right).$$

We now estimate F and its derivatives on $\{||z|| \leq 2\}$.

By the same reasoning as (4.16) and (4.17), we can obtain

$$\left(1-\frac{1}{2^{k-1}}\right) - \frac{\phi(2)}{2^k} \leq \inf_{||z||\leq 2} ||F(z)|| \leq \sup_{||z||\leq 2} ||F(z)|| \leq \phi(2).$$

By Cauchy's integral formula for derivatives, we have

$$\frac{\partial F_i}{\partial z_s}(z) = \frac{\partial \tilde{F}_i}{\partial z_s}(z) = \frac{1}{2\pi\sqrt{-1}}\int_{|\zeta_k - z_k|=1} \frac{\tilde{F}_i(z_0,\ldots,\zeta_s,\ldots,z_m)}{(z_s-\zeta_s)^2}d\zeta_s \quad (\forall\, ||z||\leq 2).$$

Hence by (4.13)

$$\left|\frac{\partial F_i}{\partial z_s}(z)\right| \leq \max_{|\zeta_s - z_s|=1} |\tilde{F}_i(z_0,\ldots,\zeta_s,\ldots,z_m)| \leq \frac{\phi(2+1)}{2^k} \quad (\forall\, ||z||\leq 2).$$

Hence the (n,n) form $\mathrm{d}\,\mathrm{d}^c\log\left(\sum_{i=0}^m |F_i(z)|^2\right) \wedge \alpha^{n-1}$ on the ball $\mathbb{B}(2) \subset \mathbb{C}^n$ is bounded from above by $O(1)\cdot\frac{1}{4^k}\cdot\alpha^n$ for some $O(1)$ depending on ϕ. Thus for any $r \in [1,2]$ we have

$$\begin{aligned} T_f(r) &= \int_1^r \frac{\mathrm{d}\,t}{t^{2n-1}}\int_{\mathbb{B}(t)} \mathrm{d}\,\mathrm{d}^c\log\left(\sum_{i=0}^m |F_i|^2\right)\wedge\alpha^{n-1} \leq \int_1^r \frac{\mathrm{d}\,t}{t^{2n-1}}\int_{\mathbb{B}(t)} O(1)\cdot\frac{1}{4^k}\cdot\alpha^n \\ &= O(1)\cdot\frac{1}{4^k}(r^2-1) \leq O(1)\cdot\frac{1}{4^k}\cdot\log r \quad \text{[caution: two } O(1)'\text{ s are different].} \end{aligned} \tag{4.20}$$

Choose $M_2 > 0$ such that $O(1)\cdot\frac{1}{4^{M_2}} \leq \psi(1)$. Setting $k \geq \max\{M_1, M_2\}$, by (4.19) and (4.20), we conclude the proof of (1.7). □

Remark 4.2 When $n > m$, if one considers universal meromorphic maps or correspondences between $\mathbb{C}^n$ and $\mathbb{CP}^m$ (cf. [27, p. 50]), then a likewise estimate (1.7) still holds true by much the same construction. However, the more essential problem is about minimal growth of universal holomorphic maps, and we are interested in whether the shapes depend on n or not.

4.4 Proof of Theorem C

Given countably many directions $\{\theta_i\}_{i\geq 1}$ in the unit sphere $\mathbb{S}^{2n-1}\subset\mathbb{C}^n$ and a continuous nondecreasing function $\psi:\mathbb{R}_{\geq 1}\to\mathbb{R}_+$ tending to infinity. Define a continuous positive function $\phi:\mathbb{R}_{\geq 0}\to\mathbb{R}_+$ by

$$\phi(r)=\begin{cases}1 & (0\leq r<1),\\ r^{\frac{\psi(r)}{2}} & (r\geq 1).\end{cases}$$

By Theorem A, we receive some universal holomorphic map $\tilde{F}:\mathbb{C}^n\to\mathbb{C}^m$ with slow growth $||\tilde{F}(\bullet)||\leq\phi(||\bullet||)$, which is hypercyclic for translation operators along these directions $\{\theta_i\}_{i\geq 1}$.

It is clear that $F=\pi\circ\tilde{F}$ satisfies our desired universality and hypercyclicity. Now we check the growth. By Jensen's formula [27, Lemma 2.1.33], for any $r>1$, we have

$$\begin{aligned}T_F(r)&=\int_1^r\frac{\mathrm{d}\,t}{t^{2n-1}}\int_{\mathbb{B}(t)}\mathrm{d}\,\mathrm{d}^c||\tilde{F}||^2\wedge\alpha^{n-1}\\&=\frac{1}{2}\int_{||z||=r}||\tilde{F}(z)||^2\,\gamma(z)-\frac{1}{2}\int_{||z||=1}||\tilde{F}(z)||^2\,\gamma(z).\end{aligned}$$

Noting that $\int_{||z||=r}\gamma=1$ and that $||\tilde{F}(z)||^2\leq\phi(||z||)^2\leq r^{\psi(r)}$ for $||z||=r$, we conclude the proof. □

Remark 4.3 The slow growth rate (1.8) is optimal, since $T_F(\bullet)$ must grow faster than any polynomial due to the universality of F.

4.5 Proof of Theorem D

Consider the set of hypercyclic angles of F

$$\hat{I}:=\{[\theta]\in\mathbb{R}/2\pi\cdot\mathbb{Z}: F\text{ is hypercyclic for }\mathsf{T}_{e^{\sqrt{-1}\theta}}\}.$$

The angular circle $\mathbb{R}/2\pi\cdot\mathbb{Z}$ has a natural metric induced from the Euclidean norm of $\mathbb{R}$, and our goal is to show that, for any $\alpha>0$ and $\delta>0$, there holds

$$H^{\alpha}_{\delta}(\hat{I})=0,\tag{4.21}$$

where we use the standard notation

$$H_\delta^\alpha(\hat{I}) = \inf\left\{\sum_{i\geq 1}(\operatorname{diam} U_i)^\alpha \;:\; \cup_{i\geq 1} U_i \supset \hat{I},\ \operatorname{diam} U_i < \delta\right\},$$

the infimum being taken over all countable covers $\{U_i\}_{i\geq 1}$ of $\hat{I}$ having diameters less than δ.

Our strategy of proving (4.21) is by using the slow growth of the zeros-counting function of F

$$\begin{aligned} n_F(t,0) := &\text{ cardinality of } \{z\in\mathbb{D}_t : F(z)=0\}\\ &\text{counting multiplicities} \quad {\scriptstyle(\forall t>0)}. \end{aligned} \tag{4.22}$$

The insight comes from the First Main Theorem in Nevanlinna theory.

Suppose F is hypercyclic for $\mathsf{T}_{e^{\sqrt{-1}\theta}}$. It is clear that F cannot be any translation of the identity map $z\mapsto z$. Hence for any $\epsilon>0$, there is some positive integer r_ϵ such that $\mathsf{T}_{e^{\sqrt{-1}\theta}}^{(r_\epsilon)}(F)$ approximates the identity map on the unit disc

$$\sup_{|z|\leq 1} |F(z+r_\epsilon\cdot e^{\sqrt{-1}\theta})-z| < \epsilon.$$

If ϵ is sufficiently small, say $\epsilon<1$, then by Rouché's theorem, F has exactly one zero in the disc $\mathbb{D}(r_\epsilon\cdot e^{\sqrt{-1}\theta},1)$. By taking an infinite shrinking sequence $\{\epsilon_i\}_{i\geq 1}\searrow 0$, we see the corresponding $\{r_{\epsilon_i}\}_{i\geq 1}\nearrow +\infty$. Thus F must have infinitely many zeros within Euclidean distance 1 to the ray $\mathbb{R}_+\cdot e^{\sqrt{-1}\theta}$.

For a fixed universal holomorphic function F in Theorem C, set

$$\begin{aligned} E_k := \{[\theta]\in\mathbb{R}/2\pi\cdot\mathbb{Z} : \exists\, r\in[2^k,2^{k+1}) \text{ such that } F\\ \text{has zeros in } \mathbb{D}(r\cdot e^{\sqrt{-1}\theta},1)\} \quad {\scriptstyle(\forall k\geq 1)}. \end{aligned} \tag{4.23}$$

Then $\hat{I}\subset\cap_{\ell\geq 1}\cup_{k\geq\ell} E_k$.

We are going to show that, if ϕ grows sufficiently slow, there must hold

$$\sum_{k\geq 1} H_\delta^\alpha(E_k) < +\infty, \tag{4.24}$$

hence (4.21) follows immediately by the Borel–Cantelli lemma.

Now Nevanlinna theory plays a key rôle. Recall Nevanlinna's inequality [27, Theorem 1.1.18] that the growth of F "controls" the number of zeros of F (see (4.22)):

$$T_F(r)+O(1)\geq N_F(r,0) := \int_{t=1}^{r} n_F(t,0)\,\frac{\mathrm{d}\,t}{t}, \quad {\scriptstyle(\forall r>1)}, \tag{4.25}$$

where we constantly abuse the notation $O(1)$ for uniformly bounded terms. For an auxiliary positive increasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ tending to infinity $\lim_{r\to+\infty}\psi(r)=$

$+\infty$, we consider the transcendental growth rate

$$\phi(r) := r^{\psi(r)}. \tag{4.26}$$

Then, by Cartan's formula (1.5), the growth condition (1.9) implies that the left-hand side of (4.25)

$$T_F(r) + O(1) \leq \psi(r) \cdot \log r + O(1) \quad (\forall\, r > 1).$$

Combing the estimates

$$N_F(r^2, 0) \;\leq\; T_F(r^2) + O(1) \;\leq\; \psi(r^2) \cdot \log (r^2) + O(1) \quad (\forall\, r > 1)$$

and

$$N_F(r^2, 0) \;\geq\; \int_r^{r^2} n_F(r, 0) \frac{\mathrm{d}\, t}{t} \;=\; n_F(r, 0) \cdot \log r \quad (\forall\, r > 1),$$

we receive that

$$n_F(r, 0) \;\leq\; 2 \cdot \psi(r^2) + o(1) \quad (\forall\, r > 1). \tag{4.27}$$

Now we choose ψ in (4.26) to grow extremely slowly, say

$$\psi((2^{k+2})^2) \leq k \quad (\forall\, k \geq 1). \tag{4.28}$$

Whence the number of zeros of F in each annulus $\mathscr{A}_k := \{2^k - 1 < |z| < 2^{k+1} + 1\}$ (for $k \geq 1$) is at most (see (4.27), (4.28))

$$n_F(2^{k+1}+1, 0) - n_F(2^k - 1, 0) \leq n_F(2^{k+2}, 0) \leq 2 \cdot \psi((2^{k+2})^2) + o(1) \leq 2k + o(1).$$

Hence by (4.23) and by plane geometry, E_k can be covered by intervals of the shape $[\theta_j - \frac{\pi}{r_j}, \theta_j + \frac{\pi}{r_j}]$ (mod $2\pi \cdot \mathbb{Z}$), where $r_j \cdot e^{\sqrt{-1}\theta_j}$ is a zero of F in $\mathscr{A}_k$. Thus

$$\begin{aligned} H_\delta^\alpha(E_k) \;&\leq\; \Big(2k + o(1)\Big) \cdot H_\delta^\alpha\Big(\Big[-\tfrac{\pi}{2^k - 1}, \tfrac{\pi}{2^k - 1}\Big]\Big) \\ &\leq\; \big(2k + o(1)\big) \cdot \Big(\tfrac{2\pi}{2^k - 1}\Big)^\alpha \quad \Big(\forall\, k \gg 1, \text{ s.t. } \tfrac{2\pi}{2^k - 1} < \delta\Big). \end{aligned}$$

Therefore

$$\sum_{k \geq 1} H_\delta^\alpha(E_k) < +\infty$$

is guaranteed for any $\alpha > 0$.

Lastly, we use Lemma 2.1 to replace ϕ in (4.26) by an inferior real analytic function of the shape (1.4). This conclude the proof. □

4.6 Proof of Theorem E

The insight is that certain Cantor set in the interval [0, 1] has zero Lebesgue measure but is uncountable. We now modify our algorithm to construct some direction set I likewise.

Notice that each positive integer n has a finite length $l(n)$ in binary expansion

$$n = \sum_{k=0}^{l(n)} a_k\, 2^k, \quad a_0, a_1, a_2, ..., a_{l(n)} \in \{0, 1\}, \quad a_{l(n)} = 1.$$

We reset the sequence $\{(g_j, \hat{r}_j)\}_{j\geq 1}$ in (3.3) as $g_j := f_{\varphi_1(l(j))}$, $\hat{r}_j := r_{\varphi_1(l(j))}$. Thus each (f_i, r_i) for $i = 1, 2, 3, \ldots$ still repeats infinitely many times in $(g_j, \hat{r}_j)_{j\geq 1}$, while

$$\begin{aligned} g_{2^k} = g_{2^k+1} = \cdots = g_{2^{k+1}-1} = f_{\varphi_1(l(2^k))}, \quad \hat{r}_{2^k} = \hat{r}_{2^k+1} \\ = \cdots = \hat{r}_{2^{k+1}-1} \quad (\forall\, k \geq 0). \end{aligned} \tag{4.29}$$

The second place we adjust is the choice of $c_k = r_k \cdot e^{\sqrt{-1}\theta_k}$ in polar coordinates $r_k \in \mathbb{Z}_+, \theta_k \in \mathbb{R}/2\pi \cdot \mathbb{Z}$. Since each G_k obtained by our Algorithm is continuous and satisfies (3.7), we can retain the modulus r_k and slightly perturb the argument θ_k of c_k within a small angle $0 < \delta_k \ll 1$ so that a similar estimate

$$\begin{aligned} |G_k(r_k \cdot e^{\sqrt{-1}\eta} + z) - g_k(z)| \leq 2\epsilon_k \\ (\forall\, z \in \mathbb{D}_{\hat{r}_k};\ \forall\, \eta \in (\theta_k - \delta_k, \theta_k + \delta_k) \bmod 2\pi \cdot \mathbb{Z}) \end{aligned} \tag{4.30}$$

still holds true. By our Algorithm, we can subsequently choose $r_k \in \mathbb{R}_+$, $\theta_k \in \mathbb{R}/2\pi \cdot \mathbb{Z}$ and $0 < \delta_k \ll 1$ for $k = 1, 2, 3, \ldots$ with the additional requirements that $(\theta_{2k} - \delta_{2k}, \theta_{2k} + \delta_{2k})$ and $(\theta_{2k+1} - \delta_{2k+1}, \theta_{2k+1} + \delta_{2k+1})$ are disjoint subsets of $(\theta_k - \delta_k, \theta_k + \delta_k)$ and that $1 \gg \delta_1 \gg \delta_2 \gg \delta_3 \gg \cdots$.

Gathering all possible angles

$$E_k = \bigcup_{i=2^k}^{2^{k+1}-1} (\theta_i - \delta_i, \theta_i + \delta_i) \quad (\forall\, k \geq 0)$$

obtained in adjacent steps, we see a fast decreasing pattern $E_0 \supset E_1 \supset E_2 \supset \cdots$. Thus the Cantor-like set $\cap_{k=0}^{+\infty} E_k$ is uncountable.

On the other hand, by (4.29), (4.30) and by the arguments in Sect. 3.1, we can show that all arguments $\theta \in \cap_{k=0}^{+\infty} E_k$ produce $a = e^{\sqrt{-1}\cdot\theta}$ such that that F is hypercyclic for T_a.

It suffices to check that for any pair (f_i, r_i) in Sect. 3.1 and any $\epsilon > 0$, there is some large integer $n \gg 1$ such that

$$||F(z + n \cdot \theta) - f_i(z)|| \leq \epsilon \tag{4.31}$$

uniformly for $z \in \mathbb{B}(r_i)$. Using the sequence $\{\epsilon_\ell\}_{\ell\geq 1}$ defined in (3.5), we can find a large integer M such that $\sum_{\ell\geq M}\epsilon_\ell < \epsilon$. Then, recalling (4.29) and (3.21), we select a step number $k > M$ such that $\varphi_1 \circ l(k) = i$ and $\theta \in (\theta_k - \delta_k, \theta_k + \delta_k)$. It is easy to check that $n = n_k$ satisfies the proximity requirement (4.31) by chasing the algorithm and by noticing (4.30).

Lastly, by Lemma 1.5, we conclude the proof. □

5 Reflections

Constructing interesting holomorphic objects is one central theme in complex geometry. Let us recall the modern Oka principle [15, p. 369]:

Analytic problems on Stein manifolds which can be formulated in terms of maps to Oka manifolds, or liftings with respect to Oka maps, have only topological obstructions.

In light of Nevanlinna theory we shall test the above slogan by seeking interesting holomorphic maps into Oka manifolds Y with slow growth. See e.g. [11] in this line of thought. In particular, we are interested in minimal growths of universal holomorphic maps from various source spaces into Y. In general, such questions are very difficult, because certain effective version of Runge's approximation theorem ought to be required. For instance, from $\mathbb{C}^n$ to $\mathbb{C}^m$, it is our Algorithm. Nevertheless, we would like to ask

Question 5.1 Is it true that for any compact Oka manifold Y, there exist some universal entire curves $f : \mathbb{C} \to Y$ having slow growth no more than the shape (1.8)?

Question 5.2 If the answer to the above question is yes, is the same statement holds true for any compact Oka-1 manifold?

Recall Kusakabe's result [24, Theorem 1.4] that an Oka manifold Y admits universal holomorphic maps $f : X \to Y$ from various source spaces X with mild symmetry.

Question 5.3 What can we say about minimal growth of f from other source space $X \neq \mathbb{C}$? Here if Y is not compact, we shall find appropriate metrics on Y for defining Nevanlinna characteristic functions T_f.

The above questions seek quantitative invariants of Oka (resp. Oka-1) manifolds. They could also provide new insight for certain classification problems, e.g., whether any K3 surface or Fano manifold is Oka or not. Moreover, the techniques developed along this line shall shed light for constructing entire and rational curves on Fano manifolds *via analytic methods*, which has been a long standing challenge in complex geometry.

Acknowledgements We were inspired by an insight of Sibony [31]. This paper would never happen without Forstnerič's excellent online lectures on Oka manifolds theory in AMSS SCV seminar, November 2022. We thank Zhangchi Chen and Dinh Tuan Huynh for stimulating conversations and valuable suggestions which improved this manuscript.

Funding The second named author is partially supported by National Key R&D Program of China Grant no. 2021YFA1003100 and NSFC Grant no. 12288201.

Declarations

Conflict of interest On behalf of the two authors, the corresponding author states that there is no conflict of interest.

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