

VANISHING OF INVARIANT 2-JET DIFFERENTIALS AND IMPROVED HYPERBOLICITY DEGREE BOUNDS IN DIMENSION TWO

LEI HOU, DINH TUAN HUYNH, JOËL MERKER, AND SONG-YAN XIE

ABSTRACT. This paper establishes new degree bounds for Kobayashi hyperbolicity in dimension two. Our main results are:

- A *very generic* surface in $\mathbb{P}^3$ of degree at least 17 is Kobayashi hyperbolic.
- The complement of a *generic* curve in $\mathbb{P}^2$ of degree at least 12 is Kobayashi hyperbolic.

These bounds improve the long-standing records in the field, lowering the threshold from 18 to 17 for surfaces (Păun) and from 14 to 12 for complements (Rousseau).

Central to the proofs are new vanishing results for certain negatively twisted invariant 2-jet differentials, obtained through a novel combination of algebraic reduction and computer algebra. Since Demailly’s Santa Cruz lectures in 1995, the thresholds for the existence of such differentials—and consequently the limits of what 2-jet techniques can accomplish toward the Kobayashi conjecture in dimension two—have been recognized as $d = 15$ in the compact case and $d = 11$ in the logarithmic case. While previous approaches were unable to reach these targets, the present work provides both the theoretical foundations and the algorithmic framework required to access them, and has already improved the known bounds to $d = 17$ and $d = 12, 13$, respectively.

As an unexpected byproduct, our computational method reveals the existence of nonzero negatively twisted invariant 2-jet differentials with weighted degree 3 for hyperelliptic-type equations of degree at least 11 in the logarithmic case and degree at least 15 in the compact case.

Moreover, in the logarithmic setting, we establish an effective quantitative refinement via a Second Main Theorem in Nevanlinna theory. Specifically, for every generic smooth curve $C \subset \mathbb{P}^2$ of degree $d \geq 12$ and any nonconstant holomorphic entire curve $f: \mathbb{C} \rightarrow \mathbb{P}^2$, we prove the inequality

$$T_f(r) \leq C_d N_f^{[1]}(r, C) + o(T_f(r)) \quad \parallel,$$

where C_d is an explicit constant depending only on d . The notation “ $\parallel$ ” signifies that the estimate holds for all $r > 1$ outside a set of finite Lebesgue measure.

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1. Introduction

1.1. Motivation. In 1967, Shoshichi Kobayashi introduced an intrinsically defined pseudo-distance for every connected complex manifold X . The infinitesimal version of this pseudo-distance was later characterized by Royden [59]. It is given by

$$F_p(v) := \inf \left\{ \lambda > 0 \mid \exists f: \{ |z| < 1 \} \rightarrow X \text{ holomorphic, } f(0) = p, f'(0) = v/\lambda \right\} \quad (p \in X, v \in T_p X).$$

If this Finsler pseudometric is nondegenerate everywhere, X is called *Kobayashi hyperbolic*. Throughout, an *entire curve* in X means a nonconstant holomorphic map $f: \mathbb{C} \rightarrow X$. For compact X , Brody’s criterion [4] states that Kobayashi hyperbolicity is equivalent to the absence of entire curves.

In dimension one, the Uniformization Theorem and Liouville’s Theorem imply that all Riemann surfaces of genus ≥ 2 are hyperbolic. It is natural to expect that, in higher dimensions, hyperbolic complex manifolds should likewise represent the “most” case (see the preface of [39]). This leads to the following celebrated conjecture:

Kobayashi Hyperbolicity Conjecture. *A generic hypersurface H in the projective n -space $\mathbb{P}^n$ of degree $d \geq 2n - 1$ is hyperbolic, for every $n \geq 3$.*

Here and throughout the paper, we work over the complex number field $\mathbb{C}$. The term *generic* means that the corresponding property holds outside a proper algebraic subvariety of the relevant parameter space.

On the other hand, the Little Picard theorem may be viewed as stating that the complement of 3 distinct points in the Riemann sphere $\mathbb{P}^1$ is hyperbolic. This leads naturally to the following conjecture in higher dimensions:

Logarithmic Kobayashi Hyperbolicity Conjecture. *The complement of a generic hypersurface $H \subset \mathbb{P}^n$ of degree $d \geq 2n + 1$ is hyperbolic, for every $n \geq 2$.*

The expected optimal degree bounds $2n - 1$ in the compact case and $2n + 1$ in the logarithmic case were suggested by Zaidenberg [76]. Fundamental work of Clemens [11], Ein [22, 23], and Voisin [70] shows that, for very generic hypersurfaces of sufficiently high degree, all subvarieties are of general type; see also the logarithmic counterpart due to Pacienza and Rousseau [51]. Here, a *very generic* hypersurface means one lying in the complement of a countable union of proper Zariski-closed subsets of the parameter space. Assuming the Green–Griffiths–Lang conjecture and its logarithmic analogue, these results would imply the above Kobayashi hyperbolicity conjectures in the expected optimal degree range.

Since then, the study of hyperbolicity has attracted considerable attention, in part due to its deep connections with Diophantine geometry. In particular, Lang [40] conjectured that an algebraic variety $V_{\mathbb{K}}$ defined

over a number field $\mathbb{K}$ contains only finitely many $\mathbb{K}$ -rational points provided that, after a base change $\mathbb{K} \hookrightarrow \mathbb{C}$, the complex variety $V_{\mathbb{C}}$ is hyperbolic.

Substantial progress has been made over the past decades toward the above conjectures, especially in the regime where the degree of the hypersurface is sufficiently large compared with the dimension (see [15] for an overview).

The Kobayashi hyperbolicity conjecture for surfaces in $\mathbb{P}^3$ was first verified by McQuillan [45] for degrees $d \geq 36$, and subsequently improved by Demailly–El Goul [18] to $d \geq 21$, and by Păun [52] to $d \geq 18$. In the logarithmic setting, hyperbolicity of the complement of a generic plane algebraic curve was established by Siu–Yeung [67] for sufficiently high degree, then improved by El Goul [24] to $d \geq 15$, and later refined by Rousseau [57] to $d \geq 14$ using Siu’s slanted vector fields and Păun’s strategy (see also [24]).

In higher dimensions, the Kobayashi conjectures are known to hold for generic hypersurfaces of sufficiently large degree. We refer the reader to several key milestones [63, 20, 65, 6, 54, 1] and the extensive references therein for further background on the topic.

From a different perspective, constructions of hyperbolic hypersurfaces were developed. For general n , the existence of such hypersurfaces in $\mathbb{P}^n$ was first established by Masuda and Noguchi [43], who also gave examples for $n = 3, 4$ (with verification by “Mathematica” code) and considered the complement case. Subsequently, quadratic degree bounds were obtained by Siu and Yeung [68] ($d(n) \geq 16(n-1)^2$) and later improved by Shiffman and Zaidenberg [62] ($d(n) \geq 4(n-1)^2$). The current best asymptotic bound is $d \geq \left(\frac{n+2}{2}\right)^2$, due to Huynh [33]. In low dimensions, using a deformation method of Zaidenberg [77], examples of hyperbolic hypersurfaces of minimal degree $d = 2n$ were constructed by Duval [21] for $n = 3$ and by Huynh [34] for $4 \leq n \leq 6$.

1.2. Fundamental Vanishing Theorem for Entire Curves. Most progress in the study of hyperbolicity problems has relied heavily on the theory of jet differentials, a framework originally introduced by André Bloch [2]. Building on Bloch’s ideas, Green and Griffiths [27] introduced, for every projective manifold X , the bundle $E_{k,m}^{GG}T_X^* \rightarrow X$ of jet polynomials of order k and weighted degree m (see Section 2.1).

Focusing on properties intrinsic to the entire curve itself, Demailly [16] further refined the Green–Griffiths construction by introducing the subbundle $E_{k,m}T_X^* \subset E_{k,m}^{GG}T_X^*$ (see Subsection 2.2). In this work, we will primarily use $E_{2,m}T_X^*$ with $k = 2$ and $m \geq 3$.

Every global section of such a jet differential bundle, when twisted by the dual of an ample line bundle, gives rise to an *algebraic differential equation* that must be satisfied by every nonconstant entire holomorphic curve $f: \mathbb{C} \rightarrow X$.

Fundamental vanishing theorem ([27, 66, 16]). *Let X be a projective complex manifold and $\mathcal{A}$ an ample line bundle on X . For any nonzero global holomorphic jet differential with negative twist*

$$0 \neq \mathcal{P} \in H^0(X, E_{k,m}^{GG}T_X^* \otimes \mathcal{A}^{-1}),$$

every nonconstant entire curve $f: \mathbb{C} \rightarrow X$ satisfies the associated differential equation

$$\mathcal{P}(f^{(1)}, \dots, f^{(k)}) \equiv 0.$$

The principal challenge in the jet differential approach to the hyperbolicity problem lies in constructing a sufficient number of algebraically independent, negatively twisted jet differentials. The goal of such a construction is to eliminate all derivatives, thereby yielding *purely algebraic equations* that constrain entire curves. For a generic hypersurface $H \subset \mathbb{P}^{n+1}$ of dimension $n \geq 3$, this challenge has been addressed in the works cited above [64, 20, 65, 6, 54, 1], albeit only under the condition that the degree of H is sufficiently large relative to its dimension.

In dimension 2, for a smooth surface $X \subset \mathbb{P}^3$, a result of Sakai [61] shows that nontrivial symmetric differentials do not exist; equivalently, $H^0(X, E_{1,m}T_X^*) = 0$ for all $m \geq 1$. Hence, one must consider jet differentials of order $k \geq 2$.

By combining standard Riemann–Roch computations with the Bogomolov vanishing theorem [3], Demailly [16] established the existence of nonzero negatively twisted 2-jet differentials for smooth surfaces $X \subset \mathbb{P}^3$ of degree $d \geq 15$, thereby identifying $d = 15$ as the threshold down to which the 2-jet approach has the potential to address the Kobayashi conjecture in the compact case.

In the logarithmic setting, an analogous Riemann–Roch computation (see [16, 24]) shows that nontrivial negatively twisted invariant logarithmic 2-jet differentials exist for complements of smooth curves $\mathcal{C} \subset \mathbb{P}^2$

of degree $d \geq 11$, marking $d = 11$ as the threshold down to which the 2-jet approach has the potential to address the logarithmic Kobayashi conjecture.

Motivated by these thresholds, the present work introduces a new method that lays the theoretical and algorithmic groundwork for reaching them. Before presenting our approach, we first recall the classical strategy developed by Păun [52], which forms the starting point of our investigation.

1.3. Păun’s Strategy. The proof of Păun [52] relies on four essential ingredients.

Denote by $\mathbb{P}^{N_d}$ the parameter space of surfaces of degree d in $\mathbb{P}^3$, where $N_d = \binom{d+3}{3} - 1$. The first key result is the following celebrated theorem.

Theorem 1.1 (Clemens [11]). *A very generic surface $X_a \subset \mathbb{P}^3$ of degree $d \geq 5$ contains no rational curves. Furthermore, a very generic surface $X_a \subset \mathbb{P}^3$ of degree $d \geq 6$ contains no elliptic curves.*

The degree bound for elliptic curves was subsequently improved from $d \geq 6$ to $d \geq 5$ by Xu [75].

The second technique originates from McQuillan’s deep work on entire curves tangent to foliations on surfaces of general type [44].

Theorem 1.2 (McQuillan [44]). *Every parabolic leaf of an algebraic (multi-) foliation on a surface X of general type is algebraically degenerate.*

Consequently, the problem reduces to constructing two algebraically independent families of negatively twisted invariant 2-jet differentials $\omega_{1,a}$ and $\omega_{2,a}$ on X_a , as a varies over the parameter space of surfaces of degree ≥ 5 in $\mathbb{P}^3$. By the fundamental vanishing theorem for entire curves, any entire curve $f: \mathbb{C} \rightarrow X_a$ must satisfy $f^*\omega_{1,a} \equiv 0$ and $f^*\omega_{2,a} \equiv 0$. This implies that f is tangent to a foliation, and McQuillan’s Theorem then forces f to be algebraically degenerate. Finally, Theorem 1.1 guarantees that a very generic surface $X_a \subset \mathbb{P}^3$ of degree ≥ 5 contains no rational or elliptic curves. Combining these results yields the hyperbolicity of X_a for very generic parameters a .

In practice, however, proving the existence of two such independent families of differentials $\omega_{1,a}$ and $\omega_{2,a}$ presents a substantial difficulty. The remaining two techniques are precisely designed to circumvent this obstacle.

The third technique uses Miyaoka’s method [49] (see also Lu–Yau [42]) in the form developed by Demailly and El Goul [18]. It employs a Riemann–Roch type computation on the zero locus $\{\omega_1 = 0\}$ inside the second level X_2 (here we omit the parameter a for brevity) of the Demailly–Semple tower.

More precisely, let c_1 and c_2 denote the first and second Chern classes of X , respectively. Suppose there exists a nonzero irreducible negatively twisted invariant 2-jet differential

$$\omega_1 \in H^0(X, E_{2,m}T_X^* \otimes \mathcal{O}_X(-t(d-4)))$$

that satisfies

$$t < \frac{13c_1^2 - 9c_2}{12c_1^2} m.$$

Then one can show that the line bundle $\mathcal{O}_{X_2}(1)$ is big on the “nontrivial” component of the zero locus $\{\omega_1 = 0\} \subset X_2$; see Proposition 6.1. By the bigness of $\mathcal{O}_{X_2}(1)$ restricted to this component, one can produce a second nonzero negatively twisted differential ω_2 defined only there. Because the entire curve f satisfies $f^*\omega_1 \equiv 0$, its lift $f_{[2]}$ lies entirely inside $\{\omega_1 = 0\}$; consequently f also satisfies $f^*\omega_2 \equiv 0$. Hence f is tangent to the foliation defined by the pair (ω_1, ω_2) , and McQuillan’s theorem can be applied as before.

One therefore focuses on the remaining case where

$$t \geq \frac{13c_1^2 - 9c_2}{12c_1^2} m.$$

For this situation, Păun developed a fourth key technique: the construction of *slanted vector fields*, a method originally envisioned by Siu [64] in the spirit of Clemens [11]. The method works effectively when the vanishing order $t(d-4)$ of the initial nonzero irreducible negatively twisted 2-jet differential ω_1 exceeds the pole order of a carefully chosen slanted vector field V_1 . Differentiating ω_1 by V_1 then yields a second algebraically independent negatively twisted jet differential ω_2 , thereby completing the proof.

When $d \geq 18$ and $m \geq 6$, it is assumed that the vanishing order of the initial 2-jet differential ω_1 satisfies

$$t(d-4) \geq \frac{13c_1^2 - 9c_2}{12c_1^2} m(d-4) > 7.$$

Thus slanted vector fields of pole order 7 can be applied directly, as they are abundant and (generically) globally generated.

In the complementary range $3 \leq m \leq 5$ (still with $d \geq 18$), the same assumption ensures that

$$t(d-4) > 3,$$

so that, at least formally, only slanted vector fields of pole order 3 are accessible. Their effective construction, which ultimately succeeds, rests on a key technical contribution of Păun [52]. The principal difficulty lies in ensuring that the resulting differential ω_2 is algebraically independent of ω_1 . Păun proceeds by contradiction: assuming otherwise leads to a certain ODE, and he shows that this ODE admits no nontrivial solution. This ODE-based argument, however, breaks down when $m \geq 6$, leaving the extension to that case an open problem.

In the logarithmic case, by adapting the three key techniques described above, Rousseau [57] proved that the complement $\mathbb{P}^2 \setminus \mathcal{C}$ is hyperbolic for generic plane curves $\mathcal{C}$ of degree $d \geq 14$, thus improving the previous bounds of Siu–Yeung [67] and El Goul [24].

1.4. Our Main Results. In our earlier work [31], we introduced a method for proving the vanishing of negatively twisted logarithmic 2-jet differentials, motivated by an effective Second Main Theorem in Nevanlinna theory. That paper treated three generic conics in $\mathbb{P}^2$, a setting where the computations simplify considerably. Writing the three conics as homogeneous polynomials A, B, C , one can work on a suitable large Zariski-open set where the logarithmic 2-jet differential frame is transparently generated by $(\log \frac{A}{C})'$, $(\log \frac{B}{C})'$ and their Wronskian determinant. This convenient semi-global frame drastically reduces the algebraic complexity.

In the present paper we study two settings in which such a frame is no longer available: the complement of a single smooth curve in $\mathbb{P}^2$, and a smooth surface $X = \{R = 0\}$ in $\mathbb{P}^3$. For surfaces, the defining equation $R = 0$ itself imposes an additional relation among the jet coordinates, which must be taken into account throughout the reduction; this introduces further algebraic complications. In both cases, substantial extra effort is required to control the jet differentials across all affine charts and to convert the holomorphicity conditions into a tractable linear system. The earlier three-conics approach was tailored to that specific configuration, but its conceptual framework is the one we adapt and significantly extend here.

Building on this framework, we develop a new method aimed at reaching the 2-jet thresholds $d = 15$ in the compact case and $d = 11$ in the logarithmic case for the Kobayashi conjectures in dimension two. For more than two decades, a major obstacle to lowering the degree bounds below 18 (compact) and 14 (logarithmic) has been the lack of essential vanishing results for certain invariant (logarithmic) 2-jet differentials. In this work, we make a decisive advance by proving the following two key vanishing lemmas, which allow us to lower the bounds to $d = 17$ in the compact case and to $d = 12, 13$ in the logarithmic case.

Lemma 1.3 (Key Vanishing Lemma – Logarithmic Case). *Let $\mathcal{C} \subset \mathbb{P}^2$ be a generic curve of degree d . Then:*

- For $d = 13$, $H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}(-t)) = 0$ for all (m, t) in $\{(3, 3), (4, 4), (5, 5), (6, 5), (6, 6), (7, 6), (7, 7), (8, 7)\}$.
- For $d = 12$, $H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}(-t)) = 0$ for all (m, t) in $\{(3, 2), (4, 2), (5, 3), (6, 3), (7, 4), (8, 4), (9, 5), (10, 5), (11, 6), (12, 6), (13, 7), (14, 7)\}$.

The proof of Lemma 1.3 is given in Section 3. It relies on a delicate adaptation and generalization of the method introduced in our previous work [31], following a three-step strategy.

- **Algebraic Reduction.** Working in affine coordinates (x, y) of $\mathbb{C}^2 \subset \mathbb{P}^2$, let R be the polynomial defining the curve $\mathcal{C}$ in this affine chart. We first derive the most general local expression of a global section of the negatively twisted logarithmic invariant 2-jet differential bundle on an open set where a given partial derivative of R is nonvanishing, say $\{R_y \neq 0\}$. Such sections are expressed as polynomials in the jet variables divided by powers of the nonvanishing partial derivative. Imposing holomorphicity on the alternative chart $\{R_x \neq 0\}$ yields severe restrictions on the admissible pole

orders. A key innovation of the present paper, compared to our earlier work [31], is the discovery and systematic exploitation of the triangular structure of the transition formulas; this allows the restrictions to be determined recursively. The analysis reduces an a priori infinite-dimensional problem to a finite-dimensional linear system satisfied by the coefficients of the polynomials appearing in the local expression. Indeed, for generic R , the presumed negatively twisted logarithmic invariant 2-jet differentials must take a finite form (see Proposition 3.1 and (3.22)). The remaining task is to determine whether this finite-dimensional linear system admits nontrivial solutions. The same algebraic reduction strategy will be applied, with the necessary modifications, to the compact case treated later in this paper. The ideas and methods developed in this step constitute a foundational contribution of our paper; they will serve as a key tool in future work on the lower degree case, in the study of higher-order jet differentials, and more broadly in other related problems.

- **Computational Verification.** Solving the resulting linear systems directly for a generic polynomial R would be computationally prohibitive, as the number of variables easily exceeds ten thousand; even extracting the linear system from the geometric constraints is itself a nontrivial task. A key technical innovation lies in our choice of a special polynomial R . The trick is to select an equation that is a deformation of a Fermat-type polynomial, so that the relevant partial derivative simplifies drastically to $R_y = dy^{d-1}$. This makes the divisibility conditions particularly simple to implement: requiring divisibility by $(y^{d-1})^i$ means that after clearing denominators, all coefficients of y^k for $k < (d-1)i$ must vanish identically. These conditions translate into explicit linear systems on the coefficients of the polynomials appearing in the local expressions. Solving them by computer algebra confirms that, for each specified (m, t) with $d = 12, 13$, the only solution is the trivial one.¹
- **Geometric Extension.** Finally, by the semicontinuity theorem in algebraic geometry, the vanishing established for this single carefully chosen example automatically extends to generic curves. This standard argument allows us to deduce the generic vanishing statements from computations performed on a specific smooth curve $\mathcal{C}$.

Lemma 1.4 (Key Vanishing Lemma – Compact Case). *Let $X \subset \mathbb{P}^3$ be a generic surface of degree $d = 17$. Then $H^0(X, E_{2,m}T_X^* \otimes \mathcal{O}_X(-t)) = 0$ for all (m, t) in*

$$\{(3, 3), (4, 4), (5, 5), (6, 6), (7, 7)\}.$$

The proof of Lemma 1.4 is given in Section 4. Building upon the techniques developed for the logarithmic case, a similar argument allows us to control the structure of the presumed negatively twisted invariant 2-jet differentials, showing that they must take a finite form; see Proposition 4.1 and (4.28). Nevertheless, the compact case presents additional algebraic challenges. In affine coordinates (x, y, z) of $\mathbb{C}^3 \subset \mathbb{P}^3$, let $R \in \mathbb{C}[x, y, z]$ be the polynomial defining the surface X . The key difference from the logarithmic case lies in the local coordinate ring of X : there it is the polynomial ring $\mathbb{C}[x, y]$, whereas here it is the quotient ring $\mathbb{C}[x, y, z]/(R)$. To convert divisibility conditions in this quotient ring into concrete constraints in $\mathbb{C}[x, y, z]$, we introduce a new theoretical framework that ensures effective computability (see Lemmas 4.2).

- **An algorithmic reduction.** We select the surface $X \subset \mathbb{P}^3$ to be a deformation of the Fermat type:

$$(X^d + Y^d + Z^d + T^d) + X^a T^b = 0,$$

where the positive integers a, b satisfy $a + b = d$ and are chosen so that $|a - b|$ is small. The Fermat type is chosen for its simplicity, which will significantly reduce the subsequent computational complexity. We must include the extra term $X^a T^b$ precisely because, as shown by Brotbek’s Wronskian construction for 1-jet differentials obtained via cohomological computations [5], the pure Fermat surface itself (with $d \geq 9$) is known to carry nonvanishing negatively twisted invariant jet differentials. The observation that this construction extends to the 2-jet case was noted, in a geometric way, by Xie [72, Proposition 6.10] (see also [73, 7, 6]).

Working in the affine chart $\{T \neq 0\}$, the equation dehomogenizes as

$$R(x, y, z) = 1 + x^d + y^d + z^d + x^a,$$

where a will be chosen as $\lfloor d/2 \rfloor$. This particular form of the defining equation yields two crucial simplifications.

¹The Maple file, along with accompanying explanatory and output PDFs, is available at: <https://xiesongyan.github.io/>.

The first is that the relevant partial derivative simplifies to $R_z = dz^{d-1}$, which is independent of x and y . This renders the divisibility conditions—requiring an expression to be divisible by $(R_z)^i$ in the quotient ring $\mathbb{C}[x, y, z]/(R)$ —far more tractable (see Lemma 4.2).

Second, since the extra term $X^a T^b$ involves no variable Y , we can use the relation

$$y^d = -1 - x^d - z^d - x^a$$

on the affine chart $X \cap \{T \neq 0\}$ to express every element in the coordinate ring $\mathbb{C}[x, y, z]/(R)$ uniquely and efficiently as a polynomial in $\mathbb{C}[x, y, z]$ with $\deg_y < d$. The reduction is achieved by directly eliminating all higher powers y^k with $k > d$: performing Euclidean division on the exponent k by d gives $k = dp + q$, and substituting $(y^d)^p = (-1 - x^d - z^d - x^a)^p$ reduces the term to y^q multiplied by an expression in x and z . The strategic choice of y —rather than the seemingly more natural choice z —as the reduction variable is essential to our algorithm (see again Lemma 4.2).

This careful choice of the defining equation enables us to rapidly set up a finite linear system, transforming a computationally prohibitive problem into a tractable one. As a further technical simplification, one may work modulo a prime — say $p = 19$ — since a linear system with integer coefficients that has no nontrivial solution modulo p also has none over $\mathbb{Z}$ (and hence over $\mathbb{C}$). To demonstrate the effectiveness of our method, we focus here on the degree 17 case. Finally, we implement this procedure in Maple to verify all the required vanishing results.²

Equipped with the two Key Vanishing Lemmas, we are now ready to state our main results. A notable methodological departure from the earlier works of Păun [52] and Rousseau [57] is that our proof avoids the subtle and somewhat exceptional slanted vector fields of pole order 3. For the qualitative hyperbolicity bounds in Theorems 1.7 and 1.8, our proof avoids the exceptional pole-order-3 vector fields and uses only the abundant pole-order-7 fields. The pole-order-3 construction is retained only in the quantitative optimization for Theorem 1.9 in the $d \geq 14$ case.

The later pole-lowering arguments of [13, 32] also show that the present proof can be reorganized with pole order 4. We retain pole order 7 here because it gives the cleanest presentation of the argument developed in this paper; the lower pole order changes the finite Key Vanishing list but introduces no new principle needed for the degree bounds proved below.

Theorem 1.5. *Let $d \geq 17$. There exists a Zariski-open subset $U \subset \mathbb{P}^{N_d^3}$ in the parameter space of degree- d surfaces in $\mathbb{P}^3$ with the following property: for each $a \in U$, there exists a proper subvariety S_a in the first level $X_{a,1}$ of the Demailly–Semple tower for the surface $X_a \subset \mathbb{P}^3$ such that:*

- (1) *The family $\{S_a\}_{a \in U}$ depends algebraically on the parameter $a \in U$;*
- (2) *For every entire curve $f: \mathbb{C} \rightarrow X_a$, its lift $f_{[1]}: \mathbb{C} \rightarrow X_{a,1}$ satisfies $f_{[1]}(\mathbb{C}) \subset S_a$.*

The formulation of Theorem 1.5 is intricate, as our ultimate goal is to establish the Kobayashi hyperbolicity of a *generic* degree- d surface in $\mathbb{P}^3$ for $d \geq 17$ (see Question 7.5 and Remark 7.6). Achieving this requires additional efforts by means of holomorphic foliation theory (see Remark 7.7), which lie beyond the scope of the present paper. Nonetheless, the theorem above can serve as a first step in this direction.

The case $d \geq 18$ was essentially established by Păun [52] (earlier, Demailly–El Goul [18] obtained a similar conclusion for $d \geq 21$). Their results, together with our application of the Grauert’s direct image theorem (an idea suggested to us by Siu, see Section 6.2), already yield Theorem 1.5 for $d \geq 18$. Consequently, we will focus on the remaining cases $d = 17$. The proof of Theorem 1.5 for $d = 17$ will be presented in Sections 6.2 and 6.3.

Theorem 1.5 shows that for a generic surface X_a of degree $d \geq 17$, every entire curve $f: \mathbb{C} \rightarrow X_a$ is tangent to a multi-foliation. Combined with McQuillan’s Theorem 1.2, this immediately yields the following conclusion.

Theorem 1.6. *For a generic surface $X \subset \mathbb{P}^3$ of degree $d \geq 17$, every entire curve $f: \mathbb{C} \rightarrow X$ is algebraically degenerate.* $\square$

For a very generic surface X_a of degree ≥ 17 , Clemens’ Theorem 1.1 asserts the absence of rational and elliptic curves. Hence, Theorem 1.6 yields the following hyperbolicity result.

²The Maple file, together with an explanatory PDF and a PDF of the run results, is available at: <https://xiesongyan.github.io/>.

Theorem 1.7. *A very generic surface $X \subset \mathbb{P}^3$ of degree $d \geq 17$ is Kobayashi hyperbolic.* $\square$

This strengthens Păun’s result [52] by lowering the degree threshold from $d \geq 18$ to $d = 17$.

In the logarithmic setting, we prove the analogous result:

Theorem 1.8. *Let $C \subset \mathbb{P}^2$ be a generic algebraic curve of degree $d \geq 12$. Then the complement $\mathbb{P}^2 \setminus C$ is Kobayashi hyperbolic and hyperbolically embedded.*

This statement is formulated for “generic” rather than “very generic” curves, as the algebraic degeneracy of entire curves in $\mathbb{P}^2 \setminus C$ for a generic C already implies Kobayashi hyperbolicity for a generic C .

The case $d \geq 14$ was already established by Rousseau in [57] (see Remark 6.4 for an alternative approach). Consequently, we will focus on the remaining cases $d = 12$ and 13 . The proof of Theorem 1.8 will be given in Section 6.4.

Moreover, we strengthen the above hyperbolicity result by establishing a Second Main Theorem in Nevanlinna theory.

Theorem 1.9. *For a generic curve C of degree $d \geq 12$ in $\mathbb{P}^2$ and for any entire curve $f: \mathbb{C} \rightarrow \mathbb{P}^2$, the following inequality holds:*

$$(1.1) \quad T_f(r) \leq C_d N_f^{[1]}(r, C) + o(T_f(r)) \quad \parallel,$$

where C_d is a constant depending only on d . Explicitly,

- For $d \geq 14$,

$$C_d = \frac{72}{32d - 117 - \sqrt{640d^2 - 1248d + 1017}}.$$

- For $d = 13$,

$$C_{13} = \frac{18}{113 - \sqrt{12653}} \approx 35.0.$$

- For $d = 12$,

$$C_{12} = \frac{30}{173 - \sqrt{29809}} \approx 86.4.$$

We recall the standard notation of Nevanlinna theory in Section 2.5. A related result due to Campana, Darondeau, and Rousseau [9, Proposition 4.8] shows that every entire curve ramifying over a smooth curve of degree $d \geq 12$ with sufficiently high order must satisfy an algebraic differential equation of order 2. Here Theorem 1.9 provides a quantitative estimate in Nevanlinna theory, valid for any entire curve. Its proof follows the strategy introduced in our previous work [31] and is given in Section 6.5 (see also Remark 7.2 for the case $d = 11$).

The numerical coefficient in the restricted zero-locus branch is $3/\tau$: the residual factor left after removing the double vertical Wronskian factor can vanish and must not be divided out. This correction enlarges only the explicit constants above; the degree bounds and the Key Vanishing Lemmas are unchanged.

1.5. Overview of the Paper’s Approach. The story begins with André Bloch’s foundational work a century ago [2], which established that complex hyperbolicity is governed by the positivity of k -jet differential bundles. This positivity carries two distinct meanings: abstract existence of negatively twisted k -jet differentials (guaranteed by Riemann–Roch) and effective control of their base locus. The latter is far more delicate and lies at the heart of the field. To address it, one adopts a two-pronged strategy: the intrinsic side uses abstract algebro-geometric structures, while the extrinsic side introduces explicit constructions such as Siu’s slanted vector fields. Yet even this combination falls short for low-degree cases. The present paper supplies the missing piece: the key vanishing lemmas, which prove that the weighted degree m and negative twist t of certain abstractly obtained jet differentials cannot fall into certain critical ranges, thereby bridging the gap between abstract existence and effective vanishing.

Our proof of the Kobayashi hyperbolicity conjectures in dimension two for degrees ≥ 17 (compact case) and ≥ 12 (logarithmic case) weaves together intrinsic and extrinsic algebro-geometric arguments. On the intrinsic side, we rely on the Demailly–Semple tower, Riemann–Roch computations, and the direct image theorem. On the extrinsic side — for those (m, t) beyond intrinsic reach — we employ Siu’s method of slanted vector fields of pole order 7 and develop a systematic approach that culminates in the key vanishing lemmas. A useful analogy is the Riemann–Roch theorem for curves, which admits both an intrinsic

sheaf-theoretic proof and an extrinsic proof based on coordinate changes and pole–zero counting (cf. [28]); our approach follows a similar extrinsic philosophy, though the algebraic complexity is vastly higher in dimension two.

The linear algebra techniques of Siu and Yeung [67, 65] were designed to construct sections when the degree is sufficiently high. In this paper, we develop a method that, in principle, computes all sections for a given (m, t) . However, such nonzero sections exist only when m is taken sufficiently large — far beyond current computational reach — leaving explicit constructions in low-degree settings elusive. Nonetheless, by proving vanishing results, we establish hyperbolicity without ever exhibiting an explicit section.

Our first key contribution is to determine the finite algebraic structure of negatively twisted invariant 2-jet differentials: we prove that any such global section must take a specific finite form (see Proposition 3.1 together with (3.21), and Proposition 4.1 together with (4.27)), reducing an infinite-dimensional problem to a finite-dimensional linear system. While a similar local description appeared in our previous work [31], it was tailored to three conics; the present general form applies to arbitrary curves and surfaces.

Our second contribution is the design of efficient algorithms to solve these linear systems. For the logarithmic case, we adapted the algorithmic framework of [31]. The compact case required a fundamentally new theoretical approach, because the local coordinate ring is no longer $\mathbb{C}[x, y]$ but the quotient ring $\mathbb{C}[x, y, z]/(R)$, necessitating a novel reduction procedure to transform divisibility conditions into concrete algebraic constraints (see Lemmas 4.2).

Our third contribution is the implementation of these algorithms in Maple, which systematically verifies the vanishing results for $d = 12, 13$ (logarithmic) and $d = 17$ (compact). The scale is substantially larger than in our previous work: the hardest case in [31] involved “only” 12,550 variables, whereas here the largest linear systems involve over 300,000 variables (logarithmic) and over 600,000 variables (compact). Solving systems of this size required careful algorithm design and optimization within Maple’s capabilities.

These vanishing lemmas improve the degree bounds for the Kobayashi conjecture, achieving 17 in the compact case and 12, 13 in the logarithmic case.

The Kobayashi hyperbolicity conjectures conceal a vast ocean of complexity. The degree of the hypersurface serves as a gauge of depth: lower degrees bring us closer to the seabed of hidden structures, where the true nature of the problem reveals itself. In dimension two, reaching the thresholds $d = 15$ (compact case) and $d = 11$ (logarithmic case) would mark the limit of what 2-jet techniques alone can achieve, and would illuminate the intrinsic complexity beneath — a complexity rooted in deep theoretical structures, yet demanding cross-disciplinary collaboration to uncover.

Some parts of the algebraic reductions and our Maple implementation are designed for a one-term perturbation of the Fermat equation. This simple choice works remarkably well for the cases we have settled — degree 17 in the compact setting and degrees 12 and 13 in the logarithmic setting — but it is too rigid to reach the ultimate 2-jet thresholds. Indeed, for the small parameters $(m, t) = (3, 1)$ both in the compact case with $d = 15$ and in the logarithmic case with $d = 11$, the algorithm produces a non-zero space of solutions, thereby exhibiting nontrivial negatively twisted jet differentials (see Remarks 3.2 and 4.4). This outcome is actually reassuring: it confirms that the code correctly detects the existence of such differentials when they are present, and it shows that the hyperelliptic-type equation we have used is not sufficiently generic for those lowest degrees.

To prove the Key Vanishing Lemma for degree 15 (compact) and degree 11 (logarithmic) one must start from a more involved defining equation, for instance a hypersurface that contains two carefully chosen extra monomials beyond the Fermat equation. The resulting linear systems will be substantially larger — the number of unknown coefficients easily exceeds 60 million in the compact case and 20 million in the logarithmic case — and they lie far beyond the capabilities of the present Maple implementation. Realising such a computation therefore poses a formidable engineering challenge: it will require an efficient C++ implementation, massive parallelisation, and likely a more refined choice of the defining equation to keep the systems within manageable bounds. Nevertheless, we are optimistic that with sustained efforts these two bottom cases will eventually be settled.

1.6. Structure of the Paper. Section 2 provides the necessary background material. Sections 3 and 4 form the core of the paper, where we establish two key vanishing lemmas for invariant 2-jet differentials with low vanishing orders. For completeness, Section 5 recalls Siu’s slanted vector field method, following the constructions of Păun and Rousseau. Section 6 contains the proofs of the main theorems. Finally, Section 7 outlines several directions for future research.

2. Preliminaries

2.1. Jet Bundles and Jet Differentials. Let X be an n -dimensional complex manifold. The k -jet bundle $J_k X$ consists of equivalence classes of holomorphic germs $f: (\mathbb{C}, 0) \rightarrow (X, x)$, where two germs f and g are equivalent if their Taylor expansions agree up to order k in some local coordinates around x . The equivalence class of a germ f is called the k -jet of f and is denoted by $j_k(f)$.

For a point $x \in X$, denote by $J_k(X)_x$ the space of all k -jets of holomorphic germs $(\mathbb{C}, 0) \rightarrow (X, x)$. The total k -jet bundle is then defined as the disjoint union

$$J_k(X) := \coprod_{x \in X} J_k(X)_x.$$

Together with the natural projection

$$\pi_k: J_k(X) \longrightarrow X, \quad j_k(f) \longmapsto x = f(0),$$

the set $J_k(X)$ carries the structure of a holomorphic fibre bundle over X .

When $k = 1$, the fibre $J_1(X)_x$ is naturally identified with the tangent space $T_x X$, and $J_1(X)$ is canonically isomorphic to the holomorphic tangent bundle T_X . For $k \geq 2$, however, the fibres $J_k(X)_x$ no longer carry a uniform vector space structure that varies holomorphically with x ; consequently, $J_k(X)$ is in general only a (non-linear) fibre bundle, not a vector bundle.

Let $U \subset X$ be an open subset. Given a holomorphic 1-form $\omega \in H^0(U, T_X^*)$ and a k -jet $j_k(f) \in J_k(X)|_U$, the pullback $f^*\omega$ can be written as $A(z)dz$ for some holomorphic function A . Because the derivatives $A^{(j)}$ ($0 \leq j \leq k-1$) depend only on the equivalence class $j_k(f)$ and not on the choice of representative f , the form ω induces a well-defined holomorphic map

$$(2.1) \quad \tilde{\omega}: J_k(X)|_U \longrightarrow \mathbb{C}^k, \quad j_k(f) \longmapsto (A(z), A'(z), \dots, A^{(k-1)}(z)).$$

Now suppose $\omega_1, \dots, \omega_n$ is a local holomorphic coframe on U , i.e. $\omega_1 \wedge \dots \wedge \omega_n \neq 0$. Using the maps $\tilde{\omega}_i$ defined as in (2.1), we obtain a trivialisation

$$\begin{aligned} H^0(U, J_k(X)) &\longrightarrow U \times (\mathbb{C}^k)^n, \\ \sigma &\longmapsto (\pi_k \circ \sigma; \tilde{\omega}_1 \circ \sigma, \dots, \tilde{\omega}_n \circ \sigma). \end{aligned}$$

The components $x_i^{(j)}$ ($1 \leq i \leq n$, $1 \leq j \leq k$) of the vectors $\tilde{\omega}_i \circ \sigma$ are called *jet-coordinates*.

Now let D be a normal crossing divisor on X . Following Iitaka [37], the *logarithmic cotangent bundle of X along D* , denoted by $T_X^*(\log D)$, is the locally free sheaf that is locally generated by the forms

$$\frac{dz_1}{z_1}, \dots, \frac{dz_\ell}{z_\ell}, dz_{\ell+1}, \dots, dz_n,$$

where $z_1, \dots, z_\ell, z_{\ell+1}, \dots, z_n$ are local coordinates centred at a point x in which D is given by

$$D = \{z_1 \cdots z_\ell = 0\}.$$

Here $\ell = \ell(x)$ denotes the number of local branches of D passing through x .

A holomorphic section $s \in H^0(U, J_k(X))$ over an open set $U \subset X$ is called a *logarithmic k -jet field* if, for every open subset $U' \subset U$ and every logarithmic 1-form $\omega \in H^0(U', T_X^*(\log D))$, the composition $\tilde{\omega} \circ s$ is holomorphic on U' , where $\tilde{\omega}$ is the map induced by ω as in (2.1). The collection of all such logarithmic k -jet fields forms a subsheaf of $J_k(X)$; this subsheaf is precisely the sheaf of sections of a holomorphic fibre bundle over X , called the *logarithmic k -jet bundle of X along D* and denoted by $J_k(X, -\log D)$ [50].

A *jet differential* (resp. *logarithmic jet differential*) of order k and weighted degree m at a point $x \in X$ is a polynomial $Q(f', \dots, f^{(k)})$ on the fibre of $J_k(X)$ (resp. $J_k(X, -\log D)$) over x that is homogeneous with respect to the $\mathbb{C}^*$ -action

$$Q(\lambda f', \lambda^2 f'', \dots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \dots, f^{(k)}) \quad (\forall \lambda \in \mathbb{C}^*).$$

Denote by $E_{k,m}^{GG} T_X^*|_x$ (resp. $E_{k,m}^{GG} T_X^*(\log D)|_x$) the vector space of such polynomials and set

$$E_{k,m}^{GG} T_X^* := \bigcup_{x \in X} E_{k,m}^{GG} T_X^*|_x, \quad E_{k,m}^{GG} T_X^*(\log D) := \bigcup_{x \in X} E_{k,m}^{GG} T_X^*(\log D)|_x.$$

By Faà di Bruno's formula [12, 48], both $E_{k,m}^{GG} T_X^*$ and $E_{k,m}^{GG} T_X^*(\log D)$ carry the structure of holomorphic vector bundles over X ; they are called the *Green–Griffiths bundle* and the *logarithmic Green–Griffiths bundle*, respectively.

The classical result below states that every (logarithmic) jet differential twisted negatively by an ample line bundle yields an algebraic differential equation satisfied by all entire curves.

Fundamental Vanishing Theorem for Entire Curves ([27, Section 2], [66, Lemma 2.1], [16, Corollary 7.9]). *Let X be a smooth complex projective variety and $\mathcal{A}$ an ample line bundle on X .*

(1) **Compact case.** *If*

$$0 \neq \mathcal{P} \in H^0(X, E_{k,m}^{GG} T_X^* \otimes \mathcal{A}^{-1}),$$

then every nonconstant holomorphic curve $f: \mathbb{C} \rightarrow X$ satisfies

$$\mathcal{P}(f', \dots, f^{(k)}) \equiv 0.$$

(2) **Logarithmic case.** *Let $D \subset X$ be a normal crossing divisor. If*

$$0 \neq \mathcal{P} \in H^0(X, E_{k,m}^{GG} T_X^*(\log D) \otimes \mathcal{A}^{-1}),$$

then every nonconstant holomorphic curve $f: \mathbb{C} \rightarrow X \setminus D$ satisfies

$$\mathcal{P}(f', \dots, f^{(k)}) \equiv 0.$$

2.2. Invariant Jet Differentials. Geometrically, only the image of an entire curve in X matters, not its specific parametrization. This motivates the notion of *invariant jets differentials*, introduced by Demailly [16] in the compact case and later extended to the logarithmic setting by Dethloff–Lu [19].

Let $\mathbb{G}_k$ be the group of germs of k -jets of biholomorphisms of $(\mathbb{C}, 0)$:

$$\mathbb{G}_k := \left\{ \varphi: t \mapsto \sum_{j=1}^k c_j t^j \mid c_1 \in \mathbb{C}^*, c_j \in \mathbb{C} (j \geq 2) \right\},$$

where the group operation is composition modulo t^{k+1} . The group $\mathbb{G}_k$ acts on $J_k(X)$ by reparametrization on the right:

$$\varphi \cdot j_k(f) := j_k(f \circ \varphi).$$

A (logarithmic) jet differential P over an open set $U \subset X$ is called *invariant under $\mathbb{G}_k$* if for every $\varphi \in \mathbb{G}_k$ it satisfies

$$(2.2) \quad P((f \circ \varphi)', \dots, (f \circ \varphi)^{(k)}) = (\varphi')^m P(f', \dots, f^{(k)}).$$

This invariance under reparametrization is the crucial property that distinguishes (Demailly–Semple) invariant jet differentials from Green–Griffiths jet differentials.

The subbundle

$$E_{k,m} T_X^* \subset E_{k,m}^{GG} T_X^* \quad (\text{resp. } E_{k,m} T_X^*(\log D) \subset E_{k,m}^{GG} T_X^*(\log D))$$

whose fibres consist of all such invariant polynomials is called the *Demailly–Semple bundle* (resp. the *logarithmic Demailly–Semple bundle*). A key advantage of these bundles, especially in dimension two, is their favourable positivity properties. For example, they allow one to apply Riemann–Roch techniques at lower degrees, which in turn yields the existence of negatively twisted global sections that are crucial for establishing hyperbolicity.

Throughout this article we work exclusively with jets of order $k = 2$. When $\dim_{\mathbb{C}} X = 2$, the bundle of invariant (logarithmic) 2-jet differentials admits a filtration whose associated graded bundle is given by [16, Section 12]:

$$(2.3) \quad \text{Gr}^\bullet E_{2,m} T_X^* = \bigoplus_{0 \leq j \leq \lfloor m/3 \rfloor} S^{m-3j} T_X^* \otimes K_X^j,$$

$$(2.4) \quad \text{Gr}^\bullet E_{2,m} T_X^*(\log D) = \bigoplus_{0 \leq j \leq \lfloor m/3 \rfloor} S^{m-3j} T_X^*(\log D) \otimes \overline{K}_X^j,$$

where K_X is the canonical bundle and $\overline{K}_X := K_X \otimes \mathcal{O}_X(D)$ is the logarithmic canonical bundle.

This grading reflects the fact that an invariant 2-jet differential $P(f', f'')$ (or its logarithmic counterpart outside D) can be written locally as

$$(2.5) \quad P(f', f'') = \sum_{0 \leq j \leq \lfloor m/3 \rfloor} \sum_{\substack{\alpha_1 + \alpha_2 = m - 3j \\ \alpha_1, \alpha_2 \geq 0}} R_{\alpha_1, \alpha_2, j}(f_1, f_2) (f_1')^{\alpha_1} (f_2')^{\alpha_2} (f_1' f_2'' - f_2' f_1'')^j,$$

where $f = (f_1, f_2)$ are local coordinates of the holomorphic germ, and the coefficients $R_{\alpha_1, \alpha_2, j}$ are holomorphic functions. For higher-order invariant jet differentials, analogous local expressions exist, though they are considerably more complicated, and the bases are related by intricate syzygies; see [46].

2.3. Demailly–Semple Tower. We now recall the construction of the Demailly–Semple tower [16, Section 6].

A *directed manifold* is a pair (X, V) where X is a complex manifold and $V \subset T_X$ is a holomorphic sub-bundle of the tangent bundle.

In the compact two-dimensional setting, we begin with the directed manifold $(X_0, V_0) := (X, T_X)$, where X is a generic algebraic surface in $\mathbb{P}^3$. Set

$$X_1 := \mathbb{P}(V_0),$$

equipped with the natural projection $\pi_{1,0}: X_1 \rightarrow X_0$. The sub-bundle $V_1 \subset T_{X_1}$ is defined fibre-wise by

$$(2.6) \quad V_{1,(x,[v])} := \left\{ \xi \in T_{X_1,(x,[v])} \mid (\pi_{1,0})_* \xi \in \mathbb{C} \cdot v \right\},$$

where $\mathbb{C} \cdot v \subset V_{0,x} = T_{X,x}$. This yields the first level (X_1, V_1) of the Demailly–Semple tower over (X_0, V_0) .

Iterating the same construction gives the second level, which fits into the tower

$$(X_2, V_2) \longrightarrow (X_1, V_1) \longrightarrow (X_0, V_0),$$

with projections $\pi_{2,1}: X_2 \rightarrow X_1$ and $\pi_{2,0} := \pi_{1,0} \circ \pi_{2,1}: X_2 \rightarrow X_0$.

Let $\mathcal{O}_{X_1}(-1)$ denote the tautological line bundle of $X_1 = \mathbb{P}(V_0)$; thus

$$\mathcal{O}_{X_1}(-1)_{(x,[v])} = \mathbb{C} \cdot v.$$

From the definition of V_1 we obtain the short exact sequence

$$(2.7) \quad 0 \longrightarrow T_{X_1/X_0} \longrightarrow V_1 \xrightarrow{(\pi_{1,0})^*} \mathcal{O}_{X_1}(-1) \longrightarrow 0,$$

where T_{X_1/X_0} is the relative tangent bundle of the fibration $\pi_{1,0}: X_1 \rightarrow X_0$. Over the fibre $\mathbb{P}(V_{0,x})$ this construction restricts to the usual Euler sequence, giving the relative Euler sequence

$$(2.8) \quad 0 \longrightarrow \mathcal{O}_{X_1} \longrightarrow \pi_{1,0}^* V_0 \otimes \mathcal{O}_{X_1}(1) \longrightarrow T_{X_1/X_0} \longrightarrow 0.$$

At the second level there is a canonical inclusion $\mathcal{O}_{X_2}(-1) \hookrightarrow \pi_{2,1}^* V_1$. Composing it with $\pi_{2,1}^*(\pi_{1,0})^*$ yields a morphism of line bundles

$$(2.9) \quad \mathcal{O}_{X_2}(-1) \longrightarrow \pi_{2,1}^* \mathcal{O}_{X_1}(-1),$$

whose zero divisor is the hyperplane sub-bundle

$$(2.10) \quad \Gamma_2 := \mathbb{P}(T_{X_1/X_0}) \subset \mathbb{P}(V_1) = X_2.$$

Consequently,

$$(2.11) \quad \mathcal{O}_{X_2}(\Gamma_2) \cong \pi_{2,1}^* \mathcal{O}_{X_1}(-1) \otimes \mathcal{O}_{X_2}(1).$$

In the logarithmic case, following Dethloff–Lu [19], we consider the notion of a *log-directed manifold*, a triple (X, D, V) where X is a complex manifold, D is a simple normal crossing divisor, and V is a sub-bundle of the logarithmic tangent bundle $T_X(-\log D)$.

We start with the initial log-directed manifold

$$(X_0, D_0, V_0) := (X, D, T_X(-\log D)).$$

Set $X_1 := \mathbb{P}(V_0)$ with the natural projection $\pi_1: X_1 \rightarrow X_0$, and define $D_1 := \pi_1^* D_0$, so that π_1 becomes a log-morphism. The sub-bundle $V_1 \subset T_{X_1}(-\log D_1)$ is then given fibre-wise by

$$V_{1,(x,[v])} := \left\{ \xi \in T_{X_1,(x,[v])}(-\log D_1) \mid (\pi_1)_* \xi \in \mathbb{C} \cdot v \right\},$$

where $\mathbb{C} \cdot v \subset V_{0,x} = T_{X,x}(-\log D_0)$. Again, let $\mathcal{O}_{X_1}(-1)$ be the tautological line bundle of $\mathbb{P}(V_0)$, so that $\mathcal{O}_{X_1}(-1)_{(x,[v])} = \mathbb{C} \cdot v$.

Iterating the construction produces the *logarithmic Demailly–Semple tower*

$$(X_2, D_2, V_2) \longrightarrow (X_1, D_1, V_1) \longrightarrow (X_0, D_0, V_0),$$

together with projections $\pi_{k,0}: X_k \rightarrow X_0$ for $k = 1, 2$.

The same short exact sequences (2.7) and (2.8) remain valid in the logarithmic setting. Moreover, with a slight abuse of notation, the divisor Γ_2 on the second level X_2 of the logarithmic tower can be defined exactly as in (2.10).

From the construction, the direct image sheaf

$$(\pi_{2,0})_* \mathcal{O}_{X_2}(m), \quad \mathcal{O}_{X_2}(m) := \mathcal{O}_{X_2}(1)^{\otimes m},$$

on the second level X_2 of the Demailly–Semple tower (resp. logarithmic Demailly–Semple tower) coincides precisely with the bundle $E_{2,m}T_X^*$ (resp. $E_{2,m}T_X^*(\log D)$). Consequently, we obtain the following direct-image formula.

Direct image formula (cf. [16, Theorem 6.8], [19, Proposition 3.9]). *Let $\mathcal{A}$ be an ample line bundle on X .*

(1) *If X_2 denotes the second level of the Demailly–Semple tower for (X, T_X) , then*

$$(2.12) \quad H^0(X, E_{2,m}T_X^* \otimes \mathcal{A}^{-1}) \cong H^0(X_2, \mathcal{O}_{X_2}(m) \otimes \pi_{2,0}^* \mathcal{A}^{-1}).$$

(2) *If X_2 denotes the second level of the logarithmic Demailly–Semple tower for $(X, D, T_X(-\log D))$, then*

$$H^0(X, E_{2,m}T_X^*(\log D) \otimes \mathcal{A}^{-1}) \cong H^0(X_2, \mathcal{O}_{X_2}(m) \otimes \pi_{2,0}^* \mathcal{A}^{-1}).$$

2.4. Existence of Negatively Twisted Invariant 2-Jet Differentials. Classical vanishing theorems in algebraic geometry imply that, for any smooth hypersurface in $\mathbb{P}^n$ with $n \geq 3$, there exist no nonzero negatively twisted symmetric differentials (i.e. the case $k = 1$), regardless of the degree. This forces one to consider jet differentials of order $k \geq 2$.

For smooth hypersurfaces of general type in $\mathbb{P}^n$ (i.e., degree $\geq n + 2$), the existence of such negatively twisted jet differentials has been established for large orders $k \gg 1$ by Merker [48] (see also [17, 8]). However, for $k \geq 3$ the control of the base locus remains a serious technical obstacle, rendering these higher-order differentials unsuitable for sharpening the hyperbolicity degree bounds in $\mathbb{P}^3$.

In this paper we therefore focus on the case $k = 2$, which provides a viable intermediate regime — the first-order theory being obstructed, and higher orders being computationally inaccessible. We now specialize to surfaces in $\mathbb{P}^3$.

Proposition 2.1 ([16, Corollary 13.9]). *Let X be a smooth projective surface of general type and A an ample line bundle on X . Then*

$$h^0(X, E_{2,m}T_X^* \otimes \mathcal{O}(-A)) \geq \frac{m^4}{648} (13c_1^2 - 9c_2) - O(m^3),$$

where c_i are the Chern numbers of X . □

The proof of Proposition 2.1 relies on a Riemann–Roch calculation based on the filtration (2.3) and the following Bogomolov vanishing theorem.

Theorem 2.2. *Let X be a smooth projective surface. Assume that X is of general type (i.e. K_X is big) and that K_X is nef. Then*

$$H^0(X, S^p T_X \otimes \mathcal{O}_X(qK_X)) = 0,$$

for all integers p, q satisfying $p > 2q$. □

Bogomolov [3] proved the original vanishing theorem under the assumption that the tangent bundle T_X is semistable. Enoki [25] later showed that for a minimal Kähler space X with big canonical bundle K_X , its tangent bundle T_X is K_X -semistable. Independently, Tsuji [69] established the K_X -semistability of T_X for smooth minimal algebraic varieties (here “minimal” means that K_X is nef). Combining these results yields a Bogomolov-type vanishing theorem for projective varieties whose canonical bundle K_X is both big and nef; see [16, Theorem 14.1].

If X is a smooth surface in $\mathbb{P}^3$ of degree d , one can compute the Chern numbers:

$$(2.13) \quad c_1^2 = d(d-4)^2, \quad c_2 = d(d^2 - 4d + 6).$$

Thus the leading coefficient satisfies

$$13c_1^2 - 9c_2 = d(4d^2 - 68d + 154),$$

which is positive for every $d \geq 15$. Hence Proposition 2.1 yields:

Corollary 2.3 ([16]). *For a smooth surface $X \subset \mathbb{P}^3$ of degree $d \geq 15$ and any ample line bundle A on X ,*

$$H^0(X, E_{2,m}T_X^* \otimes \mathcal{O}(-A)) \neq 0$$

for all sufficiently large $m \gg 1$. □

Similar statements hold in the logarithmic setting by essentially the same arguments (cf. [24]).

Proposition 2.4 ([24, Theorem 1.2.1]). *Let X be a smooth projective surface and $\mathcal{C} \subset X$ a simple normal crossing divisor such that $(X, \mathcal{C})$ is of log-general type. Let A be an ample line bundle on X . Then*

$$h^0(X, E_{2,m}T_X^*(\log \mathcal{C}) \otimes \mathcal{O}(-A)) \geq \frac{m^4}{648} (13\bar{c}_1^2 - 9\bar{c}_2) - O(m^3),$$

where $\bar{c}_i$ are the logarithmic Chern numbers of $(X, \mathcal{C})$. □

When $X = \mathbb{P}^2$ and $\mathcal{C} \subset \mathbb{P}^2$ is a smooth plane curve of degree d , one can compute the logarithmic Chern numbers

$$(2.14) \quad \bar{c}_1^2 = (d-3)^2, \quad \bar{c}_2 = d^2 - 3d + 3.$$

Hence the leading coefficient satisfies

$$13\bar{c}_1^2 - 9\bar{c}_2 = 4d^2 - 51d + 90 > 0 \quad \text{for } d \geq 11.$$

Consequently, we obtain

Corollary 2.5. *For a smooth algebraic curve $\mathcal{C} \subset \mathbb{P}^2$ of degree $d \geq 11$ and any ample line bundle A on $\mathbb{P}^2$,*

$$H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}(-A)) \neq 0$$

for all sufficiently large $m \gg 1$. □

2.5. Nevanlinna Theory. We begin by recalling the standard notation of Nevanlinna theory. Let $\mathcal{C} \subset \mathbb{P}^2$ be an algebraic curve and let $f: \mathbb{C} \rightarrow \mathbb{P}^2$ be an entire curve with $f(\mathbb{C}) \not\subset \mathcal{C}$. For $k \in \mathbb{N} \cup \{\infty\}$, the k -truncated counting function of f with respect to $\mathcal{C}$ is defined as

$$N_f^{[k]}(r, \mathcal{C}) := \int_1^r \sum_{|z| < t} \min\{k, \text{ord}_z f^*\mathcal{C}\} \frac{dt}{t} \quad (r > 1),$$

which measures the frequency of intersections $f(\mathbb{C}) \cap \mathcal{C}$. The *order function* of f is given by

$$T_f(r) := \int_1^r \frac{dt}{t} \int_{|z| < t} f^* \omega_{\text{FS}} \quad (r > 1),$$

where ω_{FS} denotes the Fubini–Study metric on $\mathbb{P}^2$. This function captures the average growth of the area of the holomorphic discs $f(\{|z| < t\})$ with respect to ω_{FS} as t tends to infinity.

The pursuit of a Second Main Theorem faces a fundamental obstacle: entire curves exhibit an inherent and wild flexibility in large-scale geometry, as evidenced by the exotic examples constructed in [36, 74, 71, 10]. Our strategy to overcome this difficulty is to constrain this flexibility by leveraging additional algebro-geometric structures, thereby extracting the underlying rigidity of entire curves. The proof relies on the following two key results.

The first result can be viewed as a quantitative refinement of the Fundamental Vanishing Theorem for entire curves (cf. [27, Section 2], [66, Lemma 1.2], and [16, Corollary 7.9]).

Theorem 2.6 ([35, Theorem 3.1]). *Let $\mathcal{C} = \sum_{i=1}^q \mathcal{C}_i$ be a simple normal crossing divisor in $\mathbb{P}^2$. For any logarithmic jet differential*

$$\omega \in H^0(\mathbb{P}^2, E_{k,m}^{GG}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)) \quad (k, m, t \geq 1),$$

and any entire holomorphic curve $f: \mathbb{C} \rightarrow \mathbb{P}^2$ with $f^\omega \neq 0$, the following Second Main Theorem type estimate holds:*

$$(2.15) \quad T_f(r) \leq \frac{m}{t} \sum_{i=1}^q N_f^{[1]}(r, \mathcal{C}_i) + O(\log T_f(r) + \log r) \quad \parallel.$$

Here, the symbol “ $\|$ ” indicates that the inequality (2.15) holds for all $r > 1$ outside a set of finite Lebesgue measure. This notation will be used consistently throughout this paper.

The second key ingredient is a theorem of McQuillan (see also [58]), which plays a crucial role in the proof of Theorem 1.9.

Theorem 2.7 (McQuillan [44]). *Let $\mathcal{C} = \sum_{i=1}^q \mathcal{C}_i$ be a simple normal crossing divisor in $\mathbb{P}^2$, where each $\mathcal{C}_i$ is a smooth algebraic curve of degree d_i and $d = \sum_{i=1}^q d_i \geq 4$. Let $f: \mathbb{C} \rightarrow \mathbb{P}^2$ be an algebraically nondegenerate entire curve. If the lift $f_{[1]}: \mathbb{C} \rightarrow X_1 = \mathbb{P} T_{\mathbb{P}^2}(-\log \mathcal{C})$ is algebraically degenerate, then*

$$(d-3)T_f(r) \leq N_f^{[1]}(r, \mathcal{C}) + o(T_f(r)) \quad \|.$$

3. Proof of Key Vanishing Lemma 1.3

In this section, we prove Lemma 1.3 by executing the three-step strategy outlined in the introduction.

3.1. Local Description of Invariant Logarithmic Jet Differentials. Now, working in affine coordinates (x, y) on $\mathbb{P}^2$ with

$$[T : X : Y] \in \mathbb{P}^2, \quad [1 : x : y] := [1 : \frac{X}{T} : \frac{Y}{T}],$$

consider an algebraic curve $\mathcal{C} \subset \mathbb{P}^2$ defined by

$$R(x, y) = 0,$$

where $R \in \mathbb{C}[x, y]$ is a polynomial of degree

$$\deg R = d \quad (d \in \mathbb{N}).$$

We take R to be a perturbation of the Fermat curve:

$$R(x, y) = 1^d + x^d + y^d + \dots,$$

with generic coefficients, so that in particular $\mathcal{C}$ is smooth at every point of $\mathbb{P}^2$. Consequently, in the affine (x, y) -chart we have

$$\emptyset = \{R = 0\} \cap \{R_x = 0\} \cap \{R_y = 0\}.$$

On the open set $\{R_y \neq 0\}$ (understood to be taken within the affine chart $\mathbb{C}^2$), the implicit function theorem allows us to solve locally for y as a function of x . On $\{R_x \neq 0\}$, similarly, x can be solved locally as a function of y .

In $\{R_y \neq 0\}$, where $\{R = 0\}$ is the divisor that entire curves $f: \mathbb{C} \rightarrow \mathbb{P}^2 \setminus \mathcal{C}$ must avoid, the fibers of the logarithmic Green–Griffiths 1-jet bundle are polynomially generated by

$$\langle x', (\log R)' \rangle = \left\langle x', \frac{R'}{R} \right\rangle,$$

with

$$R' := x'R_x + y'R_y.$$

Likewise, on $\{R_x \neq 0\}$, the logarithmic 1-jets are generated by

$$\langle (\log R)', y' \rangle = \left\langle \frac{R'}{R}, y' \right\rangle.$$

Concerning logarithmic 2-jets, the generators in $\{R_y \neq 0\}$ are:

$$\langle x', x'', (\log R)', (\log R)'' \rangle = \left\langle x', x'', \frac{R'}{R}, \frac{R''}{R} - \frac{R'R'}{R^2} \right\rangle,$$

where

$$R'' := x''R_x + y''R_y + x'x'R_{xx} + 2x'y'R_{xy} + y'y'R_{yy}.$$

Similarly, the generators in $\{R_x \neq 0\}$ are:

$$\langle (\log R)', (\log R)'', y', y'' \rangle = \left\langle \frac{R'}{R}, \frac{R''}{R} - \frac{R'R'}{R^2}, y', y'' \right\rangle.$$

For the invariant logarithmic 2-jet bundles, instead of plain 2-jets, one considers the logarithmic Wronskian. In $\{R_y \neq 0\}$ it is given by

$$\Delta_{xR}''' := \begin{vmatrix} x' & (\log R)' \\ x'' & (\log R)'' \end{vmatrix} = \begin{vmatrix} x' & \frac{R'}{R} \\ x'' & \frac{R''}{R} - \frac{R'^2}{R^2} \end{vmatrix},$$

and in $\{R_x \neq 0\}$ by

$$\Delta'''_{Ry} := \begin{vmatrix} (\log R)' & y' \\ (\log R)'' & y'' \end{vmatrix} = \begin{vmatrix} \frac{R'}{R} & y' \\ \frac{R''}{R} - \frac{R'^2}{R^2} & y'' \end{vmatrix}.$$

The transition from $\{R_x \neq 0\}$ to $\{R_y \neq 0\}$ consists in solving for y' and y'' from the relations obtained by differentiating the curve equation $R(x, y) = 0$ once and twice:

$$(3.1) \quad \begin{aligned} 0 &= -R' + x'R_x + y'R_y, \\ 0 &= -R'' + x''R_x + y''R_y + x'x'R_{xx} + 2x'y'R_{xy} + y'y'R_{yy}. \end{aligned}$$

Solving the first equation yields

$$(3.2) \quad y' = -x' \frac{R_x}{R_y} + \frac{R'}{R_y}.$$

Substituting this into the second equation and solving for y'' gives

$$\begin{aligned} y'' &= -x'' \frac{R_x}{R_y} + \frac{R''}{R_y} \\ &\quad - x'x' \frac{R_x^2 R_{yy}}{R_y^3} + 2x'R' \frac{R_x R_{yy}}{R_y^3} - R'R' \frac{R_{yy}}{R_y^3} + 2x'x' \frac{R_x R_{xy}}{R_y^2} - 2x'R' \frac{R_{xy}}{R_y^2} - x'x' \frac{R_{xx}}{R_y}. \end{aligned}$$

The transition of a jet differential is then obtained by replacing y' and y'' with these expressions.

In particular, the logarithmic Wronskian Δ'''_{Ry} transforms under the change of chart as follows:

$$(3.3) \quad \begin{aligned} \Delta'''_{Ry} &= \frac{R_x}{R_y} \Delta'''_{xR} + \frac{R}{R_y^3} (R_y^2 - R_{yy}R) \left(\frac{R'}{R} \right)^3 + \frac{2R}{R_y^3} (R_x R_{yy} - R_{xy} R_y) x' \left(\frac{R'}{R} \right)^2 \\ &\quad + \frac{1}{R_y^3} (-R_x^2 R_{yy} + 2R_x R_{xy} R_y - R_{xx} R_y^2) x'^2 \frac{R'}{R}. \end{aligned}$$

Now consider a global section of the invariant logarithmic 2-jet differential bundle $E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C})$ of weighted degree m . By the GAGA principle, such a section is algebraic, and it is a basic fact of algebraic geometry that on the open set $\{R_y \neq 0\}$ it takes the general form

$$(3.4) \quad J_{xR} := \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{F_{j,k}}{R_y^{f_{j,k}}} (x')^{m-3k-j} \left(\frac{R'}{R} \right)^j (\Delta'''_{xR})^k,$$

where $F_{j,k} \in \mathbb{C}[x, y]$ are polynomials and $f_{j,k} \in \mathbb{N}$ are nonnegative integers indicating the admissible powers of R_y in the denominator (division by R_y being permissible on $\{R_y \neq 0\}$).

Question 3.5. What are the maximal possible values of $f_{j,k} \in \mathbb{N}$?

The following proposition answers this question by showing that any global section of the invariant logarithmic 2-jet differential bundle must take a prescribed finite form, with undetermined polynomial coefficients.

Proposition 3.1. Let $\hat{R}(T, X, Y) \in \mathbb{C}[T, X, Y]$ be a homogeneous polynomial defining a smooth curve $\mathcal{C} \subset \mathbb{P}^2$, and assume that the divisors in $\mathbb{P}^2$ given by $\hat{R}$ and its partial derivatives $\hat{R}_T, \hat{R}_X, \hat{R}_Y$ are in general position (i.e., any intersection of them has the expected codimension). Dehomogenizing by setting $T = 1$ yields an affine polynomial $R(x, y) \in \mathbb{C}[x, y]$ such that $\mathcal{C} \cap \{T \neq 0\} = \{R = 0\}$. For any weighted order m , every global section of the invariant logarithmic 2-jet differential bundle $E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C})$ on $\mathbb{P}^2$, when restricted to the affine open set $\{R_y \neq 0\} \subset \{T \neq 0\}$, admits the general expression

$$J_{xR} = \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{F_{j,k}}{R_y^{m-2k}} (x')^{m-3k-j} \left(\frac{R'}{R} \right)^j (\Delta'''_{xR})^k,$$

where $F_{j,k} \in \mathbb{C}[x, y]$ are polynomials.

Proof. In the standard affine coordinates $(x, y) = (X/T, Y/T)$ on $\{T \neq 0\} = \mathbb{C}^2$, the invariant logarithmic 2-jet differential is algebraic on $\{R_x \neq 0\}$. By the same reasoning as in (3.4), on this open set it admits an expression of the form

$$(3.6) \quad J_{Ry} := \sum_{\substack{0 \leq q \leq \lfloor m/3 \rfloor \\ 0 \leq p \leq m-3q}} \bar{F}_{p,q} \left(\frac{R'}{R} \right)^p (y')^{m-3q-p} (\Delta'''_{Ry})^q,$$

where the coefficients $\bar{F}_{p,q}$ are rational functions whose denominators may involve only powers of R_x .

We now examine the compatibility of the two expressions (3.4) and (3.6) for the jet differential. Substituting the transition formulas (3.2) and (3.3) into (3.6) and comparing with (3.4), we find that the coefficients in (3.4) become rational expressions whose denominators are products of powers of R_x (which must ultimately cancel) and powers of R_y . The exponent of R_y in the denominator arises from two sources:

- Each of the $m - 3q - p$ factors of y' contributes at most one factor of R_y^{-1} via (3.2).
- Each of the q factors of the Wronskian Δ'''_{Ry} contributes at most one factor of R_y^{-1} from the leading term in (3.3), and at most three factors of R_y^{-1} from the remaining quadratic terms.

Moreover, the term $(x')^{m-3k-j} \left(\frac{R'}{R} \right)^j (\Delta'''_{xR})^k$ in expression (3.4) arises from expanding terms of the form

$$(3.7) \quad \left(\frac{R'}{R} \right)^p (y')^{m-3q-p} (\Delta'''_{Ry})^q \quad \text{with } q \geq k.$$

When expanding such a term, $(\Delta'''_{Ry})^q$ contributes a factor of $(\Delta'''_{xR})^k$ together with k powers of R_y^{-1} from the leading term in (3.3) and an additional $q - k$ powers of R_y^{-3} from the remaining quadratic terms. The factor $(y')^{m-3q-p}$ contributes $m - 3q - p$ powers of R_y^{-1} via (3.2). Hence, the total exponent of R_y in the denominator of the coefficient multiplying $(x')^{m-3k-j} \left(\frac{R'}{R} \right)^j (\Delta'''_{xR})^k$ is at most

$$k + 3(q - k) + (m - 3q - p) = m - 2k - p \leq m - 2k.$$

This upper bound $m - 2k$ can be achieved by expanding the term in (3.7) corresponding to $q = k$ and $p = 0$, namely $(y')^{m-3k} (\Delta'''_{Ry})^k$.

By our assumption on $\hat{R}$, the divisors defined by R and its partial derivatives R_x, R_y are in general position. Under this genericity condition, after cancelling common factors between numerator and denominator, the maximal possible pole order $f_{j,k}$ of the coefficients in (3.4) with respect to R_y satisfies

$$f_{j,k} \leq m - 2k.$$

Moreover, since the expression (3.4) must be regular on $\{R_y \neq 0\}$, any factor of R_x in the denominator must cancel. Consequently, the coefficients $F_{j,k}$ may be taken as polynomials, and we may set $f_{j,k} = m - 2k$ as the maximal admissible exponent, as claimed. $\square$

3.2. Denominator Exponent Constraints from Holomorphicity. On $\mathbb{P}^2$, the affine coordinates (x, y) arise from homogeneous coordinates $[T : X : Y]$ by setting $T = 1$:

$$(1, x, y) = \left(\frac{T}{T}, \frac{X}{T}, \frac{Y}{T} \right).$$

Two further affine charts are obtained by normalizing with X or Y instead of T . Normalizing by X yields coordinates $(\tilde{t}, \tilde{y})$ via $[\tilde{t} : 1 : \tilde{y}] = [T : X : Y] = [\frac{T}{X} : 1 : \frac{Y}{X}]$, namely

$$(3.8) \quad \tilde{t} := \frac{1}{x}, \quad \tilde{y} := \frac{y}{x},$$

with inverse transformation

$$(3.9) \quad x = \frac{1}{\tilde{t}}, \quad y = \frac{\tilde{y}}{\tilde{t}}.$$

To study the behavior along the line at infinity $\mathbb{P}_\infty^1 := \{T = 0\}$, we work in this new chart, where $\mathbb{P}_\infty^1$ is locally given by $\{\tilde{t} = 0\}$. This chart covers the open subset $\mathbb{P}_\infty^1 \setminus \mathbb{P}_\infty^0$, with $\mathbb{P}_\infty^0 := \{T = 0, X = 0\}$ a single point of $\mathbb{P}_\infty^1$. The remaining part of $\mathbb{P}_\infty^1$ would be covered by normalizing with Y ; however, since all objects considered are polynomial or rational, their vanishing or pole order along $\mathbb{P}_\infty^1$ is already determined on the dense open set $\mathbb{P}_\infty^1 \setminus \mathbb{P}_\infty^0$. It therefore suffices to examine the chart transitions (3.8) and (3.9).

Under the change $(x, y) \mapsto (\tilde{t}, \tilde{y})$ given by (3.9), the 1-jets and 2-jets transform according to

$$(3.10) \quad \begin{aligned} x' &= -\frac{\tilde{t}'}{\tilde{t}^2}, & y' &= \frac{\tilde{y}'}{\tilde{t}} - \frac{\tilde{y}\tilde{t}'}{\tilde{t}^2}, \\ x'' &= -\frac{\tilde{t}''}{\tilde{t}^2} + 2\frac{(\tilde{t}')^2}{\tilde{t}^3}, & y'' &= \frac{\tilde{y}''}{\tilde{t}} - 2\frac{\tilde{t}'\tilde{y}'}{\tilde{t}^2} - \frac{\tilde{y}\tilde{t}''}{\tilde{t}^2} + 2\frac{\tilde{y}(\tilde{t}')^2}{\tilde{t}^3}. \end{aligned}$$

To understand its behavior along the line at infinity $\mathbb{P}_\infty^1$, given locally by $\{\tilde{t} = 0\}$, we examine its transformation under (3.9). The polynomial R transforms as

$$R(x, y) = R\left(\frac{1}{\tilde{t}}, \frac{\tilde{y}}{\tilde{t}}\right) = \frac{1}{\tilde{t}^d} R^*(\tilde{t}, \tilde{y}),$$

where $d = \deg R$ and $R^*(\tilde{t}, \tilde{y}) := \tilde{t}^d R(1/\tilde{t}, \tilde{y}/\tilde{t})$ is a polynomial with $R^*(0, \tilde{y}) \neq 0$ for generic R .

Taking the logarithmic derivative of both sides yields the transformation rules

$$(3.11) \quad (\log R)' = (\log R^*)' - d \frac{\tilde{t}'}{\tilde{t}},$$

and consequently for the second derivative,

$$(\log R)'' = (\log R^*)'' - d \frac{\tilde{t}''}{\tilde{t}} + d \frac{(\tilde{t}')^2}{\tilde{t}^2}.$$

Finally, the logarithmic Wronskian transforms as

$$(3.12) \quad \Delta_{xR} = \begin{vmatrix} x' & (\log R)' \\ x'' & (\log R)'' \end{vmatrix} = -\frac{1}{\tilde{t}^2} \begin{vmatrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{vmatrix} + d \frac{(\tilde{t}')^3}{\tilde{t}^4} - 2 \frac{(\tilde{t}')^2}{\tilde{t}^3} (\log R^*)'.$$

Similarly, under the inverse change of chart $(\tilde{t}, \tilde{y}) \mapsto (x, y)$ given by (3.8), the 1-jets and 2-jets transform according to the following formulas:

$$(3.13) \quad \begin{aligned} \tilde{t}' &= -\frac{x'}{x^2}, & \tilde{y}' &= \frac{y'}{x} - \frac{y x'}{x^2}, \\ \tilde{t}'' &= -\frac{x''}{x^2} + 2\frac{(x')^2}{x^3}, & \tilde{y}'' &= \frac{y''}{x} - 2\frac{x' y'}{x^2} - \frac{y x''}{x^2} + 2\frac{y (x')^2}{x^3}. \end{aligned}$$

The logarithmic derivative transforms as

$$(3.14) \quad \begin{aligned} (\log R^*)' &= (\log R)' - d \frac{x'}{x}, \\ (\log R^*)'' &= (\log R)'' - d \frac{x''}{x} + d \frac{(x')^2}{x^2}. \end{aligned}$$

Consequently, the logarithmic Wronskian transforms as

$$(3.15) \quad \begin{vmatrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{vmatrix} = -\frac{1}{x^2} \begin{vmatrix} x' & (\log R)' \\ x'' & (\log R)'' \end{vmatrix} + d \frac{(x')^3}{x^4} - 2 \frac{(x')^2}{x^3} (\log R)'.$$

To simplify computations, we now make a specific choice for the polynomial defining the curve $\mathcal{C} \subset \mathbb{P}^2$. We take its homogeneous equation to be a perturbation of the Fermat curve:

$$\hat{R} := T^d + X^d + Y^d + P(T, X) = 0,$$

where the perturbation term $P(T, X) \in \mathbb{C}[T, X]$ is homogeneous of degree d and involves only the variables T and X . In the affine chart $T = 1$, this becomes

$$R(x, y) = 1 + x^d + y^d + S(x),$$

with $S(x) \in \mathbb{C}[x]$ a polynomial of degree at most $d - 1$, to be chosen later in a suitably generic way.

Convention 3.16. In what follows, we work under the following generic assumptions:³

- The curve $\mathcal{C}$ defined by $\hat{R} = 0$ is smooth.

³The equation $\hat{R}$ to be used later, with $P(T, X) = X^6 T^{d-6}$ for $d = 12, 13$, satisfies Convention 3.16. The associated Maple code is available at <https://github.com/LeiHou-code/Hyperbolicity-Dimension-2-Conventions-Facts-2026>.

- The divisors defined by $\widehat{R}$ and its partial derivatives $\widehat{R}_T, \widehat{R}_X, \widehat{R}_Y$ are in general position; i.e., any intersection of them has the expected (co)dimension.
- The coordinate line $\{Y = 0\}$ and $\{T = 0\}$ intersect $\mathcal{C}$ transversely.

The advantage of this choice is that, up to the nonzero constant factor $\frac{1}{d}$ (which causes no essential complication, especially for computer algebra calculations), we have simply

$$\frac{1}{d}R_y = y^{d-1}.$$

Recall that for an algebraic invariant logarithmic 2-jet differential initially defined on $\{R_y \neq 0\}$ to be also holomorphic on $\{R_x \neq 0\}$, certain divisibility conditions by R_y must be satisfied. With the above choice $R_y = dy^{d-1}$, these divisibility conditions translate directly into the requirement that, after expanding with respect to y , the polynomial coefficients of $y^0, y^1, \dots, y^{d-2}$ all vanish identically.

In contrast, if one starts with a truly generic polynomial $R(x, y)$ of degree d , its partial derivative R_y would contain many monomials involving y , and the required divisibility by R_y would involve a Euclidean division — a computationally intensive task even on powerful electronic equipment. It is therefore highly advantageous for us to have made the preliminary choice

$$R_y = dy^{d-1}.$$

On $\{T \neq 0\} \cap \{\widehat{R}_Y \neq 0\}$, every invariant logarithmic 2-jet differential of weighted order m on $\mathbb{P}^2$ admits the general expression

$$(3.17) \quad J_{X\widehat{R}} := \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{\widehat{G}_{j,k}}{T^{g_{j,k}} \widehat{R}_Y^{m-2k}} (x')^{m-3k-j} \left(\frac{R'}{R}\right)^j (\Delta_{xR}''')^k,$$

where $\widehat{G}_{j,k}(T, X, Y)$ are homogeneous polynomials in $\mathbb{C}[T, X, Y]$ satisfying

$$\deg \widehat{G}_{j,k} = g_{j,k} + (m-2k)(d-1),$$

and $\widehat{R}_Y$ is the partial derivative of the homogeneous polynomial $\widehat{R}$ as in Convention 3.16. Here the exponent of $\widehat{R}_Y$ in the denominator is taken to be $m-2k$, which coincides with the maximal admissible exponent of R_y in the denominators of the coefficients $F_{j,k}/R_y^{m-2k}$ from Proposition 3.1.

Similarly, on $\{X \neq 0\} \cap \{\widehat{R}_Y \neq 0\}$, every invariant logarithmic 2-jet differential of weighted order m on $\mathbb{P}^2$ admits the general expression

$$(3.18) \quad J_{T\widehat{R}} := \sum_{\substack{0 \leq q \leq \lfloor m/3 \rfloor \\ 0 \leq p \leq m-3q}} \frac{\widehat{E}_{p,q}}{X^{e_{p,q}} \widehat{R}_Y^{m-2q}} (\tilde{t}')^{m-3q-p} ((\log R^*)')^p \left| \frac{\tilde{t}'}{\tilde{t}''} \frac{(\log R^*)'}{(\log R^*)''} \right|^q,$$

where $\widehat{E}_{p,q}(T, X, Y)$ are homogeneous polynomials in $\mathbb{C}[T, X, Y]$ satisfying

$$\deg \widehat{E}_{p,q} = e_{p,q} + (m-2q)(d-1).$$

Question 3.19. What happens to the general form $J_{T\widehat{R}}$ given in (3.18) after the change of affine chart from $\{X \neq 0\} \cap \{\widehat{R}_Y \neq 0\}$ to $\{T \neq 0\} \cap \{\widehat{R}_Y \neq 0\}$?

Substituting (3.13), (3.14), and (3.15) into (3.18) yields the following transformation for the term:

$$\begin{aligned} & \frac{\widehat{E}_{p,q}}{X^{e_{p,q}} \widehat{R}_Y^{m-2q}} (\tilde{t}')^{m-3q-p} ((\log R^*)')^p \left| \frac{\tilde{t}'}{\tilde{t}''} \frac{(\log R^*)'}{(\log R^*)''} \right|^q \\ &= \frac{\widehat{E}_{p,q}}{X^{e_{p,q}} \widehat{R}_Y^{m-2q}} \left(-\frac{x'}{x^2}\right)^{m-3q-p} ((\log R)')^p - d \frac{x'}{x} \left(-\frac{1}{x^2} \left| \frac{x'}{x''} \frac{(\log R)'}{(\log R)''} \right| + d \frac{1}{x^4} (x')^3 - 2 \frac{1}{x^3} (x')^2 (\log R)'\right)^q \\ &= \frac{(-1)^{m-3q-p}}{x^{2(m-3q-p)+p+4q}} \frac{\widehat{E}_{p,q}}{X^{e_{p,q}} \widehat{R}_Y^{m-2q}} (x')^{m-3q-p} (x(\log R)' - dx')^p \\ & \quad \cdot \left(-x^2 \left| \frac{x'}{x''} \frac{(\log R)'}{(\log R)''} \right| + d(x')^3 - 2x(x')^2 (\log R)'\right)^q, \end{aligned}$$

where the factor $(x(\log R)' - dx')^p$ can be expanded as

$$(x(\log R)' - dx')^p = \sum_{r=0}^p (-1)^{p-r} \binom{p}{r} d^{p-r} x^r (x')^{p-r} ((\log R)')^r.$$

The last factor expands as

$$\begin{aligned} & \left(-x^2 \left| \begin{matrix} x' & (\log R)' \\ x'' & (\log R)'' \end{matrix} \right| + d(x')^3 - 2x(x')^2(\log R)' \right)^q \\ &= \sum_{\alpha+\beta+\gamma=q} \frac{q!}{\alpha!\beta!\gamma!} d^\alpha (x')^{3\alpha} (-1)^\beta 2^\beta x^\beta (x')^{2\beta} ((\log R)')^\beta (-1)^\gamma x^{2\gamma} \left| \begin{matrix} x' & (\log R)' \\ x'' & (\log R)'' \end{matrix} \right|^\gamma \\ &= \sum_{\alpha+\beta+\gamma=q} (-1)^{\beta+\gamma} d^\alpha 2^\beta \frac{q!}{\alpha!\beta!\gamma!} x^{\beta+2\gamma} (x')^{3\alpha+2\beta} ((\log R)')^\beta \left| \begin{matrix} x' & (\log R)' \\ x'' & (\log R)'' \end{matrix} \right|^\gamma \\ &= \sum_{\gamma=0}^q \sum_{\beta=0}^{q-\gamma} (-1)^{\beta+\gamma} d^{q-\beta-\gamma} 2^\beta \frac{q!}{\beta!\gamma!(q-\beta-\gamma)!} x^{\beta+2\gamma} (x')^{3(q-\beta-\gamma)+2\beta} ((\log R)')^\beta \left| \begin{matrix} x' & (\log R)' \\ x'' & (\log R)'' \end{matrix} \right|^\gamma. \end{aligned}$$

Combining the expansions, the term becomes

$$\begin{aligned} & \frac{\widehat{E}_{p,q}}{X^{e_{p,q}} \widehat{R}_Y^{m-2q}} (\tilde{t}')^{m-3q-p} ((\log R^*)')^p \left| \begin{matrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{matrix} \right|^q \\ &= \sum_{r=0}^p \sum_{\gamma=0}^q \sum_{\beta=0}^{q-\gamma} (-1)^{m-3q-r+\beta+\gamma} 2^\beta d^{p-r+q-\beta-\gamma} \binom{p}{r} \frac{q!}{\beta!\gamma!(q-\beta-\gamma)!} \\ & \quad \cdot \frac{x^{r+\beta+2\gamma}}{x^{2(m-3q-p)+p+4q}} \frac{\widehat{E}_{p,q}}{X^{e_{p,q}} \widehat{R}_Y^{m-2q}} (x')^{m-3q-r+3(q-\beta-\gamma)+2\beta} ((\log R)')^{r+\beta} \left| \begin{matrix} x' & (\log R)' \\ x'' & (\log R)'' \end{matrix} \right|^\gamma. \end{aligned}$$

Introducing new indices $k := \gamma$ and $j := r + \beta$, we rewrite the expression as

$$\begin{aligned} & \frac{\widehat{E}_{p,q}}{X^{e_{p,q}} \widehat{R}_Y^{m-2q}} (\tilde{t}')^{m-3q-p} ((\log R^*)')^p \left| \begin{matrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{matrix} \right|^q \\ &= \sum_{r=0}^p \sum_{k=0}^q \sum_{j=r}^{q+r-k} (-1)^{m-3q-2r+j+k} 2^{j-r} d^{p+q-j-k} \binom{p}{r} \frac{q!}{(j-r)! k! (q+r-j-k)!} \\ & \quad \cdot \frac{x^{j+2k}}{x^{2(m-3q-p)+p+4q}} \frac{\widehat{E}_{p,q}}{X^{e_{p,q}} \widehat{R}_Y^{m-2q}} (x')^{m-3k-j} ((\log R)')^j \left| \begin{matrix} x' & (\log R)' \\ x'' & (\log R)'' \end{matrix} \right|^k. \end{aligned}$$

Consequently, the full expression $J_{T\hat{R}}$ transforms as

$$\begin{aligned}
J_{T\hat{R}} &= \sum_{q=0}^{\lfloor m/3 \rfloor} \sum_{p=0}^{m-3q} \sum_{r=0}^p \sum_{k=0}^q \sum_{j=r}^{q+r-k} (-1)^{m-3q-2r+j+k} 2^{j-r} d^{p+q-j-k} \binom{p}{r} \frac{q!}{(j-r)!k!(q+r-j-k)!} \\
&\quad \cdot \frac{x^{j+2k}}{x^{2(m-3q-p)+p+4q}} \frac{\hat{E}_{p,q}}{X^{e_{p,q}} \hat{R}_Y^{m-2q}} (x')^{m-3k-j} ((\log R)')^j \left| \frac{x'}{x''} \frac{(\log R)'}{(\log R)''} \right|^k \\
&= \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{j=0}^{m-3k} \sum_{q=k}^{\lfloor m/3 \rfloor} \sum_{r=\max(0, j+k-q)}^{\min(j, m-3q)} \sum_{p=r}^{m-3q} (-1)^{m-3q-2r+j+k} 2^{j-r} d^{p+q-j-k} \binom{p}{r} \frac{q!}{(j-r)!k!(q+r-j-k)!} \\
&\quad \cdot \frac{x^{j+2k+p+2q}}{x^{2m}} \frac{\hat{E}_{p,q}}{X^{e_{p,q}} \hat{R}_Y^{m-2q}} (x')^{m-3k-j} ((\log R)')^j \left| \frac{x'}{x''} \frac{(\log R)'}{(\log R)''} \right|^k \\
&= \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{j=0}^{m-3k} \frac{T^{m-j-2k}}{X^{m-j-2k}} \sum_{q=k}^{\lfloor m/3 \rfloor} \sum_{r=\max(0, j+k-q)}^{\min(j, m-3q)} \sum_{p=r}^{m-3q} \\
&\quad \cdot (-1)^{m-3q-2r+j+k} 2^{j-r} d^{p+q-j-k} \binom{p}{r} \frac{q!}{(j-r)!k!(q+r-j-k)!} \frac{T^{m-p-2q}}{X^{m-p-2q}} \frac{\hat{E}_{p,q}}{X^{e_{p,q}} \hat{R}_Y^{m-2q}} \\
&\quad \cdot (x')^{m-3k-j} ((\log R)')^j \left| \frac{x'}{x''} \frac{(\log R)'}{(\log R)''} \right|^k.
\end{aligned}$$

Observe that $p+2q \leq m-3q+2q = m-q \leq m-k$, which implies $m-p-2q \geq k$. Hence we may write

$$\begin{aligned}
J_{T\hat{R}} &= \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{j=0}^{m-3k} \frac{T^{m-j-k}}{X^{m-j-k}} \sum_{q=k}^{\lfloor m/3 \rfloor} \sum_{r=\max(0, j+k-q)}^{\min(j, m-3q)} \sum_{p=r}^{m-3q} \\
&\quad \cdot (-1)^{m-3q-2r+j+k} 2^{j-r} d^{p+q-j-k} \binom{p}{r} \frac{q!}{(j-r)!k!(q+r-j-k)!} \frac{T^{m-k-p-2q}}{X^{m-k-p-2q}} \frac{\hat{E}_{p,q}}{X^{e_{p,q}} \hat{R}_Y^{m-2q}} \\
&\quad \cdot (x')^{m-3k-j} ((\log R)')^j \left| \frac{x'}{x''} \frac{(\log R)'}{(\log R)''} \right|^k.
\end{aligned}$$

The coefficients

$$\frac{T^{m-j-k}}{X^{m-j-k}} (-1)^{m-3q-2r+j+k} 2^{j-r} d^{p+q-j-k} \binom{p}{r} \frac{q!}{(j-r)!k!(q+r-j-k)!} \frac{T^{m-k-p-2q}}{X^{m-k-p-2q}} \frac{\hat{E}_{p,q}}{X^{e_{p,q}} \hat{R}_Y^{m-2q}}$$

in the last line, as rational functions, not only have no pole along the divisor $\{T=0\}$, but also vanish with multiplicity at least $m-j-k$.

Comparing with the coefficients of (3.17), it follows that, on $\{T \neq 0\} \cap \{\hat{R}_Y \neq 0\}$, every invariant logarithmic 2-jet differential of weighted order m on $\mathbb{P}^2$ admits the general expression

$$(3.20) \quad J_{X\hat{R}} = \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{T^{m-j-k} \hat{H}_{j,k}}{\hat{R}_Y^{m-2k}} (x')^{m-3k-j} \left(\frac{R'}{R} \right)^j (\Delta_{xR}''')^k,$$

where $\hat{H}_{j,k}(T, X, Y)$ are homogeneous polynomials in $\mathbb{C}[T, X, Y]$ satisfying

$$\deg \hat{H}_{j,k} = (m-2k)(d-1) - m + j + k.$$

If we now require the logarithmic jet differential as (3.20) to vanish to order at least $t \geq 1$ along the divisor $\{X=0\}$, i.e.,

$$\text{Twist} := -t \leq -1,$$

then this corresponds to considering sections of the jet differential bundle twisted by $\mathcal{O}_{\mathbb{P}^2}(-t)$, i.e.,

$$(\cdot) \otimes \mathcal{O}_{\mathbb{P}^2}(-t).$$

Therefore, on $\{T \neq 0\} \cap \{\widehat{R}_Y \neq 0\}$, every invariant logarithmic 2-jet differential of weighted order m and vanishing order at least t along $\{X = 0\}$ on $\mathbb{P}^2$ admits the general expression

$$(3.21) \quad J_{X\widehat{R}} = \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{T^{m-j-k} X^t \widehat{K}_{j,k}}{\widehat{R}_Y^{m-2k}} (x')^{m-3k-j} \left(\frac{R'}{R}\right)^j (\Delta_{xR}''')^k,$$

where $\widehat{K}_{j,k}(T, X, Y)$ are homogeneous polynomials in $\mathbb{C}[T, X, Y]$ satisfying

$$\deg \widehat{K}_{j,k} = (m-2k)(d-1) - m + j + k - t.$$

When the coefficients of the above expression are expressed as regular functions in the affine coordinates (x, y) , we obtain

$$(3.22) \quad J_{xR} := x^t \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{K_{j,k}}{R_y^{m-2k}} (x')^{m-3k-j} \left(\frac{R'}{R}\right)^j (\Delta_{xR}''')^k,$$

where $K_{j,k}(x, y)$ are polynomials in $\mathbb{C}[x, y]$ satisfying

$$\deg K_{j,k} \leq (m-2k)(d-1) - m + j + k - t.$$

To prove Lemma 1.3, we focus on the range $m \leq 14$. For these low weights, the above expression simplifies considerably. Indeed, the exponent k of the logarithmic Wronskian Δ_{xR}''' satisfies $k \leq \lfloor m/3 \rfloor \leq 4$. Terms involving a power of the logarithmic Wronskian exceeding $\lfloor m/3 \rfloor$ are understood to be identically zero.

It remains to check the holomorphicity of the jet differential (3.21) along the divisor $\{T = 0\}$ at infinity.

Question 3.23. What happens to the general form $J_{X\widehat{R}}$ given in (3.21) after the change of affine chart (3.9)?

Substituting (3.10), (3.11), and (3.12) into (3.21), the term transforms as follows:

$$\begin{aligned} & \frac{T^{m-j-k} X^t \widehat{K}_{j,k}}{\widehat{R}_Y^{m-2k}} (x')^{m-3k-j} \left(\frac{R'}{R}\right)^j (\Delta_{xR}''')^k \\ &= \frac{T^{m-j-k} X^t \widehat{K}_{j,k}}{\widehat{R}_Y^{m-2k}} \left(-\frac{\tilde{t}'}{\tilde{t}^2}\right)^{m-3k-j} ((\log R^*)' - d\frac{\tilde{t}'}{\tilde{t}})^j \\ & \quad \cdot \left(-\frac{1}{\tilde{t}^2} \begin{vmatrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{vmatrix} + d\frac{1}{\tilde{t}^4}(\tilde{t}')^3 - 2\frac{1}{\tilde{t}^3}(\tilde{t}')^2(\log R^*)'\right)^k \\ &= \frac{(-1)^{m-3k-j}}{\tilde{t}^{2(m-3k-j)+j+4k}} \frac{T^{m-j-k} X^t \widehat{K}_{j,k}}{\widehat{R}_Y^{m-2k}} (\tilde{t}')^{m-3k-j} (\tilde{t}(\log R^*)' - d\tilde{t}')^j \\ & \quad \cdot \left(-\tilde{t}^2 \begin{vmatrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{vmatrix} + d(\tilde{t}')^3 - 2\tilde{t}(\tilde{t}')^2(\log R^*)'\right)^k, \end{aligned}$$

where the factor $(\tilde{t}(\log R^*)' - d\tilde{t}')^j$ can be expanded as

$$(\tilde{t}(\log R^*)' - d\tilde{t}')^j = \sum_{r=0}^j (-1)^{j-r} \binom{j}{r} d^{j-r} \tilde{t}^r (\tilde{t}')^{j-r} ((\log R^*)')^r.$$

The last factor expands as

$$\begin{aligned}
& \left(-\tilde{t}^2 \left| \begin{smallmatrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{smallmatrix} \right| + d(\tilde{t}')^3 - 2\tilde{t}(\tilde{t}')^2 (\log R^*)' \right)^k \\
&= \sum_{\alpha+\beta+\gamma=k} \frac{k!}{\alpha!\beta!\gamma!} d^\alpha (\tilde{t}')^{3\alpha} (-1)^\beta 2^\beta \tilde{t}^\beta (\tilde{t}')^{2\beta} ((\log R^*)')^\beta (-1)^\gamma \tilde{t}^{2\gamma} \left| \begin{smallmatrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{smallmatrix} \right|^\gamma \\
&= \sum_{\alpha+\beta+\gamma=k} (-1)^{\beta+\gamma} d^\alpha 2^\beta \frac{k!}{\alpha!\beta!\gamma!} \tilde{t}^{\beta+2\gamma} (\tilde{t}')^{3\alpha+2\beta} ((\log R^*)')^\beta \left| \begin{smallmatrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{smallmatrix} \right|^\gamma \\
&= \sum_{\gamma=0}^k \sum_{\beta=0}^{k-\gamma} (-1)^{\beta+\gamma} d^{k-\beta-\gamma} 2^\beta \frac{k!}{\beta!\gamma!(k-\beta-\gamma)!} \tilde{t}^{\beta+2\gamma} (\tilde{t}')^{3(k-\beta-\gamma)+2\beta} ((\log R^*)')^\beta \left| \begin{smallmatrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{smallmatrix} \right|^\gamma.
\end{aligned}$$

Combining the expansions, the term becomes

$$\begin{aligned}
& \frac{T^{m-j-k} X^t \widehat{K}_{j,k}}{\widehat{R}_Y^{m-2k}} (x')^{m-3k-j} \left(\frac{R'}{R} \right)^j (\Delta_{xR}''')^k \\
&= \sum_{r=0}^j \sum_{\gamma=0}^k \sum_{\beta=0}^{k-\gamma} (-1)^{m-3k-r+\beta+\gamma} 2^\beta d^{j-r+k-\beta-\gamma} \binom{j}{r} \frac{k!}{\beta!\gamma!(k-\beta-\gamma)!} \frac{\tilde{t}^{r+\beta+2\gamma}}{\tilde{t}^{2(m-3k-j)+j+4k}} \frac{T^{m-j-k} X^t \widehat{K}_{j,k}}{\widehat{R}_Y^{m-2k}} \\
& \quad \cdot (\tilde{t}')^{m-3k-r+3(k-\beta-\gamma)+2\beta} ((\log R^*)')^{r+\beta} \left| \begin{smallmatrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{smallmatrix} \right|^\gamma \\
&= \sum_{r=0}^j \sum_{q=0}^k \sum_{p=r}^{k+r-q} (-1)^{m-3k-2r+p+q} 2^{p-r} d^{j+k-p-q} \binom{j}{r} \frac{k!}{(p-r)!q!(k+r-p-q)!} \\
& \quad \cdot \frac{\tilde{t}^{p+2q}}{\tilde{t}^{2(m-3k-j)+j+4k}} \frac{T^{m-j-k} X^t \widehat{K}_{j,k}}{\widehat{R}_Y^{m-2k}} (\tilde{t}')^{m-3q-p} ((\log R^*)')^p \left| \begin{smallmatrix} \tilde{t}' & (\log R^*)' \\ \tilde{t}'' & (\log R^*)'' \end{smallmatrix} \right|^q,
\end{aligned}$$

where the last line is obtained by setting $q := \gamma$ and $p := r + \beta$.

Consequently, the full expression transforms as

$$\begin{aligned}
J_{T\hat{R}} &= \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{j=0}^{m-3k} \sum_{r=0}^j \sum_{q=0}^k \sum_{p=r}^{k+r-q} (-1)^{m-3k-2r+p+q} 2^{p-r} d^{j+k-p-q} \binom{j}{r} \frac{k!}{(p-r)!q!(k+r-p-q)!} \\
&\quad \cdot \frac{\tilde{t}^{p+2q}}{\tilde{t}^{2(m-3k-j)+j+4k}} \frac{T^{m-j-k} X^t \hat{K}_{j,k}}{\hat{R}_Y^{m-2k}} (\tilde{t}')^{m-3q-p} ((\log R^*)')^p \left| \frac{\tilde{t}'}{\tilde{t}''} \frac{(\log R^*)'}{(\log R^*)''} \right|^q \\
&= \sum_{q=0}^{\lfloor m/3 \rfloor} \sum_{p=0}^{m-3q} \sum_{k=q}^{\lfloor m/3 \rfloor} \sum_{r=\max(0, p+q-k)}^{\min(p, m-3k)} \sum_{j=r}^{m-3k} (-1)^{m-3k-2r+p+q} 2^{p-r} d^{j+k-p-q} \binom{j}{r} \frac{k!}{(p-r)!q!(k+r-p-q)!} \\
&\quad \cdot \frac{\tilde{t}^{j+2k+p+2q}}{\tilde{t}^{2m}} \frac{T^{m-j-k} X^t \hat{K}_{j,k}}{\hat{R}_Y^{m-2k}} (\tilde{t}')^{m-3q-p} ((\log R^*)')^p \left| \frac{\tilde{t}'}{\tilde{t}''} \frac{(\log R^*)'}{(\log R^*)''} \right|^q \\
&= \sum_{q=0}^{\lfloor m/3 \rfloor} \sum_{p=0}^{m-3q} \sum_{k=q}^{\lfloor m/3 \rfloor} \sum_{r=\max(0, p+q-k)}^{\min(p, m-3k)} \sum_{j=r}^{m-3k} \\
&\quad (-1)^{m-3k-2r+p+q} 2^{p-r} d^{j+k-p-q} \binom{j}{r} \frac{k!}{(p-r)!q!(k+r-p-q)!} \\
&\quad \cdot X^{m-j-2k} \frac{X^{m-p-2q}}{T^{m-p-2q}} T^k \frac{X^t \hat{K}_{j,k}}{\hat{R}_Y^{m-2k}} (\tilde{t}')^{m-3q-p} ((\log R^*)')^p \left| \frac{\tilde{t}'}{\tilde{t}''} \frac{(\log R^*)'}{(\log R^*)''} \right|^q \\
&= \sum_{q=0}^{\lfloor m/3 \rfloor} \sum_{p=0}^{m-3q} \frac{X^{m-p-q} X^t}{\hat{R}_Y^{m-2q} T^{m-3q-p}} \sum_{k=q}^{\lfloor m/3 \rfloor} \sum_{r=\max(0, p+q-k)}^{\min(p, m-3k)} \sum_{j=r}^{m-3k} \\
&\quad (-1)^{m-3k-2r+p+q} 2^{p-r} d^{j+k-p-q} \binom{j}{r} \frac{k!}{(p-r)!q!(k+r-p-q)!} X^{m-q-j-2k} T^{k-q} \hat{R}_Y^{2(k-q)} \hat{K}_{j,k} \\
&\quad \cdot (\tilde{t}')^{m-3q-p} ((\log R^*)')^p \left| \frac{\tilde{t}'}{\tilde{t}''} \frac{(\log R^*)'}{(\log R^*)''} \right|^q,
\end{aligned}$$

where the exponent $m - q - j - 2k \geq 0$ since $j + 2k \leq (m - 3k) + 2k = m - k$ and $q \leq k$.

To guarantee the holomorphicity of the jet differential (3.21) along the divisor at infinity $\{T = 0\}$, each coefficient

$$\begin{aligned}
&\frac{X^{m-p-q} X^t}{\hat{R}_Y^{m-2q} T^{m-3q-p}} \sum_{k=q}^{\lfloor m/3 \rfloor} \sum_{r=\max(0, p+q-k)}^{\min(p, m-3k)} \sum_{j=r}^{m-3k} \\
&\quad (-1)^{m-3k-2r+p+q} 2^{p-r} d^{j+k-p-q} \binom{j}{r} \frac{k!}{(p-r)!q!(k+r-p-q)!} X^{m-q-j-2k} T^{k-q} \hat{R}_Y^{2(k-q)} \hat{K}_{j,k} \\
&= (-1)^{m+p+q} \frac{X^{m-p-q} X^t}{\hat{R}_Y^{m-2q} T^{m-3q-p}} \sum_{k=q}^{\lfloor m/3 \rfloor} \sum_{r=\max(0, p+q-k)}^{\min(p, m-3k)} \sum_{j=r}^{m-3k} \\
&\quad (-1)^{-3k} 2^{p-r} d^{j+k-p-q} \binom{j}{r} \frac{k!}{(p-r)!q!(k+r-p-q)!} X^{m-q-j-2k} T^{k-q} \hat{R}_Y^{2(k-q)} \hat{K}_{j,k},
\end{aligned}$$

where the last equality uses $(-1)^{2r} = 1$ for any integer r , must be regular on $\{X \neq 0\} \cap \{\hat{R}_Y \neq 0\}$. Consequently, the holomorphicity conditions translate into the following divisibility requirements in the polynomial ring $\mathbb{C}[T, X, Y]$:

$$\begin{aligned}
(3.24) \quad &T^{m-3q-p} \left| \sum_{k=q}^{\lfloor m/3 \rfloor} \sum_{r=\max(0, p+q-k)}^{\min(p, m-3k)} \sum_{j=r}^{m-3k} \right. \\
&\quad \left. (-1)^{-3k} 2^{p-r} d^{j+k-p-q} \binom{j}{r} \frac{k!}{(p-r)!q!(k+r-p-q)!} X^{m-q-j-2k} T^{k-q} \hat{R}_Y^{2(k-q)} \hat{K}_{j,k}, \right.
\end{aligned}$$

for every $q = 0, \dots, \lfloor \frac{m}{3} \rfloor$ and $p = 0, \dots, m - 3q$.

3.3. End of the Proof of the Key Vanishing Lemma 1.3. For a smooth curve $\mathcal{C} \subset \mathbb{P}^2$, we consider the bundle $E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C})$ of invariant logarithmic 2-jet differentials and its negative twists:

$$E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t),$$

with integer $t \geq 1$.

We deliberately choose the affine equation of the curve $\mathcal{C}$ to be

$$R := (x^d + y^d + 1) + x^6 \in \mathbb{C}[x, y],$$

where the extra monomial x^6 in this deformed Fermat-type polynomial is determined experimentally: lower-order perturbations such as $x^1, x^2, \dots$ fail to yield the desired vanishing for certain pairs (m, t) (nontrivial solutions persist). Moreover, its projectivization $\hat{R} := T^d + X^d + Y^d + X^6 T^{d-6}$ has degree d . One can verify the smoothness of $\mathcal{C}$ as well as the general position of the four divisors R, R_X, R_Y, R_T ; we omit these routine computations.

For instance, when $m = 3$, by Proposition 3.1, an invariant logarithmic 2-jet differential of weighted order 3 in $E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C})$ takes the following general local form on $\{R_y \neq 0\}$:

$$J_{xR} = \sum_{0 \leq j \leq 3} \frac{A_j}{R_y^3} (x')^{3-j} \left(\frac{R'}{R} \right)^j + \frac{B_0}{R_y} \Delta_{xR}''''.$$

The jet bundle transition from $\{R_y \neq 0\}$ to $\{R_x \neq 0\}$ consists in solving for x' and x'' from the relations (3.1) which gives the solutions:

$$\begin{aligned} (3.25) \quad x' &= -y' \frac{R_y}{R_x} + \frac{R'}{R_x}, \\ x'' &= -y'' \frac{R_y}{R_x} + \frac{R''}{R_x} \\ &\quad - y'y' \frac{R_y^2 R_{xx}}{R_x^3} + 2y'R' \frac{R_y R_{xx}}{R_x^3} - R'R' \frac{R_{xx}}{R_x^3} + 2y'y' \frac{R_y R_{xy}}{R_x^2} - 2y'R' \frac{R_{xy}}{R_x^2} - y'y' \frac{R_{yy}}{R_x}, \end{aligned}$$

and in replacing x' and x'' in the considered jet differential J_{xR} .

Performing the transition from $\{R_y \neq 0\}$ to $\{R_x \neq 0\}$, we obtain the transformed expression:

$$\begin{aligned} J_{xR} &= \Delta_{R_y}'''' \frac{B_0}{R_x} \\ &\quad - y'^3 \frac{A_0}{R_x^3} \\ &\quad + R'y'^2 \left(\frac{3A_0}{R_y R_x^3} + \frac{A_1}{R_y R R_x^2} + B_0 \left[\frac{R_{xx} R_y}{R R_x^3} - \frac{2R_{xy}}{R R_x^2} + \frac{R_{yy}}{R_y R R_x} \right] \right) \\ &\quad + R'^2 y' \left(-\frac{3A_0}{R_y^2 R_x^3} - \frac{2A_1}{R_y^2 R R_x^2} - \frac{A_2}{R_y^2 R^2 R_x} + B_0 \left[\frac{2R_{xy}}{R_y R R_x^2} - \frac{2R_{xx}}{R R_x^3} \right] \right) \\ &\quad + R'^3 \left(\frac{A_0}{R_y^3 R_x^3} + \frac{A_1}{R_y^3 R R_x^2} + \frac{A_2}{R_y^3 R^2 R_x} + \frac{A_3}{R_y^3 R^3} + B_0 \left[\frac{R_{xx}}{R_y R R_x^3} - \frac{1}{R_y R^2 R_x} \right] \right). \end{aligned}$$

In the first two lines, only R_x appears in denominators — which is allowed on the open set $\{R_x \neq 0\}$ — so no constraints are imposed on B_0 or A_0 from these terms.

On the third line, however, R_y appears to the first power in denominators. More precisely, the coefficient of $R'y'^2$ in J_{xR} , which we denote by $[R'y'^2](J_{xR})$, factors as:

$$[R'y'^2](J_{xR}) = \frac{3A_0 R + A_1 R_x + B_0 [R_y^2 R_{xx} - 2R_y R_x R_{xy} - R_x^2 R_{yy}]}{R R_y R_x^3}.$$

To ensure holomorphicity after transitioning to the chart $\{R_x \neq 0\}$ and after clearing denominators involving R_x and R , it is necessary that R_y divides the numerator in $\mathbb{C}[x, y]$:

$$R_y \mid 3A_0 R + A_1 R_x + B_0 [R_y^2 R_{xx} - 2R_y R_x R_{xy} - R_x^2 R_{yy}] \quad \text{in } \mathbb{C}[x, y].$$

Since the first two terms inside the brackets are already divisible by R_y , this condition reduces to

$$R_y \mid 3 A_0 R + A_1 R_x + B_0 R_x^2 R_{yy} \quad \text{in } \mathbb{C}[x, y].$$

Next, the coefficients of $R'^2 y'$ and R'^3 in J_{xR} are respectively:

$$\begin{aligned} [R'^2 y'] (J_{xR}) &= \frac{-3 A_0 R^2 - 2 A_1 R R_x - A_2 R_x^2 + 2 B_0 [R_y R R_x R_{xy} - R_y^2 R R_{xx}]}{R^2 R_y^2 R_x^3}, \\ [R'^3] (J_{xR}) &= \frac{A_0 R^3 + A_1 R^2 R_x + A_2 R R_x^2 + A_3 R_x^3 + B_0 [R_y^2 R^2 R_{xx} - R_y^2 R R_x^2]}{R^3 R_y^3 R_x^3}. \end{aligned}$$

After clearing the admissible powers of R_x and R from the denominators, the holomorphicity conditions on the chart $\{R_x \neq 0\}$ translate into the following divisibility requirements in $\mathbb{C}[x, y]$:

$$\begin{aligned} R_y^2 \mid -3 A_0 R^2 - 2 A_1 R R_x - A_2 R_x^2 + 2 B_0 [R_y R R_x R_{xy} - R_y^2 R R_{xx}] \quad \text{in } \mathbb{C}[x, y], \\ R_y^3 \mid A_0 R^3 + A_1 R^2 R_x + A_2 R R_x^2 + A_3 R_x^3 + B_0 [R_y^2 R^2 R_{xx} - R_y^2 R R_x^2] \quad \text{in } \mathbb{C}[x, y]. \end{aligned}$$

In general, for any $m \geq 3$, returning to the expression of J_{xR} given in (3.22) and applying the jet bundle transition — which replaces x' and x'' via formula (3.25) — we obtain an expansion of the form

$$J_{xR} = x^t \sum_{i+j+3k=m} \left(\frac{R'}{R}\right)^i (y')^j (\Delta'''_{Ry})^k \frac{\text{Eq}_{i,j,k}(K_{j,k}; j_{x,y}^m R)}{(R_y)^i R_x^{m-2k}},$$

where $\text{Eq}_{i,j,k}$ are certain complicated polynomials in the coefficient functions $K_{j,k}$ appearing in the initial expression of J_{xR} (see (3.22)) and in the m -jet $j_{x,y}^m R(x, y)$.

Since division by R_x^{m-2k} is allowed on the open set $\{R_x \neq 0\}$, the only remaining constraints arise from the powers $(R_y)^i$ appearing in the denominator. Holomorphicity on $\{R_x \neq 0\}$ therefore requires, for every triple $(i, j, k) \in \mathbb{N}^3$ with $i + j + 3k = m$,

$$(R_y)^i \mid \text{Eq}_{i,j,k}(K_{j,k}; j_{x,y}^m R) \quad \text{in } \mathbb{C}[x, y].$$

Having chosen the defining polynomial to be of the Fermat-type perturbation

$$R := x^d + y^d + x^6 + 1,$$

for which $R_y = dy^{d-1}$, these divisibility conditions become

$$(y^{d-1})^i \mid \text{Eq}_{i,j,k}(K_{j,k}; j_{x,y}^m R) \quad \text{in } \mathbb{C}[x, y].$$

We then expand each $\text{Eq}_{i,j,k}$ as a polynomial in y and discard all terms of degree at least $(d-1)i$, keeping only the remainder modulo $y^{(d-1)i}$. This yields the truncated expression

$$\text{Eq}_{i,j,k}^{(\text{trunc})} := \text{Eq}_{i,j,k} \pmod{y^{(d-1)i}},$$

i.e., the polynomial obtained by removing any factor of $y^{(d-1)i}$ from $\text{Eq}_{i,j,k}$. All remaining terms must vanish identically for the original divisibility condition to hold, thereby converting the geometric constraints into an explicit linear system.

Next, we proceed inductively with respect to $i = 1, 2, \dots, m$ (starting naturally from $i = 1$). For each fixed i , we collect all divisibility equations sharing the same power of R_y , namely:

$$(3.26) \quad (y^{d-1})^i \mid \text{Eq}_{i,j,k}^{(\text{trunc})} \quad \text{in } \mathbb{C}[x, y], \quad \text{for all } (j, k) \text{ with } j + 3k = m - i.$$

The computer then extracts, for each triple $(i, j, k) \in \mathbb{N}^3$ with $j + 3k = m - i$, all coefficients

$$[x^p y^q] (\text{Eq}_{i,j,k}^{(\text{trunc})}), \quad q \leq (d-1)i - 1,$$

and sets them to zero in order to satisfy the divisibility conditions (3.26). Here, the notation $[x^p y^q](\cdot)$ denotes the coefficient of the monomial $x^p y^q$ in the polynomial expansion. These vanishing conditions yield a linear system that must be satisfied by the coefficients of the polynomials $\{K_{j,k}\}$ appearing in the local expression of the jet differential.

For each fixed i , this linear system is solved symbolically. The solutions are stored in memory before proceeding to the next value of i , up to $i = m$. At each stage, intermediate feedback — such as the number of nonzero terms remaining in the full expansions of $\{K_{j,k}\}$ — is monitored to track progress.

It remains to check the holomorphicity of the jet differential (3.22) along the divisor $\{T = 0\}$ at infinity.

To this end, we rewrite the polynomials $K_{j,k} \in \mathbb{C}[x, y]$ appearing in (3.22) as homogeneous polynomials $\widehat{K}_{j,k} \in \mathbb{C}[T, X, Y]$, where each $\widehat{K}_{j,k}(T, X, Y)$ satisfies

$$\deg \widehat{K}_{j,k} = (m - 2k)(d - 1) - m + j + k - t,$$

for $m \leq 14$, $k = 0, \dots, 4$, and $j = 0, \dots, m - 3k$.

Then, as in (3.24), we must verify the following divisibility conditions in the polynomial ring $\mathbb{C}[T, X, Y]$:

$$T^{m-3q-p} \mid \text{HEq}_{p,q}(\widehat{K}_{j,k}, T, X) \quad \text{in } \mathbb{C}[T, X, Y], \quad \text{for every } q = 0, \dots, 4 \text{ and } p = 0, \dots, m - 3q,$$

where $\text{HEq}_{p,q}(\widehat{K}_{j,k}, T, X)$ are certain complicated polynomials in the coefficient functions $\widehat{K}_{j,k}$ (from the initial expression of $J_{X\widehat{R}}$ in (3.21)) and in T, X . Explicitly, as in (3.24),

$$\begin{aligned} \text{HEq}_{p,q}(\widehat{K}_{j,k}, T, X) &= \sum_{k=q}^{\lfloor m/3 \rfloor} \sum_{r=\max(0, p+q-k)}^{\min(p, m-3k)} \sum_{j=r}^{m-3k} \\ &\quad (-1)^{-3k} 2^{p-r} d^{j+k-p-q} \binom{j}{r} \frac{k!}{(p-r)! q! (k+r-p-q)!} X^{m-q-j-2k} T^{k-q} \widehat{R}_Y^{2(k-q)} \widehat{K}_{j,k}. \end{aligned}$$

Proceeding in a completely analogous manner — expanding $\text{HEq}_{p,q}$ in T and discarding higher-degree terms — the divisibility conditions in T translate into a second linear system, which is constructed and solved symbolically by the computer following a similar inductive procedure.

We then expand each $\text{HEq}_{p,q}$ as a polynomial in T and discard all terms of degree at least $m - 3q - p$, keeping only the remainder modulo T^{m-3q-p} . This yields the truncated expression

$$\text{HEq}_{p,q}^{(\text{trunc})} := \text{HEq}_{p,q} \pmod{T^{m-3q-p}},$$

i.e., the polynomial obtained from $\text{HEq}_{p,q}$ by removing any factor of T^{m-3q-p} . For the original divisibility condition to hold, all remaining terms must vanish identically, thereby converting the geometric constraints into another explicit linear system.

Next, we proceed inductively with respect to $i = 1, 2, \dots, m$ (starting naturally from $i = 1$). For each fixed i , we collect all divisibility equations sharing the same power of T , namely:

$$(3.27) \quad T^i \mid \text{HEq}_{p,q}^{(\text{trunc})} \quad \text{in } \mathbb{C}[T, X, Y], \quad \text{for all } (p, q) \text{ with } p + 3q = m - i.$$

The computer then extracts, for each triple $(i, p, q) \in \mathbb{N}^3$ with $p + 3q = m - i$, all coefficients

$$[T^\alpha X^\beta Y^\gamma] (\text{HEq}_{p,q}^{(\text{trunc})}), \quad \alpha \leq m - 3q - p - 1 = i - 1,$$

and sets them to zero in order to satisfy the divisibility conditions (3.27). Here, the notation $[T^\alpha X^\beta Y^\gamma](\cdot)$ denotes the coefficient of the monomial $T^\alpha X^\beta Y^\gamma$ in the polynomial expansion. These vanishing conditions yield a linear system that must be satisfied by the coefficients of the polynomials $\widehat{K}_{j,k}$ in (3.21) (which correspond to the coefficients of $\{K_{j,k}\}$ in (3.22)) appearing in the local expression of the jet differential.

Finally, combining the two linear systems obtained from the divisibility conditions in y and in T , a computer algebra verification shows that all coefficients of $\{K_{j,k}\}$ are forced to vanish. This confirms that for the specified pairs (m, t) with $d = 12, 13$, only the trivial solution exists. The Key Vanishing Lemma 1.3 for a generic curve $\mathcal{C} \subset \mathbb{P}^2$ then follows by a standard semicontinuity argument.

For execution, a single Maple file handles all cases in the logarithmic setting.⁴ □

Remark 3.2. Our computer-assisted computations yield several nonvanishing examples that further confirm the reliability of the code. For the parameter $(m, t) = (3, 1)$, the dimension of $H^0(\mathbb{P}^2, E_{2,3} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-1))$ is:

- 1 for the degree 7 Fermat curve $X^7 + Y^7 + Z^7 = 0$;
- 2 for the degree 12 curve $X^{12} + Y^{12} + Z^{12} + X^6 Z^6 = 0$;

⁴The Maple code for the logarithmic case is available at <https://xiesongyan.github.io/>. The parameters (m, t) and d at the beginning of the code can be adjusted as needed to verify the required cases.

- 1 for the degree 11 curve $X^{11} + Y^{11} + Z^{11} + X^5 Z^6 = 0$.

These non-trivial sections are striking at first sight, but we have inspected them and verified that they satisfy all required conditions without error. The existence of such sections supports the correctness of our algorithm: if every test had returned zero, one might have suspected a systematic mistake. Instead, the code correctly detects negatively twisted invariant jet differentials whenever they are present.

4. Proof of Key Vanishing Lemma 1.4

4.1. Changes of Affine Chart in Jet Bundles of Orders 1 and 2. Let $X \subset \mathbb{P}^3$ be a smooth, generic algebraic surface of degree d . We fix homogeneous coordinates $[T : X : Y : Z]$ on $\mathbb{P}^3$. Although the surface and a coordinate share the symbol X , this standard convention causes no ambiguity and is maintained throughout. In the standard affine chart $\{T \neq 0\} \cong \mathbb{C}^3$ with local coordinates $(x, y, z) = (X/T, Y/T, Z/T)$, the surface X is defined by the zero set of a dehomogenized polynomial $R(x, y, z) \in \mathbb{C}[x, y, z]$ of degree d .

The complement of this affine chart is the hyperplane at infinity $\mathbb{P}_\infty^2 := \{T = 0\}$. To understand the behavior of jets on this divisor, one must consider alternative affine charts covering it. These are obtained by dividing by the other homogeneous coordinates; for instance, in the chart where $X \neq 0$, we set $X = 1$ and introduce new coordinates

$$(4.1) \quad \tilde{t} := \frac{T}{X} = \frac{1}{x}, \quad \tilde{y} := \frac{Y}{X} = \frac{y}{x}, \quad \tilde{z} := \frac{Z}{X} = \frac{z}{x},$$

so that the transformation from this new chart $(\tilde{t}, \tilde{y}, \tilde{z})$ to the original chart (x, y, z) is given by

$$(4.2) \quad x = \frac{1}{\tilde{t}}, \quad y = \frac{\tilde{y}}{\tilde{t}}, \quad z = \frac{\tilde{z}}{\tilde{t}}.$$

The charts $Y = 1$ and $Z = 1$ yield analogous transformations.

In the affine chart $\mathbb{C}^3 \subset \mathbb{P}^3$, the smoothness of $X \subset \mathbb{P}^3$ is equivalent to the condition

$$\{R = 0\} \cap \{R_x = 0\} \cap \{R_y = 0\} \cap \{R_z = 0\} = \emptyset.$$

In addition to the smoothness condition, we require R to be such that these four divisors are in general position, i.e., any intersection of them has the expected (co)dimension.

In this work, we will focus on jets of order 2. For an (entire) holomorphic curve parametrized by

$$\zeta \mapsto (x(\zeta), y(\zeta), z(\zeta)),$$

the associated jet coordinates are denoted by

$$x', y', z', \quad x'', y'', z''.$$

Since the curve lies in the surface X , substituting its parametrization into the defining equation yields the identity

$$R(x(\zeta), y(\zeta), z(\zeta)) \equiv 0.$$

Differentiating this relation with respect to ζ provides one constraint among first-order jets:

$$(4.3) \quad x' R_x + y' R_y + z' R_z = 0,$$

and a second differentiation gives one constraint among second-order jets:

$$(4.4) \quad \begin{aligned} 0 = & x'' R_x + y'' R_y + z'' R_z \\ & + x'^2 R_{xx} + 2x' y' R_{xy} + 2x' z' R_{xz} + y'^2 R_{yy} \\ & + 2y' z' R_{yz} + z'^2 R_{zz}. \end{aligned}$$

Solving the first relation (4.3) for y' yields:

$$(4.5) \quad y' := -\frac{R_x}{R_y} x' - \frac{R_z}{R_y} z'.$$

Solving the second relation (4.4) for y'' gives:

$$\begin{aligned} y'' = & -\frac{R_x}{R_y} x'' - \frac{R_z}{R_y} z'' \\ & - \frac{R_{xx}}{R_y} x'^2 - 2 \frac{R_{xy}}{R_y} x' y' - 2 \frac{R_{xz}}{R_y} x' z' \\ & - \frac{R_{yy}}{R_y} y'^2 - 2 \frac{R_{yz}}{R_y} y' z' - \frac{R_{zz}}{R_y} z'^2, \end{aligned}$$

and after substituting the expression (4.5) for y' , this becomes:

$$\begin{aligned} (4.6) \quad y'' = & -\frac{R_x}{R_y} x'' - \frac{R_z}{R_y} z'' \\ & + x'^2 \left(-\frac{R_{xx}}{R_y} + 2 \frac{R_x}{R_y} \frac{R_{xy}}{R_y} - \left(\frac{R_x}{R_y} \right)^2 \frac{R_{yy}}{R_y} \right) \\ & + 2x' z' \left(-\frac{R_{xz}}{R_y} + \frac{R_z}{R_y} \frac{R_{xy}}{R_y} + \frac{R_x}{R_y} \frac{R_{yz}}{R_y} - \frac{R_x}{R_y} \frac{R_z}{R_y} \frac{R_{yy}}{R_y} \right) \\ & + z'^2 \left(-\frac{R_{zz}}{R_y} + 2 \frac{R_z}{R_y} \frac{R_{yz}}{R_y} - \left(\frac{R_z}{R_y} \right)^2 \frac{R_{yy}}{R_y} \right). \end{aligned}$$

These two formulas for y' and y'' admit a geometric interpretation. Denote by J^2 the bundle of 2-jets of local holomorphic curves $\mathbb{C} \rightarrow X$. On the open subset of the affine chart where the partial derivative with respect to y does not vanish,

$$X \cap \mathbb{C}^3 \cap \{R_y \neq 0\},$$

the holomorphic implicit function theorem allows us to solve locally for y as a function of (x, z) . Hence, over this set, the surface X is locally graphed as a holomorphic map $(x, z) \mapsto (x, \text{function}(x, z), z)$ over the (x, z) -plane, where we deliberately omit introducing a specific symbol for this resolution $y = \text{function}(x, z)$.

Consequently, in this open set $X \cap \mathbb{C}^3 \cap \{R_y \neq 0\}$, the 2-jet bundle J^2 can be coordinatized by

$$(x, z, x', z', x'', z'').$$

Likewise, on the complementary open set $X \cap \mathbb{C}^3 \cap \{R_z \neq 0\}$, one can solve locally for z in terms of (x, y) . Accordingly, J^2 is coordinatized there by

$$(x, y, x', y', x'', y'').$$

In a similar vein, we can also obtain the following formulas:

$$\begin{aligned} (4.7) \quad z' = & -\frac{R_x}{R_z} x' - \frac{R_y}{R_z} y', \\ (4.8) \quad z'' = & -\frac{R_x}{R_z} x'' - \frac{R_y}{R_z} y'' \\ & + x'^2 \left(-\frac{R_{xx}}{R_z} + 2 \frac{R_x}{R_z} \frac{R_{xz}}{R_z} - \left(\frac{R_x}{R_z} \right)^2 \frac{R_{zz}}{R_z} \right) \\ & + 2x' y' \left(-\frac{R_{xy}}{R_z} + \frac{R_y}{R_z} \frac{R_{xz}}{R_z} + \frac{R_x}{R_z} \frac{R_{yz}}{R_z} - \frac{R_x}{R_z} \frac{R_y}{R_z} \frac{R_{zz}}{R_z} \right) \\ & + y'^2 \left(-\frac{R_{yy}}{R_z} + 2 \frac{R_y}{R_z} \frac{R_{yz}}{R_z} - \left(\frac{R_y}{R_z} \right)^2 \frac{R_{zz}}{R_z} \right). \end{aligned}$$

For brevity, in what follows we will omit the explicit intersection with $\mathbb{C}^3$ and simply write $X \cap \{R_y \neq 0\}$ and $X \cap \{R_z \neq 0\}$, it being understood that all considerations are restricted to the affine chart $\mathbb{C}^3 \subset \mathbb{P}^3$.

Observation 4.9. The two formulas (4.5) and (4.6) express the change of coordinates for the 2-jet bundle J^2 between the two charts:

$$J^2|_{\{R_z \neq 0\}} \longrightarrow J^2|_{\{R_y \neq 0\}}.$$

Conversely, the formulas (4.7) and (4.8) give the inverse change of coordinates, expressing the transition from the $\{R_y \neq 0\}$ chart back to the $\{R_z \neq 0\}$ chart:

$$J^2|_{\{R_y \neq 0\}} \longrightarrow J^2|_{\{R_z \neq 0\}}.$$

The Wronskian $y'x'' - y''x'$ expressed in the chart $X \cap \{R_z \neq 0\}$ transforms to the chart $X \cap \{R_y \neq 0\}$ by substituting y' from (4.5) and y'' from (4.6):

$$\begin{aligned} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix} &= - \begin{vmatrix} \frac{R_x}{R_y}x' + \frac{R_z}{R_y}z' & x' \\ \frac{R_x}{R_y}x'' + \frac{R_z}{R_y}z'' & x'' \end{vmatrix} \\ &+ \begin{vmatrix} 0 & x' \\ x'^2 \left(-\frac{R_{xx}}{R_y} + 2\frac{R_x}{R_y}\frac{R_{xy}}{R_y} - \frac{R_x}{R_y}\frac{R_x}{R_y}\frac{R_{yy}}{R_y} \right) & x'' \end{vmatrix} \\ &+ \begin{vmatrix} 0 & x' \\ 2x'z' \left(-\frac{R_{xz}}{R_y} + \frac{R_z}{R_y}\frac{R_{xy}}{R_y} + \frac{R_x}{R_y}\frac{R_{yz}}{R_y} - \frac{R_x}{R_y}\frac{R_z}{R_y}\frac{R_{yy}}{R_y} \right) & x'' \end{vmatrix} \\ &+ \begin{vmatrix} 0 & x' \\ z'z' \left(-\frac{R_{zz}}{R_y} + 2\frac{R_z}{R_y}\frac{R_{yz}}{R_y} - \frac{R_z}{R_y}\frac{R_z}{R_y}\frac{R_{yy}}{R_y} \right) & x'' \end{vmatrix}, \end{aligned}$$

by expanding, by reorganizing, and by letting the new Wronskian $z'x'' - z''x'$ appear:

$$\begin{aligned} (4.10) \quad \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix} &= -\frac{R_z}{R_y} \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix} \\ &+ z'z'x' \left(\frac{R_{zz}}{R_y} - 2\frac{R_z}{R_y}\frac{R_{yz}}{R_y} + \frac{R_z}{R_y}\frac{R_z}{R_y}\frac{R_{yy}}{R_y} \right) \\ &+ z'x'x' \left(2\frac{R_{xz}}{R_y} - 2\frac{R_z}{R_y}\frac{R_{xy}}{R_y} - 2\frac{R_x}{R_y}\frac{R_{yz}}{R_y} + 2\frac{R_x}{R_y}\frac{R_z}{R_y}\frac{R_{yy}}{R_y} \right) \\ &+ x'x'x' \left(\frac{R_{xx}}{R_y} - 2\frac{R_x}{R_y}\frac{R_{xy}}{R_y} + \frac{R_x}{R_y}\frac{R_x}{R_y}\frac{R_{yy}}{R_y} \right). \end{aligned}$$

This formula expresses the Wronskian $y'x'' - y''x'$ in the $X \cap \{R_z \neq 0\}$ chart as a combination of the new Wronskian $z'x'' - z''x'$ and purely algebraic terms (involving only first-order jets) that are holomorphic in the $X \cap \{R_y \neq 0\}$ chart. The key observation is that the coefficient of the new Wronskian contains a factor R_z/R_y , while the remaining terms have denominators involving only R_y .

4.2. Local Description of Invariant Jet Differentials. Recall that $R(x, y, z)$ is the defining polynomial of a generic smooth surface $X \subset \mathbb{P}^3$ in the affine chart $\mathbb{C}^3$. By the GAGA principle, a nonzero global holomorphic section of the bundle of 2-jet differentials of weighted degree m on the surface X is algebraic, and it is a basic fact of algebraic geometry that, when restricted to the Zariski open set $X \cap \{R_z \neq 0\}$, such a section admits a local expression of the form

$$(4.11) \quad J_{yx} := \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{F_{j,k}}{R_z^{f_{j,k}}} (y')^j (x')^{m-3k-j} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}^k,$$

where $R_z = \partial R / \partial z$ is nonvanishing on this set, and $F_{j,k} \in \mathbb{C}[x, y, z]$ are polynomials. All denominators and numerators are considered modulo R , i.e., as elements of the quotient ring

$$(4.12) \quad \mathfrak{R} := \mathbb{C}[x, y, z]/(R),$$

and consequently the rational functions are regarded as elements of its fraction field. The nonnegative integers $f_{j,k}$ indicate the order of the factor R_z in the denominator.

Question 4.13. Can one obtain upper bounds for the integers $f_{j,k}$?

The answer is affirmative. The following proposition reveals that every global section of the invariant 2-jet differential bundle necessarily conforms to a prescribed finite form, with undetermined polynomial coefficients.

Proposition 4.1. *Let $\widehat{R}(T, X, Y, Z) \in \mathbb{C}[T, X, Y, Z]$ be a homogeneous polynomial defining a smooth surface $X \subset \mathbb{P}^3$, and assume that the divisors in $\mathbb{P}^3$ given by $\widehat{R}$ and its partial derivatives $\widehat{R}_T, \widehat{R}_X, \widehat{R}_Y, \widehat{R}_Z$ are in general position. Dehomogenizing with respect to $T = 1$ yields an affine polynomial $R(x, y, z) \in \mathbb{C}[x, y, z]$ such that $X \cap \{T \neq 0\} = \{R = 0\}$. For any weighted order m , every invariant 2-jet differential of weighted order m on X admits, on $X \cap \{T \neq 0\} \cap \{R_z \neq 0\}$, the general expression*

$$J_{yx} = \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{F_{j,k}}{R_z^{m-2k}} (y')^j (x')^{m-3k-j} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}^k,$$

where $F_{j,k} \in \mathbb{C}[x, y, z]$ are polynomials, viewed as elements of the quotient ring $\mathfrak{R} = \mathbb{C}[x, y, z]/(R)$ (hence as regular functions on $X \cap \{T \neq 0\}$).

Proof. On the affine part $X \cap \{R_y \neq 0\}$, by the same reasoning as in (4.11), an invariant 2-jet differential of weighted order m can be expressed as

$$(4.14) \quad J_{zx} := \sum_{\substack{0 \leq q \leq \lfloor m/3 \rfloor \\ 0 \leq p \leq m-3q}} \overline{F}_{p,q} (z')^p (x')^{m-3q-p} \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix}^q,$$

where all coefficients are rational functions whose denominators may contain only powers of R_y .

We now examine the compatibility of the two expressions (4.11) and (4.14) for the jet differential by substituting the transition formulas (4.7) and (4.8) into the alternative expression (4.14). Under this substitution, the Wronskian $z'x'' - z''x'$ — initially expressed in the chart $X \cap \{R_y \neq 0\}$ — transforms to the chart $X \cap \{R_z \neq 0\}$ after expanding and reorganizing terms so that the Wronskian $y'x'' - y''x'$ emerges explicitly:

$$(4.15) \quad \begin{aligned} \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix} &= -\frac{R_y}{R_z} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix} \\ &+ \frac{1}{R_z^3} (R_{yy}R_z^2 - 2R_zR_yR_{yz} + R_y^2R_{zz})y'y'x' \\ &+ \frac{2}{R_z^3} (R_{xy}R_z^2 - R_zR_yR_{xz} - R_xR_zR_{yz} + R_xR_yR_{zz})y'x'x' \\ &+ \frac{1}{R_z^3} (R_{xx}R_z^2 - 2R_xR_zR_{xz} + R_x^2R_{zz})x'x'x'. \end{aligned}$$

Comparing corresponding terms, we observe that the coefficients in (4.11) can be expressed as rational functions whose denominators involve powers of R_y (which must ultimately cancel) and powers of R_z . The exponents of R_z appearing in the denominators arise from the following sources:

- Each of the p factors of z' contributes at most one factor of R_z^{-1} via (4.7).
- Each of the q factors of the Wronskian $\begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix}$ contributes at most one factor of R_z^{-1} from the leading term in (4.15) or, when expanded further, contributes at most one factor of R_z^{-3} from the quadratic terms.

Moreover, the term $(y')^j (x')^{m-3k-j} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}^k$ in expression (4.11) arises from expanding terms of the form

$$(4.16) \quad (z')^p (x')^{m-3q-p} \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix}^q \quad \text{with } q \geq k.$$

When expanding such a term, $\begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix}^q$ contributes a factor of $\begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}^k$ together with k powers of R_z^{-1} from the leading term in (4.15) and an additional $q - k$ powers of R_z^{-3} from the remaining quadratic terms. The

factor $(z')^p$ contributes p powers of R_z^{-1} via (4.7). Hence, the total exponent of R_z in the denominator of the coefficient multiplying $(y')^j (x')^{m-3k-j} \left| \frac{y'}{y''} \frac{x'}{x''} \right|^k$ is at most

$$k + 3(q - k) + p = 3q + p - 2k \leq m - 2k.$$

This upper bound $m - 2k$ can be achieved by expanding the term in (4.16) corresponding to $q = k$ and $p = m - 3k$, namely $(z')^{m-3k} \left| \frac{z'}{z''} \frac{x'}{x''} \right|^k$.

By our assumption on $\widehat{R}$, the divisors defined by R and its partial derivatives R_x, R_y, R_z are in general position. Under this condition, after cancelling common factors between numerator and denominator, the maximal possible pole order $f_{j,k}$ of the coefficients in (4.11) with respect to R_z satisfies

$$f_{j,k} \leq m - 2k.$$

Moreover, since the expression (4.11) must be regular on $X \cap \{R_z \neq 0\}$, any factor of R_y in the denominator must cancel. Consequently, the coefficients $F_{j,k}$ may be taken as polynomials, and we may set $f_{j,k} = m - 2k$ as the maximal admissible exponent. $\square$

4.3. Choice of the Surface Equation. We now set the degree of the smooth surface $X \subset \mathbb{P}^3$ to be

$$\deg X = d,$$

with $d \in \mathbb{N}$. Its defining equation is taken as a perturbation of the Fermat hypersurface:

$$(4.17) \quad \widehat{R} := X^d + Y^d + Z^d + T^d + X^a T^b = 0,$$

where the positive integers a, b satisfy $a + b = d$ and will be chosen later with $|a - b|$ small (e.g. $a - b = 0$ or 1). The specific choice of the exponents a, b will be guided by computer experiments in Maple.

Convention 4.18. In the remainder of the proof, we work under the following generic assumptions:⁵

- The surface X defined by $\widehat{R} = 0$ is smooth.
- The five divisors defined by the homogeneous polynomial $\widehat{R}$ and its partial derivatives $\widehat{R}_X, \widehat{R}_Y, \widehat{R}_Z, \widehat{R}_T$ are in general position; i.e., any intersection of them has the expected (co)dimension.
- The coordinate hyperplanes $\{Z = 0\}$ and $\{T = 0\}$ intersect X transversely.

In the affine chart $T = 1$, this defining equation becomes

$$R(x, y, z) = 1 + x^d + y^d + z^d + x^a,$$

where the exponent a will eventually be chosen as $\lfloor d/2 \rfloor$. This particular form yields two crucial simplifications. First, the relevant partial derivative simplifies to

$$R_z = dz^{d-1},$$

which is independent of x and y up to the harmless constant factor d . Second, since the extra term $X^a T^b$ involves no variable Y , we obtain the relation

$$y^d = -1 - x^d - z^d - x^a,$$

which allows for an efficient reduction procedure in the quotient ring $\mathfrak{R} = \mathbb{C}[x, y, z]/(R)$.

Recall that for an algebraic invariant 2-jet differential initially defined on $X \cap \{R_z \neq 0\}$ to be also holomorphic on $X \cap \{R_y \neq 0\}$, certain divisibility conditions by R_z must be satisfied — constraints already encountered in the transition formulas. These conditions are understood in the quotient ring $\mathfrak{R}$. A polynomial $F \in \mathbb{C}[x, y, z]$ is said to be divisible by R_z in this quotient ring if there exists $G \in \mathbb{C}[x, y, z]$ such that $F \equiv R_z \cdot G \pmod{R}$, i.e., their difference lies in the ideal generated by R . Geometrically, this means that as a rational function on the surface X , F vanishes on $X \cap \{R_z = 0\}$ with at least the same multiplicity as R_z .

To test divisibility by $R_z = dz^{d-1}$ in $\mathfrak{R}$, we employ an algorithmic reduction based on the relation $y^d = -1 - x^d - z^d - x^a$. Every element of $\mathfrak{R}$ can be uniquely represented by a polynomial in $\mathbb{C}[x, y, z]$ with $\deg_y < d$. Indeed, for any monomial y^k with $k \geq d$, writing $k = dq + r$ where $0 \leq r < d$ via Euclidean division, we may replace

$$y^k = (y^d)^q \cdot y^r = (-1 - x^d - z^d - x^a)^q \cdot y^r,$$

⁵The equation $\widehat{R}$ to be adopted later has parameters $a = 8$ and $d = 17$, satisfying Convention 4.18. The associated Maple code is available at <https://github.com/LeiHou-code/Hyperbolicity-Dimension-2-Conventions-Facts-2026>.

which involves only powers of y with exponent $r < d$. Applying this substitution to each monomial yields a reduced representative $\text{red}(F) \in \mathbb{C}[x, y, z]$ satisfying $\deg_y \text{red}(F) < d$ and $\text{red}(F) \equiv F \pmod{R}$.

The uniqueness of this reduced form follows from the observation that if two polynomials F_1, F_2 with $\deg_y < d$ satisfy $F_1 \equiv F_2 \pmod{R}$, then their difference is a multiple of R . Any nonzero multiple of R must have y -degree at least d , since R itself contains the monomial y^d with coefficient 1 and no other term can cancel it after multiplication. Hence $F_1 - F_2 = 0$, establishing uniqueness.

Lemma 4.2. *Let $R(x, y, z) = 1 + x^d + y^d + z^d + S(x)$, where $S(x) \in \mathbb{C}[x]$ is a polynomial of degree strictly less than d . Assume that the surface $X \subset \mathbb{P}^3$ defined by the homogenization of R is smooth and that the coordinate hyperplane $\{Z = 0\}$ intersects X transversely. For any polynomial $F \in \mathbb{C}[x, y, z]$, let $\text{red}(F)$ denote its unique normal form satisfying $\deg_y \text{red}(F) < d$, obtained by Euclidean reduction using the relation $y^d = -1 - x^d - z^d - S(x)$. Then, for any positive integer i , the polynomial F is divisible by $(R_z)^i = (dz^{d-1})^i$ in the quotient ring $\mathfrak{R} = \mathbb{C}[x, y, z]/(R)$ if and only if its reduction $\text{red}(F)$ is divisible by $(z^{d-1})^i$ in $\mathbb{C}[x, y, z]$.*

Proof. Since $\{Z = 0\}$ intersects X transversely, z is not a zero divisor in $\mathfrak{R}$.

($\Leftarrow$) If $\text{red}(F) = (z^{d-1})^i H$ for some $H \in \mathbb{C}[x, y, z]$, then $F \equiv \text{red}(F) \equiv (z^{d-1})^i H \pmod{R}$. Since $(R_z)^i = d^i (z^{d-1})^i$, we have $(z^{d-1})^i H = (R_z)^i \cdot (\frac{1}{d^i} H)$, so $F \equiv (R_z)^i \cdot (\frac{1}{d^i} H) \pmod{R}$, proving that F is divisible by $(R_z)^i$ in $\mathfrak{R}$.

($\Rightarrow$) Suppose $F \equiv (R_z)^i G \pmod{R}$ for some $G \in \mathbb{C}[x, y, z]$. Let $G_0 = \text{red}(G)$ be the normal form of G . Then $(R_z)^i G \equiv (R_z)^i G_0 \pmod{R}$, and $(R_z)^i G_0 = d^i (z^{d-1})^i G_0$ is already in normal form (since $\deg_y G_0 < d$ and multiplication by powers of z does not affect the y -degree). By uniqueness of the normal form, we must have $\text{red}(F) = d^i (z^{d-1})^i G_0 = (z^{d-1})^i (d^i G_0)$, which is clearly divisible by $(z^{d-1})^i$ in $\mathbb{C}[x, y, z]$. $\square$

Consequently, the divisibility conditions required for holomorphicity on $X \cap \{R_y \neq 0\}$ translate into an explicit algebraic constraint: after reducing each numerator polynomial F to its normal form $\text{red}(F)$, the coefficients of $z^0, z^1, \dots, z^{d-2}$ in $\text{red}(F)$ must all vanish identically.

This reduction crucially relies on our specific choice of the defining equation. By contrast, if one starts with a genuinely generic homogeneous polynomial of degree d — rather than the special form $X^d + Y^d + Z^d + T^d + X^a T^b$ — the quotient ring $\mathfrak{R}$ becomes significantly more complicated. Its structure no longer admits a simple degree reduction in any variable, and checking divisibility by R_z would require computing normal forms with respect to a Gröbner basis. Such computations are notoriously intensive and quickly become infeasible even on powerful computer algebra systems.

4.4. Denominator Exponent Constraints from Holomorphicity. To analyze the behavior of jets on the plane at infinity $\mathbb{P}_\infty^2 := \{T = 0\}$, we will work in the affine chart $\{X \neq 0\}$. Introducing the coordinates $(\tilde{t}, \tilde{y}, \tilde{z})$ as in (4.2), the divisor $\mathbb{P}_\infty^2$ is then locally defined by $\{\tilde{t} = 0\}$.

Recall that the jets under consideration are jets of entire holomorphic curves $\mathbb{C} \rightarrow \mathbb{P}^3$, or of local holomorphic curves $\mathbb{D} \rightarrow \mathbb{C}^3 \subset \mathbb{P}^3$, where $\mathbb{D} \subset \mathbb{C}$ is the unit disc:

$$\mathbb{D} \ni \zeta \mapsto (x(\zeta), y(\zeta), z(\zeta)) \in \mathbb{C}^3.$$

Under the change of chart $(\tilde{t}, \tilde{y}, \tilde{z}) \mapsto (x, y, z)$ given by (4.1), the 1-jets and 2-jets transform according to the following formulas (expressing the original jet coordinates in terms of the new ones):

$$(4.19) \quad \begin{aligned} \tilde{t}' &= -\frac{x'}{x^2}, & \tilde{y}' &= \frac{y'}{x} - \frac{y x'}{x^2}, \\ \tilde{t}'' &= -\frac{x''}{x^2} + 2\frac{(x')^2}{x^3}, & \tilde{y}'' &= \frac{y''}{x} - 2\frac{x' y'}{x^2} - \frac{y x''}{x^2} + 2\frac{y (x')^2}{x^3}. \end{aligned}$$

And the Wronskian transforms as:

$$(4.20) \quad \begin{vmatrix} \tilde{y}' & \tilde{t}' \\ \tilde{y}'' & \tilde{t}'' \end{vmatrix} = -\frac{1}{x^3} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}.$$

Similarly, under the change of chart $(x, y, z) \mapsto (\tilde{t}, \tilde{y}, \tilde{z})$ given by (4.2), the 1-jets and 2-jets transform according to the following formulas:

$$(4.21) \quad \begin{aligned} x' &= -\frac{\tilde{t}'}{\tilde{t}^2}, & y' &= \frac{\tilde{y}'}{\tilde{t}} - \frac{\tilde{y}\tilde{t}'}{\tilde{t}^2}, \\ x'' &= -\frac{\tilde{t}''}{\tilde{t}^2} + 2\frac{(\tilde{t}')^2}{\tilde{t}^3}, & y'' &= \frac{\tilde{y}''}{\tilde{t}} - 2\frac{\tilde{t}'\tilde{y}'}{\tilde{t}^2} - \frac{\tilde{y}\tilde{t}''}{\tilde{t}^2} + 2\frac{\tilde{y}(\tilde{t}')^2}{\tilde{t}^3}. \end{aligned}$$

Consequently, the Wronskian transforms as:

$$(4.22) \quad \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix} = -\frac{1}{\tilde{t}^3} \begin{vmatrix} \tilde{y}' & \tilde{t}' \\ \tilde{y}'' & \tilde{t}'' \end{vmatrix}.$$

Now consider a general term appearing in J_{yx} as described above in Proposition 4.1. Since the polynomials $F_{j,k}$ above will be restricted to the surface defined by $R = 0$, by the same reduction argument preceding Lemma 4.2, we may assume without loss of generality the following:

Convention 4.23. All polynomials $F_{j,k}$ are assumed to have y -degree strictly less than d ; equivalently, every monomial appearing in them satisfies $\deg_y < d$.

Furthermore, on $X \cap \{T \neq 0\} \cap \{\hat{R}_Z \neq 0\}$, every invariant 2-jet differential of weighted order m on X admits the general expression

$$(4.24) \quad J_{YX} := \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{\hat{G}_{j,k}}{T^{g_{j,k}} \hat{R}_Z^{m-2k}} (y')^j (x')^{m-3k-j} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}^k,$$

where $\hat{G}_{j,k}(T, X, Y, Z)$ are homogeneous polynomials in $\mathbb{C}[T, X, Y, Z]$ satisfying

$$\deg \hat{G}_{j,k} = g_{j,k} + (m-2k)(d-1) \quad \text{and} \quad \deg_Y \hat{G}_{j,k} < d,$$

and $\hat{R}_Z$ is the partial derivative of the homogeneous polynomial $\hat{R}$ as in Convention 4.18. Here the exponent of $\hat{R}_Z$ in the denominator is taken to be $m-2k$, which coincides with the maximal admissible exponent of R_z in the denominators of the coefficients $F_{j,k}/R_z^{m-2k}$ from Proposition 4.1.

Similarly, on $X \cap \{X \neq 0\} \cap \{\hat{R}_Z \neq 0\}$, every invariant 2-jet differential of weighted order m on X admits the general expression

$$(4.25) \quad J_{TY} = \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq \ell \leq m-3k}} \frac{\hat{E}_{\ell,k}}{X^{e_{\ell,k}} \hat{R}_Z^{m-2k}} (\tilde{t}')^{m-3k-\ell} (\tilde{y}')^\ell \begin{vmatrix} \tilde{y}' & \tilde{t}' \\ \tilde{y}'' & \tilde{t}'' \end{vmatrix}^k,$$

where $\hat{E}_{\ell,k}(T, X, Y, Z)$ are homogeneous polynomials in $\mathbb{C}[T, X, Y, Z]$ satisfying

$$\deg \hat{E}_{\ell,k} = e_{\ell,k} + (m-2k)(d-1) \quad \text{and} \quad \deg_Y \hat{E}_{\ell,k} < d.$$

Question 4.26. What happens to the general form J_{TY} given in (4.25) after the change of affine chart from $X \cap \{X \neq 0\} \cap \{\hat{R}_Z \neq 0\}$ to $X \cap \{T \neq 0\} \cap \{\hat{R}_Z \neq 0\}$?

Substituting (4.19) and (4.20) into (4.25), the term transforms as follows:

$$\begin{aligned} & \frac{\hat{E}_{\ell,k}}{X^{e_{\ell,k}} \hat{R}_Z^{m-2k}} (\tilde{t}')^{m-3k-\ell} (\tilde{y}')^\ell \begin{vmatrix} \tilde{y}' & \tilde{t}' \\ \tilde{y}'' & \tilde{t}'' \end{vmatrix}^k \\ &= \frac{\hat{E}_{\ell,k}}{X^{e_{\ell,k}} \hat{R}_Z^{m-2k}} \left(-\frac{x'}{x^2} \right)^{m-3k-\ell} \left(\frac{y'}{x} - \frac{y x'}{x^2} \right)^\ell \left(-\frac{1}{x^3} \right)^k \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}^k \\ &= \frac{1}{x^{2(m-3k)+3k}} \sum_{j=0}^{\ell} (-1)^{m-2k-j} \frac{\hat{E}_{\ell,k}}{X^{e_{\ell,k}} \hat{R}_Z^{m-2k}} \binom{\ell}{j} x^j y^{\ell-j} (y')^j (x')^{m-3k-j} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}^k, \end{aligned}$$

where the last equality is obtained by expanding $(xy' - yx')^\ell$ and simplifying.

Consequently, the general form transforms as

$$\begin{aligned}
J_{TY} &= \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{\ell=0}^{m-3k} \frac{1}{x^{2(m-3k)+3k}} \sum_{j=0}^{\ell} (-1)^{m-2k-j} \frac{\widehat{E}_{\ell,k}}{X^{e_{\ell,k}} \widehat{R}_Z^{m-2k}} \binom{\ell}{j} x^j y^{\ell-j} (y')^j (x')^{m-3k-j} \left| \begin{matrix} y' & x' \\ y'' & x'' \end{matrix} \right|^k \\
&= \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{j=0}^{m-3k} \frac{(-1)^{m-2k-j}}{x^{2(m-3k)+3k-j}} \sum_{\ell=j}^{m-3k} \frac{\widehat{E}_{\ell,k}}{X^{e_{\ell,k}} \widehat{R}_Z^{m-2k}} \binom{\ell}{j} y^{\ell-j} (y')^j (x')^{m-3k-j} \left| \begin{matrix} y' & x' \\ y'' & x'' \end{matrix} \right|^k \\
&= \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{j=0}^{m-3k} (-1)^{m-2k-j} \frac{T^{2(m-3k)+3k-j}}{X^{2(m-3k)+3k-j}} \sum_{\ell=j}^{m-3k} \frac{\widehat{E}_{\ell,k}}{X^{e_{\ell,k}} \widehat{R}_Z^{m-2k}} \binom{\ell}{j} \frac{Y^{\ell-j}}{T^{\ell-j}} (y')^j (x')^{m-3k-j} \left| \begin{matrix} y' & x' \\ y'' & x'' \end{matrix} \right|^k \\
&= \sum_{k=0}^{\lfloor m/3 \rfloor} \frac{T^m}{X^m} \sum_{j=0}^{m-3k} (-1)^{m-2k-j} \sum_{\ell=j}^{m-3k} \frac{\widehat{E}_{\ell,k}}{X^{e_{\ell,k}} \widehat{R}_Z^{m-2k}} \binom{\ell}{j} \frac{T^{m-3k-\ell} Y^{\ell-j}}{X^{m-3k-j}} (y')^j (x')^{m-3k-j} \left| \begin{matrix} y' & x' \\ y'' & x'' \end{matrix} \right|^k.
\end{aligned}$$

Note that the coefficients

$$\frac{T^m}{X^m} (-1)^{m-2k-j} \sum_{\ell=j}^{m-3k} \frac{\widehat{E}_{\ell,k}}{X^{e_{\ell,k}} \widehat{R}_Z^{m-2k}} \binom{\ell}{j} \frac{T^{m-3k-\ell} Y^{\ell-j}}{X^{m-3k-j}},$$

in the last line, as rational functions, not only have no pole along the divisor $X \cap \{T = 0\}$, but also vanish with multiplicity at least m .

Comparing with the coefficients of (4.24), it follows that, on $X \cap \{T \neq 0\} \cap \{\widehat{R}_Z \neq 0\}$, every invariant 2-jet differential of weighted order m on X admits the general expression

$$J_{YX} = \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{T^m \widehat{H}_{j,k}}{\widehat{R}_Z^{m-2k}} (y')^j (x')^{m-3k-j} \left| \begin{matrix} y' & x' \\ y'' & x'' \end{matrix} \right|^k,$$

where $\widehat{H}_{j,k}(T, X, Y, Z)$ are homogeneous polynomials in $\mathbb{C}[T, X, Y, Z]$ satisfying

$$\deg \widehat{H}_{j,k} = (m-2k)(d-1) - m \quad \text{and} \quad \deg_Y \widehat{H}_{j,k} < d.$$

To determine the explicit vanishing order of the jet differential as (4.24) along the divisor $\{x = 0\} \cap X$, we need the following lemma.

Lemma 4.3. *The rational function $\frac{T^m \widehat{H}_{j,k}}{\widehat{R}_Z^{m-2k}}$ vanishes along the divisor $X \cap \{x = 0\}$ with multiplicity at least t if and only if $\widehat{H}_{j,k} \equiv X^t \widehat{K}_{j,k} \pmod{\widehat{R}}$ for some homogeneous polynomial $\widehat{K}_{j,k}$ satisfying $\deg_Y \widehat{K}_{j,k} < d$.*

Proof. ($\Leftarrow$) If $\widehat{H}_{j,k} = X^t \widehat{K}_{j,k}$ in $\mathbb{C}[T, X, Y, Z]/(\widehat{R})$, then as a rational function, it clearly vanishes to order at least t along $\{x = 0\}$. By our choice of the surface equation (4.17), $\widehat{R}_Z$ does not vanish identically on $X \cap \{x = 0\}$, so the denominator introduces no poles or zero-canceling factors along $X \cap \{x = 0\}$, preserving the vanishing order.

($\Rightarrow$) Conversely, suppose the rational function $T^m \widehat{H}_{j,k} / \widehat{R}_Z^{m-2k}$ vanishes to order at least t along $X \cap \{x = 0\}$. The statement admits a proof by induction on t ; therefore, it suffices to establish the base case $t = 1$.

Since T^m and $\widehat{R}_Z$ do not vanish identically on $X \cap \{x = 0\}$ (by our choice of the surface equation (4.17)), it follows that $\widehat{H}_{j,k}$ itself vanishes along $X \cap \{x = 0\}$; i.e., $\widehat{H}_{j,k} \equiv 0 \pmod{(X, \widehat{R})}$. By Convention 4.23, $\widehat{H}_{j,k}$ has a unique representative with $\deg_Y < d$; we may therefore assume that $\widehat{H}_{j,k}$ is already in this normal form, satisfying $\deg_Y \widehat{H}_{j,k} < d$. Consider its restriction $\widehat{H}_{j,k}(T, 0, Y, Z) \in \mathbb{C}[T, Y, Z]$ to the hyperplane $\{X = 0\}$. This restriction satisfies:

- $\widehat{H}_{j,k}(T, 0, Y, Z) \equiv \widehat{H}_{j,k} \equiv 0 \pmod{(X, \widehat{R})}$;
- $\deg_Y \widehat{H}_{j,k}(T, 0, Y, Z) \leq \deg_Y \widehat{H}_{j,k} < d$.

By our choice of the surface equation (4.17), the first condition implies $\widehat{H}_{j,k}(T, 0, Y, Z) \equiv 0 \pmod{(T^d + Y^d + Z^d)}$ in $\mathbb{C}[T, Y, Z]$. However, the second condition — namely $\deg_Y < d$ — prevents $\widehat{H}_{j,k}(T, 0, Y, Z)$ from being a nonzero multiple of $T^d + Y^d + Z^d$, since the latter has Y -degree exactly d . Hence the restriction must be identically zero: $\widehat{H}_{j,k}(T, 0, Y, Z) = 0$ in $\mathbb{C}[T, Y, Z]$. It follows that $\widehat{H}_{j,k} \equiv X \widehat{K}_{j,k} \pmod{\widehat{R}}$ for some homogeneous polynomial $\widehat{K}_{j,k}$ satisfying $\deg_Y \widehat{K}_{j,k} < d$. $\square$

If we now require the jet differential as (4.24) to vanish to order at least $t \geq 1$ along the divisor $\{x = 0\} \cap X$ on the surface X , i.e.,

$$\text{Twist} := -t \leq -1,$$

then this corresponds to considering sections of the jet differential bundle twisted by $\mathcal{O}_X(-t)$, i.e.,

$$(\cdot) \otimes \mathcal{O}_X(-t).$$

Therefore, on $X \cap \{T \neq 0\} \cap \{\widehat{R}_Z \neq 0\}$, every invariant 2-jet differential of weighted order m and vanishing order at least t along $\{x = 0\} \cap X$ on the surface X admits the general expression

$$(4.27) \quad J_{YX} = \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{T^m X^t \widehat{K}_{j,k}}{\widehat{R}_Z^{m-2k}} (y')^j (x')^{m-3k-j} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}^k,$$

where $\widehat{K}_{j,k}(T, X, Y, Z)$ are homogeneous polynomials in $\mathbb{C}[T, X, Y, Z]$ satisfying

$$\deg \widehat{K}_{j,k} = (m-2k)(d-1) - m - t \quad \text{and} \quad \deg_Y \widehat{K}_{j,k} < d.$$

When the coefficients of the above expression are expressed as regular functions in the affine coordinates (x, y, z) , we obtain

$$(4.28) \quad J_{yx} := x^t \sum_{\substack{0 \leq k \leq \lfloor m/3 \rfloor \\ 0 \leq j \leq m-3k}} \frac{K_{j,k}}{R_Z^{m-2k}} (y')^j (x')^{m-3k-j} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}^k,$$

where $K_{j,k}(x, y, z)$ are polynomials in $\mathbb{C}[x, y, z]$ satisfying

$$\deg K_{j,k} \leq (m-2k)(d-1) - m - t \quad \text{and} \quad \deg_y K_{j,k} < d.$$

For our later applications, we only need weighted orders up to $m = 11$. For these low weights, the above expression simplifies considerably. Indeed, the exponent k of the Wronskian $\begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}$ satisfies $k \leq \lfloor m/3 \rfloor \leq 3$. Terms involving a power of the Wronskian greater than $\lfloor m/3 \rfloor$ are understood to be identically zero.

It remains to check the holomorphicity of our jet differential (4.27) along the divisor at infinity $X \cap \{T = 0\}$.

Question 4.29. What happens to the general form J_{YX} given in (4.27) after the change of affine chart (4.2)?

Substituting (4.21) and (4.22) into (4.27), the term transforms as follows:

$$\begin{aligned} & \frac{T^m X^t \widehat{K}_{j,k}}{\widehat{R}_Z^{m-2k}} (y')^j (x')^{m-3k-j} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}^k \\ &= \frac{T^m X^t \widehat{K}_{j,k}}{\widehat{R}_Z^{m-2k}} \left(\frac{\tilde{y}'}{\tilde{t}} - \frac{\tilde{y} \tilde{t}'}{\tilde{t}^2} \right)^j \left(-\frac{\tilde{t}'}{\tilde{t}^2} \right)^{m-3k-j} \begin{vmatrix} y' & x' \\ y'' & x'' \end{vmatrix}^k \\ &= \frac{(-1)^k}{\tilde{t}^{2(m-3k)+3k}} \frac{T^m X^t \widehat{K}_{j,k}}{\widehat{R}_Z^{m-2k}} (\tilde{t} \tilde{y}' - \tilde{t}' \tilde{y})^j (-\tilde{t}')^{m-3k-j} \begin{vmatrix} \tilde{y}' & \tilde{t}' \\ \tilde{y}'' & \tilde{t}'' \end{vmatrix}^k \\ &= \frac{1}{\tilde{t}^{2(m-3k)+3k}} \sum_{\ell=0}^j (-1)^{m-2k-\ell} \frac{T^m X^t \widehat{K}_{j,k}}{\widehat{R}_Z^{m-2k}} \binom{j}{\ell} \tilde{t}^\ell \tilde{y}^{j-\ell} (\tilde{t}')^{m-3k-\ell} (\tilde{y}')^\ell \begin{vmatrix} \tilde{y}' & \tilde{t}' \\ \tilde{y}'' & \tilde{t}'' \end{vmatrix}^k, \end{aligned}$$

where the last equality is obtained by expanding $(\tilde{t} \tilde{y}' - \tilde{t}' \tilde{y})^j$ and simplifying.

Consequently, the general form transforms as

$$\begin{aligned}
J_{yx} &= \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{j=0}^{m-3k} \frac{1}{\tilde{t}^{2(m-3k)+3k}} \sum_{\ell=0}^j (-1)^{m-2k-\ell} \frac{T^m X^t \widehat{K}_{j,k}}{\widehat{R}_Z^{m-2k}} \binom{j}{\ell} \tilde{t}^\ell \tilde{y}^{j-\ell} (\tilde{t}')^{m-3k-\ell} (\tilde{y}')^\ell \left| \begin{smallmatrix} \tilde{y}' & \tilde{t}' \\ \tilde{y}'' & \tilde{t}'' \end{smallmatrix} \right|^k \\
&= \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{\ell=0}^{m-3k} \frac{(-1)^{m-2k-\ell}}{\tilde{t}^{2(m-3k)+3k-\ell}} \sum_{j=\ell}^{m-3k} \frac{T^m X^t \widehat{K}_{j,k}}{\widehat{R}_Z^{m-2k}} \binom{j}{\ell} \tilde{y}^{j-\ell} (\tilde{t}')^{m-3k-\ell} (\tilde{y}')^\ell \left| \begin{smallmatrix} \tilde{y}' & \tilde{t}' \\ \tilde{y}'' & \tilde{t}'' \end{smallmatrix} \right|^k \\
&= \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{\ell=0}^{m-3k} (-1)^{m-2k-\ell} \frac{X^{2(m-3k)+3k-\ell}}{T^{2(m-3k)+3k-\ell}} \sum_{j=\ell}^{m-3k} \frac{T^m X^t \widehat{K}_{j,k}}{\widehat{R}_Z^{m-2k}} \binom{j}{\ell} \frac{Y^{j-\ell}}{X^{j-\ell}} (\tilde{t}')^{m-3k-\ell} (\tilde{y}')^\ell \left| \begin{smallmatrix} \tilde{y}' & \tilde{t}' \\ \tilde{y}'' & \tilde{t}'' \end{smallmatrix} \right|^k \\
&= \sum_{k=0}^{\lfloor m/3 \rfloor} \sum_{\ell=0}^{m-3k} (-1)^{m-2k-\ell} \sum_{j=\ell}^{m-3k} \frac{X^m X^t \widehat{K}_{j,k}}{\widehat{R}_Z^{m-2k}} \binom{j}{\ell} \frac{X^{m-3k-j} Y^{j-\ell}}{T^{m-3k-\ell}} (\tilde{t}')^{m-3k-\ell} (\tilde{y}')^\ell \left| \begin{smallmatrix} \tilde{y}' & \tilde{t}' \\ \tilde{y}'' & \tilde{t}'' \end{smallmatrix} \right|^k.
\end{aligned}$$

To guarantee the holomorphicity of our jet differential (4.27) along the divisor at infinity $X \cap \{T = 0\}$, each coefficient

$$\begin{aligned}
&(-1)^{m-2k-\ell} \sum_{j=\ell}^{m-3k} \frac{X^m X^t \widehat{K}_{j,k}}{\widehat{R}_Z^{m-2k}} \binom{j}{\ell} \frac{X^{m-3k-j} Y^{j-\ell}}{T^{m-3k-\ell}} \\
&= (-1)^{m-2k-\ell} \frac{X^m X^t}{\widehat{R}_Z^{m-2k}} \frac{\sum_{j=\ell}^{m-3k} \widehat{K}_{j,k} \binom{j}{\ell} X^{m-3k-j} Y^{j-\ell}}{T^{m-3k-\ell}},
\end{aligned}$$

must be regular on $X \cap \{X \neq 0\} \cap \{\widehat{R}_Z \neq 0\}$. Then the holomorphicity conditions translate into the following divisibility requirements in the homogeneous quotient ring $\mathbb{C}[T, X, Y, Z]/(\widehat{R})$:

$$(4.30) \quad T^{m-3k-\ell} \mid \sum_{j=\ell}^{m-3k} \widehat{K}_{j,k} \binom{j}{\ell} X^{m-3k-j} Y^{j-\ell} \quad \text{for each } k = 0, \dots, \lfloor \frac{m}{3} \rfloor \text{ and } \ell = 0, \dots, m-3k.$$

4.5. End of the Proof of the Key Vanishing Lemma 1.4. For a smooth surface $X \subset \mathbb{P}^3$, we consider the bundle $E_{2,m} T_X^*$ of invariant 2-jet differentials and its negative twists:

$$E_{2,m} T_X^* \otimes \mathcal{O}_X(-t),$$

with integer $t \geq 1$.

Take the affine equation of the surface X to be a Fermat-type polynomial with a single additional monomial x^a :

$$0 = z^d + y^d + x^d + 1 + x^a =: R(x, y, z),$$

with

$$d := 17, \quad a = \lfloor \frac{d}{2} \rfloor,$$

so that its projectivization $\widehat{R}(T, X, Y, Z) := X^d + Y^d + Z^d + T^d + X^a T^b$ has degree $d = 17$. One can verify the smoothness of X (including along $X \cap \{T = 0\}$ and $X \cap \{X = 0\}$) as well as the general position of the five divisors $\widehat{R}, \widehat{R}_X, \widehat{R}_Y, \widehat{R}_Z, \widehat{R}_T$; we omit these routine computations.

For instance, when $m = 3$, by Proposition 4.1, an invariant 2-jet differential of weighted order 3 on X takes the following most general local form on $X \cap \{R_z \neq 0\}$:

$$J_{yx} = \sum_{0 \leq j \leq 3} \frac{A_j}{R_z^3} (y')^j (x')^{3-j} + \frac{B_0}{R_z} \left| \begin{smallmatrix} y' & x' \\ y'' & x'' \end{smallmatrix} \right|.$$

After applying the jet bundle transition — which consists in replacing $y'x'' - y''x'$ by formula (4.10) and y' by formula (4.5) — the transition formula becomes:

$$\begin{aligned}
J_{yx} = & -\frac{B_0}{R_y} \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix} \\
& + z'z'z' \left(-\frac{A_3}{R_y^3} \right) \\
& + z'z'x' \left(-3\frac{A_3R_x}{R_z^1R_y^3} + \frac{A_2}{R_z^1R_y^2} + B_0 \left[\frac{R_{zz}}{R_zR_y} - 2\frac{R_z}{R_y^2} \frac{R_{yz}}{R_z} + \frac{R_z}{R_y} \frac{R_z}{R_y} \frac{R_{yy}}{R_zR_y} \right] \right) \\
& + z'x'x' \left(-3\frac{A_3R_x^2}{R_z^2R_y^3} + 2\frac{A_2R_x}{R_z^2R_y^2} - \frac{A_1}{R_z^2R_y} \right. \\
& \quad \left. + B_0 \left[2\frac{R_{xz}}{R_zR_y} - 2\frac{R_x}{R_y} \frac{R_{yz}}{R_zR_y} - 2\frac{R_z}{R_y} \frac{R_{xy}}{R_zR_y} + 2\frac{R_x}{R_y} \frac{R_z}{R_y} \frac{R_{yy}}{R_zR_y} \right] \right) \\
& + x'x'x' \left(-\frac{A_3R_x^3}{R_z^3R_y^3} + \frac{A_2R_x^2}{R_z^3R_y^2} - \frac{A_1R_x}{R_z^3R_y} + \frac{A_0}{R_z^3} \right. \\
& \quad \left. + B_0 \left[\frac{R_{xx}}{R_zR_y} - 2\frac{R_x}{R_y} \frac{R_{xy}}{R_zR_y} + \frac{R_x}{R_y} \frac{R_x}{R_y} \frac{R_{yy}}{R_zR_y} \right] \right).
\end{aligned}$$

In the first two lines, only R_y appears in denominators — which is allowed on the open set $\{R_y \neq 0\}$ — so no constraints are imposed on B_0 or A_3 from these terms.

On the third line, however, R_z appears to the first power in denominators. More precisely, the coefficient of $z'z'x'$ in J_{yx} , which we denote by $[z'z'x'](J_{yx})$, factors as:

$$[z'z'x'](J_{yx}) = \frac{-3A_3R_x + A_2R_y + B_0[R_y^2R_{zz} - 2R_yR_zR_{yz} + R_z^2R_{yy}]}{R_zR_y^3}.$$

To ensure the holomorphicity after transitioning to the set $\{R_y \neq 0\}$ and after clearing out R_y , it is necessary that R_z divides the numerator:

$$R_z \mid -3A_3R_x + A_2R_y + B_0[R_y^2R_{zz} - 2R_yR_zR_{yz} + R_z^2R_{yy}] \quad \text{in } \mathfrak{R} = \mathbb{C}[x, y, z]/(R).$$

Since the last two terms inside the brackets are already divisible by R_z , this condition reduces to

$$R_z \mid -3A_3R_x + A_2R_y + B_0R_y^2R_{zz} \quad \text{in } \mathfrak{R}.$$

Next, the coefficients of $z'x'x'$ and $x'x'x'$ in J_{yx} are respectively:

$$[z'x'x'](J_{yx}) = \frac{-3A_3R_x^2 + 2A_2R_xR_y - A_1R_y^2 + 2B_0[R_y^2R_zR_{xz} - R_xR_yR_zR_{yz} - R_yR_z^2R_{xy} + R_xR_z^2R_{yy}]}{R_z^2R_y^3},$$

$$[x'x'x'](J_{yx}) = \frac{-A_3R_x^3 + A_2R_x^2R_y - A_1R_xR_y^2 + A_0R_y^3 + B_0[R_y^2R_z^2R_{xx} - 2R_xR_yR_z^2R_{xy} + R_x^2R_z^2R_{yy}]}{R_z^3R_y^3}.$$

After clearing the admissible powers of R_y from the denominators, the holomorphicity conditions on the chart $\{R_y \neq 0\}$ translate into the following divisibility requirements in the quotient ring $\mathfrak{R}$:

$$\begin{aligned}
R_z^2 \mid & -3A_3R_x^2 + 2A_2R_xR_y - A_1R_y^2 \\
& + 2B_0[R_y^2R_zR_{xz} - R_xR_yR_zR_{yz} - R_yR_z^2R_{xy} + R_xR_z^2R_{yy}] \quad \text{in } \mathfrak{R},
\end{aligned}$$

$$\begin{aligned}
R_z^3 \mid & -A_3R_x^3 + A_2R_x^2R_y - A_1R_xR_y^2 + A_0R_y^3 \\
& + B_0[R_y^2R_z^2R_{xx} - 2R_xR_yR_z^2R_{xy} + R_x^2R_z^2R_{yy}] \quad \text{in } \mathfrak{R}.
\end{aligned}$$

In general, for any $m \geq 3$, returning to the expression of J_{yx} given as (4.28) and applying the jet bundle transition — which replaces $y'x'' - y''x'$ via formula (4.10) and y' via formula (4.5) — we obtain an

expansion of the form

$$J_{yx} = x^t \sum_{i+j+3k=m} x'^i z'^j \left| \frac{z'}{z''} \frac{x'}{x''} \right|^k \frac{\text{Eq}_{i,j,k}(K_{j,k}; j_{x,y,z}^m R)}{(R_z)^i R_y^{m-2k}},$$

where $\text{Eq}_{i,j,k}$ are certain complicated polynomials in the coefficient functions $K_{j,k}$ appearing in the initial expression of J_{yx} (see (4.28)) and in the m -jet $j_{x,y,z}^m R(x, y, z)$.

Since division by R_y^{m-2k} is allowed on the open set $\{R_y \neq 0\}$, the only remaining constraints arise from the powers $(R_z)^i$ appearing in the denominator. Holomorphicity on $\{R_y \neq 0\}$ therefore requires, for every triple $(i, j, k) \in \mathbb{N}^3$ with $i + j + 3k = m$,

$$(R_z)^i \mid \text{Eq}_{i,j,k}(K_{j,k}; j_{x,y,z}^m R) \quad \text{in } \mathfrak{R}.$$

Having chosen the defining polynomial to be of the Fermat-type perturbation

$$R := z^d + y^d + x^d + x^a + 1,$$

for which $R_z = dz^{d-1}$, these divisibility conditions become

$$(z^{d-1})^i \mid \text{Eq}_{i,j,k}(K_{j,k}; j_{x,y,z}^m R) \quad \text{in } \mathfrak{R}.$$

For computational implementation, Lemma 4.2 guarantees that we need only work with the unique normal form $\text{red}(\text{Eq}_{i,j,k})$ of each $\text{Eq}_{i,j,k}$. This normal form is obtained by repeatedly replacing any factor y^i with $i \geq d$ in $\text{Eq}_{i,j,k}$ using the relation

$$y^d = -(z^d + x^d + x^a + 1),$$

until the resulting polynomial in $\mathbb{C}[x, y, z]$ has degree in y strictly less than d .

We then expand each $\text{red}(\text{Eq}_{i,j,k})$ as a polynomial in z and discard all terms of degree $\geq (d-1)i$, keeping only the remainder modulo $z^{(d-1)i}$:

$$\text{Eq}_{i,j,k}^{(\text{trunc})} := \text{red}(\text{Eq}_{i,j,k}) \pmod{z^{(d-1)i}}.$$

All remaining terms must vanish identically for the original divisibility condition to hold, thereby converting the geometric constraints into an explicit linear system.

Next, we proceed inductively with respect to $i = 1, 2, \dots, m$ (starting naturally from $i = 1$). For each fixed i , we collect all divisibility equations sharing the same power of R_z , namely:

$$(4.31) \quad (z^{(d-1)})^i \mid \text{Eq}_{i,j,k}^{(\text{trunc})} \quad \text{in } \mathbb{C}[x, y, z], \quad \text{for all } (j, k) \text{ with } j + 3k = m - i.$$

The computer then extracts, for each triple $(i, j, k) \in \mathbb{N}^3$ with $j + 3k = m - i$, all coefficients

$$[x^p y^q z^r] (\text{Eq}_{i,j,k}^{(\text{trunc})}), \quad r \leq (d-1)i - 1,$$

and sets them to zero in order to satisfy the divisibility conditions (4.31). Here, the notation $[x^p y^q z^r](\cdot)$ denotes the coefficient of the monomial $x^p y^q z^r$ in the polynomial expansion. These vanishing conditions yield a linear system that must be satisfied by the coefficients of the polynomials $\{K_{j,k}\}$ appearing in the local expression of the jet differential.

For each fixed i , this linear system is solved symbolically. The solutions are stored in memory before proceeding to the next value of i , up to $i = m$. At each stage, intermediate feedback — such as the number of nonzero terms remaining in the full expansions of $\{K_{j,k}\}$ — is monitored to track progress.

It remains to check the holomorphicity of the jet differential (4.28) along the divisor $\{T = 0\}$ at infinity.

To this end, we rewrite the polynomials $K_{j,k} \in \mathbb{C}[x, y, z]$ appearing in (4.28) as homogeneous polynomials $\widehat{K}_{j,k} \in \mathbb{C}[T, X, Y, Z]$, where each $\widehat{K}_{j,k}(T, X, Y, Z)$ satisfies

$$\deg \widehat{K}_{j,k} = (m - 2k)(d - 1) - m + j + k - t \quad \text{and} \quad \deg_Y \widehat{K}_{j,k} < d,$$

for $m \leq 11$, $k = 0, \dots, 3$, and $j = 0, \dots, m - 3k$.

Then, as in (4.30), we must verify the following divisibility conditions in the homogeneous quotient ring $\mathbb{C}[T, X, Y, Z]/(\widehat{R})$:

$$T^{m-3k-\ell} \mid \text{HEq}_{\ell,k}(\widehat{K}_{j,k}, X, Y) \quad \text{in } \mathbb{C}[T, X, Y, Z]/(\widehat{R}), \quad \text{for every } k = 0, \dots, 3 \text{ and } \ell = 0, \dots, m - 3k,$$

where $\text{HEq}_{\ell,k}(\widehat{K}_{j,k}, X, Y)$ are certain polynomials in the coefficient functions $\widehat{K}_{j,k}$ (from the initial expression of J_{YX} in (4.27)) and in X, Y . Explicitly, as in (4.30),

$$\text{HEq}_{\ell,k}(\widehat{K}_{j,k}, X, Y) = \sum_{j=\ell}^{m-3k} \widehat{K}_{j,k} \binom{j}{\ell} X^{m-3k-j} Y^{j-\ell}.$$

Proceeding in a completely analogous manner — expanding $\text{HEq}_{\ell,k}$ in T and discarding higher-degree terms — the divisibility conditions in T translate into a second linear system, which is constructed and solved symbolically by the computer following the same inductive procedure.

For computational implementation, Lemma 4.2 guarantees that we need only work with the unique homogeneous normal form $\text{red}(\text{HEq}_{\ell,k})$ of each $\text{HEq}_{\ell,k}$. This homogeneous normal form is obtained by repeatedly replacing any factor Y^i with $i \geq d$ in $\text{HEq}_{\ell,k}$ using the relation

$$Y^d = -(Z^d + X^d + T^{d-a} X^a + T^d),$$

until the resulting polynomial in $\mathbb{C}[T, X, Y, Z]$ has degree in Y strictly less than d .

We then expand each $\text{red}(\text{HEq}_{\ell,k})$ as a polynomial in T and discard all terms of degree at least $m-3\ell-k$, keeping only the remainder modulo $T^{m-3\ell-k}$. This yields the truncated expression

$$\text{HEq}_{\ell,k}^{(\text{trunc})} := \text{red}(\text{HEq}_{\ell,k}) \pmod{T^{m-3\ell-k}},$$

i.e., the polynomial obtained from $\text{red}(\text{HEq}_{\ell,k})$ by removing any factor of $T^{m-3\ell-k}$. For the original divisibility condition to hold, all remaining terms must vanish identically, thereby converting the geometric constraints into another explicit linear system.

Next, we proceed inductively with respect to $i = 1, 2, \dots, m$ (starting naturally from $i = 1$). For each fixed i , we collect all divisibility equations sharing the same power of T , namely:

$$(4.32) \quad T^i \mid \text{HEq}_{\ell,k}^{(\text{trunc})} \quad \text{in } \mathbb{C}[T, X, Y, Z], \quad \text{for all } (\ell, k) \text{ with } \ell + 3k = m - i.$$

The computer then extracts, for each triple $(i, \ell, k) \in \mathbb{N}^3$ with $\ell + 3k = m - i$, all coefficients

$$[T^s X^p Y^q Z^r] (\text{HEq}_{\ell,k}^{(\text{trunc})}), \quad s \leq m - 3\ell - k - 1 = i - 1,$$

and sets them to zero in order to satisfy the divisibility conditions (4.32). Here, the notation $[T^s X^p Y^q Z^r](\cdot)$ denotes the coefficient of the monomial $T^s X^p Y^q Z^r$ in the homogeneous polynomial expansion. These vanishing conditions yield a linear system that must be satisfied by the coefficients of the polynomials $\widehat{K}_{j,k}$ in (4.27) (which correspond to the coefficients of $\{K_{j,k}\}$ in (4.28)) appearing in the local expression of the jet differential.

Finally, combining the two linear systems obtained from the divisibility conditions in z and in T , a computer algebra verification shows that all coefficients of $\{K_{j,k}\}$ are forced to vanish. This confirms that for the specified pairs (m, t) with $d = 17$, only the trivial solution exists. The Key Vanishing Lemma 1.4 for a generic surface $X \subset \mathbb{P}^3$ of degree $d = 17$ then follows by a standard semicontinuity argument.

For execution, a single Maple file handles all cases in the compact setting.⁶ $\square$

Remark 4.4. Our computer-assisted computations for surfaces in $\mathbb{P}^3$ produce several nonvanishing examples that further confirm the reliability of the code. For the parameter $(m, t) = (3, 1)$, the dimension of $H^0(X, E_{2,3} T_X^* \otimes \mathcal{O}_X(-1))$ is:

- 1 for the degree 9 Fermat surface $X^9 + Y^9 + Z^9 + T^9 = 0$;
- 1 for the degree 15 surface $X^{15} + Y^{15} + Z^{15} + T^{15} + X^7 T^8 = 0$;
- 5 for the degree 17 surface $X^{17} + Y^{17} + Z^{17} + T^{17} + X^8 T^9 = 0$.

The presence of these nontrivial sections supports the correctness of our algorithm: if every test had returned only zero, one might have suspected a systematic mistake. Instead, the code accurately detects negatively twisted invariant jet differentials whenever they are present.

⁶The Maple code for the compact case is available at <https://xiesongyan.github.io/>. The parameters (m, t) and d at the beginning of the code can be adjusted as needed to verify the required cases.

5. Applications of Siu's Slanted Vector Fields

To obtain an additional family of independent negatively twisted jet differentials, we employ Siu's method of slanted vector fields [64]. This technique generalizes to k -jet spaces the original construction for 1-jet spaces introduced by Clemens [11], Ein [22], and Voisin [70] in their study of rational curves.

The method was first implemented in the compact surface case by Păun [52] for surfaces in $\mathbb{P}^3$. Later, Rousseau [57] established the logarithmic counterpart for pairs $(\mathbb{P}^2, D)$ with curve configurations D , and further extended both Păun's setting and the logarithmic framework to threefolds in [56, 55]. For hypersurfaces $H \subset \mathbb{P}^n$ of arbitrary dimension, Merker [47] achieved the full generalization, thereby realizing Siu's original expectation on generic global generation. The logarithmic counterpart for pairs $(\mathbb{P}^n, H)$ was subsequently established by Darondeau [14].

This section reviews the two-dimensional case of Siu's slanted vector field construction. The presentation follows the classical treatment [52, 57] and claims no originality.

5.1. Compact Case. Let $\mathcal{X} \subset \mathbb{P}^3 \times \mathbb{P}^{N_d^3}$ denote the universal family of algebraic surfaces of degree d in $\mathbb{P}^3$, where $N_d^3 := \binom{d+3}{3} - 1$ is the dimension of the parameter space. It is defined by the homogeneous equation

$$\sum_{|\alpha|=d} A_\alpha Z^\alpha = 0.$$

Here $[A] = [A_\alpha]_{|\alpha|=d} \in \mathbb{P}^{N_d^3}$ are the coefficients and $[Z] = [Z_0 : Z_1 : Z_2 : Z_3] \in \mathbb{P}^3$ are the homogeneous coordinates. For a multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^4$ with length $|\alpha| := \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$, we write $Z^\alpha := Z_0^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$.

The vertical 2-jet bundle of the universal family is defined as

$$J_2^v(\mathcal{X}) := \ker((\text{pr}_2)_*) \subset J_2(\mathcal{X}),$$

where $\text{pr}_2: \mathcal{X} \rightarrow \mathbb{P}^{N_d^3}$ is the projection onto the parameter space and $J_2(\mathcal{X})$ is the full 2-jet bundle of $\mathcal{X}$.

On the affine chart $\{Z_0 \neq 0\} \times \{A_{0d00} \neq 0\}$ of $\mathbb{P}^3 \times \mathbb{P}^{N_d^3}$, we use the affine coordinates

$$z_i := \frac{Z_i}{Z_0} \quad (i = 1, 2, 3), \quad a_\alpha := \frac{A_{\alpha_0 \alpha_1 \alpha_2 \alpha_3}}{A_{0d00}},$$

where $\alpha_0 = d - \alpha_1 - \alpha_2 - \alpha_3$. Set

$$\mathcal{X}_0 := \mathcal{X} \cap (\{Z_0 \neq 0\} \times \{A_{0d00} \neq 0\}).$$

In these coordinates $\mathcal{X}_0$ is defined by the equation

$$z_1^d + \sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 \leq d \\ \alpha_1 < d}} a_\alpha z^\alpha = 0.$$

For jets of order $j = 1, 2$ we write $\xi_i^{(j)} := z_i^{(j)}$ for $i = 1, 2$. A local coordinate system on

$$J_2(\{Z_0 \neq 0\} \times \{A_{0d00} \neq 0\}) \subset J_2(\mathbb{P}^3 \times \mathbb{P}^{N_d^3})$$

is then given by the tuple

$$(z, a, \xi', \xi''),$$

where

- $z = (z_1, z_2, z_3)$ are the affine coordinates on $\{Z_0 \neq 0\}$,
- $a = (a_\alpha)$ with $\alpha_1 + \alpha_2 + \alpha_3 \leq d$, $\alpha_1 < d$ are the normalized coefficients,
- $\xi' = (\xi'_1, \xi'_2, \xi'_3)$ and $\xi'' = (\xi''_1, \xi''_2, \xi''_3)$ are the first and second jet coordinates.

Following Siu's strategy [64] as implemented by Păun [52], we work with the family of slanted vector fields of pole order 7.

Theorem 5.1 ([52, Proposition 1.3]). *Let Σ_0 be the common zero locus of the three 2×2 Wronskians*

$$\det \begin{pmatrix} \xi'_i & \xi'_j \\ \xi''_i & \xi''_j \end{pmatrix} \quad (1 \leq i < j \leq 3),$$

that is,

$$\Sigma_0 := \left\{ (z, a, \xi', \xi'') \in J_2^v(\mathcal{X}_0) \mid \xi' \wedge \xi'' = 0 \right\}.$$

Let Σ be the closure of Σ_0 in $J_2^v(\mathcal{X})$. Denote by

$$p: J_2^v(\mathcal{X}) \longrightarrow \mathcal{X}$$

the natural projection, and by $\text{pr}_1: \mathcal{X} \rightarrow \mathbb{P}^3$ and $\text{pr}_2: \mathcal{X} \rightarrow \mathbb{P}^{N_d^2}$ the two projections of the universal family. Then the twisted tangent bundle

$$T_{J_2^v(\mathcal{X})} \otimes p^* \text{pr}_1^* \mathcal{O}_{\mathbb{P}^3}(7) \otimes p^* \text{pr}_2^* \mathcal{O}_{\mathbb{P}^{N_d^2}}(1)$$

is generated by its global sections at every point of $J_2^v(\mathcal{X}) \setminus \Sigma$.

5.2. Logarithmic Case. Let $\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}^{N_d^2}$ denote the universal family of algebraic curves of degree d in $\mathbb{P}^2$, where $N_d^2 := \binom{d+2}{2} - 1$ is the dimension of the parameter space. It is defined by the homogeneous equation

$$\sum_{|\alpha|=d} A_\alpha Z^\alpha = 0.$$

Here $[A] = [A_\alpha]_{|\alpha|=d} \in \mathbb{P}^{N_d^2}$ are the coefficients and $[Z] = [Z_0 : Z_1 : Z_2] \in \mathbb{P}^2$ the homogeneous coordinates. For a multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}^3$ with length $|\alpha| := \alpha_0 + \alpha_1 + \alpha_2$, we write $Z^\alpha := Z_0^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2}$.

The vertical logarithmic 2-jet space associated with this family is defined as

$$\overline{J}_2^v(\mathbb{P}^2 \times \mathbb{P}^{N_d^2}) := \ker((\text{pr}_2)_*) \subset J_2(\mathbb{P}^2 \times \mathbb{P}^{N_d^2}, -\log \mathcal{C}),$$

where $\text{pr}_2: \mathbb{P}^2 \times \mathbb{P}^{N_d^2} \rightarrow \mathbb{P}^{N_d^2}$ is the projection onto the parameter space, and $J_2(\mathbb{P}^2 \times \mathbb{P}^{N_d^2}, -\log \mathcal{C})$ is the logarithmic 2-jet bundle along the universal curve $\mathcal{C}$.

We define the universal family $\mathcal{Y} \subset \mathbb{P}^3 \times \mathbb{P}^{N_d^2+1}$ of degree- d hypersurfaces in $\mathbb{P}^3$ by

$$\mathcal{Y} := \left\{ A_0 Z_3^d + \sum_{|\alpha|=d} A_\alpha Z^\alpha = 0 \right\} \subset \mathbb{P}^3 \times \mathbb{P}^{N_d^2+1}.$$

Here $\mathbb{P}^3$ has homogeneous coordinates $[Z : Z_3] = [Z_0 : Z_1 : Z_2 : Z_3]$, and the coefficient space $\mathbb{P}^{N_d^2+1}$ is equipped with homogeneous coordinates $[A : A_0] = [\cdots : A_\alpha : \cdots : A_0]_{|\alpha|=d}$.

For notational convenience, let

$$\begin{aligned} \{Z = \mathbf{0}\} &:= \{Z_0 = Z_1 = Z_2 = 0\} = [0 : 0 : 0 : 1] \times \mathbb{P}^{N_d^2+1} \subset \mathbb{P}^3 \times \mathbb{P}^{N_d^2+1}, \\ \{A = \mathbf{0}\} &:= \{A_\alpha = 0 \text{ for all } |\alpha| = d\} = \mathbb{P}^3 \times [0 : \cdots : 0 : 1] \subset \mathbb{P}^3 \times \mathbb{P}^{N_d^2+1}. \end{aligned}$$

There is a natural forgetful map

$$\pi: \mathbb{P}^3 \times \mathbb{P}^{N_d^2+1} \setminus (\{Z = \mathbf{0}\} \cup \{A = \mathbf{0}\}) \longrightarrow \mathbb{P}^2 \times \mathbb{P}^{N_d^2}$$

defined by discarding the auxiliary coordinates:

$$\pi([Z : Z_3], [A : A_0]) \mapsto ([Z_0 : Z_1 : Z_2], [A]).$$

Observe first the containment

$$(5.1) \quad \mathcal{Y} \cap (\{Z = \mathbf{0}\} \cup \{A = \mathbf{0}\}) \subset \mathcal{Y} \cap (\{A_0 = 0\} \cup \{A = \mathbf{0}\}),$$

which follows from the inclusion $\mathcal{Y} \cap \{Z = \mathbf{0}\} \subset \mathcal{Y} \cap \{A_0 = 0\}$, a direct consequence of the defining equation of $\mathcal{Y}$.

Now set

$$\mathcal{Y}^* := \mathcal{Y} \setminus (\{A_0 = 0\} \cup \{A = \mathbf{0}\}).$$

By (5.1), the restricted forgetful map

$$\pi|_{\mathcal{Y}^*}: \mathcal{Y}^* \longrightarrow \mathbb{P}^2 \times \mathbb{P}^{N_d^2}$$

is everywhere defined. Moreover, one checks that

$$(\pi|_{\mathcal{Y}^*})^{-1}(\mathcal{C}) = \{Z_3 = 0\} =: \mathcal{D}.$$

Thus π induces a logarithmic morphism from the pair $(\mathcal{Y}^*, \mathcal{D})$ to $(\mathbb{P}^2 \times \mathbb{P}^{N_d^2}, \mathcal{C})$, and consequently a dominant map between the corresponding vertical logarithmic jet spaces:

$$(5.2) \quad \pi_{[2]}: \overline{J}_2^v(\mathcal{Y}^*) \longrightarrow \overline{J}_2^v(\mathbb{P}^2 \times \mathbb{P}^{N_d^2}).$$

Here $\overline{J}_2^v(\mathcal{Y}^*)$ denotes the space of vertical logarithmic 2-jets with respect to the hyperplane divisor $\mathcal{D} = \{Z_3 = 0\}$. This construction allows us to carry out all local computations on the more explicit space $\mathcal{Y}^*$ by pulling back via π , where coordinate expressions are readily available.

On the affine chart $\{Z_0 \neq 0\} \times \{A_0 \neq 0\}$ we introduce the affine coordinates

$$z_i := \frac{Z_i}{Z_0} \quad (i = 1, 2, 3), \quad a_\alpha := \frac{A_{\alpha_0 \alpha_1 \alpha_2}}{A_0},$$

where $\alpha_0 = d - \alpha_1 - \alpha_2$. Set

$$\mathcal{Y}_0 := \mathcal{Y} \cap (\{Z_0 \neq 0\} \times \{A_0 \neq 0\}).$$

In these coordinates $\mathcal{Y}_0$ is defined by

$$z_3^d + \sum_{|\alpha|=\alpha_1+\alpha_2 \leq d} a_\alpha z^\alpha = 0.$$

To streamline notation, for jets of order $j = 1, 2$ we define

$$\xi_i^{(j)} := \begin{cases} z_i^{(j)} & (i = 1, 2), \\ (\log z_3)^{(j)} & (i = 3). \end{cases}$$

A local coordinate system for

$$\overline{J}_2(\{Z_0 \neq 0\} \times \{A_0 \neq 0\}) \subset J_2(\mathbb{P}^3 \times \mathbb{P}^{N_d^2+1}, -\log \mathcal{D})$$

is then given by the abbreviated tuple

$$(z, a, \xi', \xi''),$$

where

- $z = (z_1, z_2, z_3)$ are the affine coordinates on $\{Z_0 \neq 0\} \subset \mathbb{P}^3$;
- $a = (a_\alpha)$ with $\alpha_1 + \alpha_2 \leq d$ are the normalized coefficients in the parameter direction;
- $\xi' = (\xi'_1, \xi'_2, \xi'_3)$ and $\xi'' = (\xi''_1, \xi''_2, \xi''_3)$ are the logarithmic jet coordinates.

Following Siu's strategy [64] and Rousseau's implementation [57], we work with the family of slanted vector fields of pole order 7.

Theorem 5.2 ([57, Corollary 8]). *Let Σ_0 be the zero locus of the 2×2 Wronskian $\det(\xi_i^{(j)})_{1 \leq i, j \leq 2}$ inside the vertical logarithmic jet space:*

$$\Sigma_0 := \left\{ (z_i, a_\alpha, \xi'_i, \xi''_i) \in \overline{J}_2^v(\mathcal{Y}_0) \mid \det(\xi_i^{(j)})_{1 \leq i, j \leq 2} = 0 \right\},$$

and let Σ be the closure of Σ_0 in $\overline{J}_2^v(\mathcal{Y}^*)$. Denote by

$$p: \overline{J}_2^v(\mathbb{P}^2 \times \mathbb{P}^{N_d^2}) \longrightarrow \mathbb{P}^2 \times \mathbb{P}^{N_d^2}$$

the natural projection, and by $\text{pr}_1: \mathbb{P}^2 \times \mathbb{P}^{N_d^2} \rightarrow \mathbb{P}^2$, $\text{pr}_2: \mathbb{P}^2 \times \mathbb{P}^{N_d^2} \rightarrow \mathbb{P}^{N_d^2}$ the two projections. Then the twisted tangent bundle

$$T_{\overline{J}_2^v(\mathbb{P}^2 \times \mathbb{P}^{N_d^2})} \otimes p^* \text{pr}_1^* \mathcal{O}_{\mathbb{P}^2}(7) \otimes p^* \text{pr}_2^* \mathcal{O}_{\mathbb{P}^{N_d^2}}(1)$$

is generated by its global sections at every point of

$$\overline{J}_2^v(\mathbb{P}^2 \times \mathbb{P}^{N_d^2}) \setminus \left(\overline{\pi_{[2]}(\Sigma)} \cup p^{-1}(\mathcal{C}) \right) \quad [\text{see (5.2)}].$$

□

5.3. Extension of 2-Jet Differentials. To implement Siu's strategy of slanted vector fields, we first extend a given negatively twisted invariant (logarithmic) 2-jet differential to a holomorphic family. The argument follows Siu [64, pp. 558–559] and relies on a standard extension theorem in algebraic geometry:

Theorem 5.3 ([30, p. 288]). *Let $\tau: \mathcal{Z} \rightarrow S$ be a flat holomorphic family of compact complex spaces and $\mathcal{L} \rightarrow \mathcal{Z}$ a holomorphic vector bundle. Then there exists a proper subvariety $Z \subset S$ such that for every $s_0 \in S \setminus Z$, the restriction map*

$$H^0(\tau^{-1}(U_{s_0}), \mathcal{L}) \longrightarrow H^0(\tau^{-1}(s_0), \mathcal{L}|_{\tau^{-1}(s_0)})$$

is surjective for some Zariski-dense open neighbourhood $U_{s_0} \subset S$ of s_0 .

Moreover, the argument of Demailly–El Goul (cf. [18, p. 532], [24, Lemma 1.3.2]) guarantees the existence of *irreducible* negatively twisted invariant (logarithmic) 2-jet differentials. These facts are summarised in the following statement.

Proposition 5.4. *For some positive integer $m, t \geq 1$, there exists a Zariski-open subset $U^{m,t}$ in the parameter space $\mathbb{P}^{N_d^3}$ of the algebraic surface X of degree $d \geq 15$ in $\mathbb{P}^3$, such that:*

- (1) *For each parameter $a \in U^{m,t}$, the corresponding X_a is a smooth surface.*
- (2) *There is a holomorphic family*

$$\{\omega_{1,a} \mid 0 \neq \omega_{1,a} \in H^0(X_a, E_{2,m}T_{X_a}^* \otimes \mathcal{O}_{X_a}(-t)), a \in U^{m,t}\},$$

which varies holomorphically with respect to the variable $a \in U^{m,t}$. Moreover, each zero locus $\{\omega_{1,a} = 0\}$ on the second level $X_{a,2}$ of the Demailly–Semple tower for X_a is irreducible and reduced modulo $\Gamma_{a,2}$ (see (2.10)).

The proof follows the same line as that of [31, Proposition 4.4] and is included here for the reader's convenience.

Proof. For any integers $M, T \geq 1$, Theorem 5.3 guarantees that there exists a proper Zariski-closed subset $Z_{M,T}$ in the parameter space $\mathbb{P}^{N_d^3}$, such that for any parameter a outside $Z_{M,T}$, every nonzero global section of $E_{2,M}T_{X_a}^* \otimes \mathcal{O}_{X_a}(-T)$ extends to a holomorphic family parameterized by some Zariski open neighborhood of a . Thus, to ensure the local extendability of any nonzero negatively twisted 2-jet differential on X_{a_0} , we choose a point a_0 in $\mathbb{P}^{N_d^3} \setminus \cup_{M,T \geq 1} Z_{M,T}$ such that X_{a_0} is a smooth surface and the Picard group $\text{Pic}(X_{a_0})$ of X_{a_0} is isomorphic to $\mathbb{Z}$ (by Noether–Lefschetz Theorem).

By Corollary 2.3, for some integers $M, T \geq 1$, there exists a nonzero negatively twisted invariant 2-jet differential

$$\omega_0 \in H^0(X_{a_0}, E_{2,M}T_{X_{a_0}}^* \otimes \mathcal{O}_{X_{a_0}}(-T)).$$

Next, we decompose the zero locus $\{\omega_0 = 0\}$ in the second level $X_{a_0,2}$ of the Demailly–Semple tower of X_{a_0} into irreducible components. By the same argument as Demailly and El Goul [18, p. 532], we can extract an irreducible and reduced component Z , which in fact corresponds to some irreducible negatively twisted invariant 2-jet differential

$$(5.3) \quad \omega_1 \in H^0(X_{a_0}, E_{2,m}T_{X_{a_0}}^* \otimes \mathcal{O}_{X_{a_0}}(-t)), \quad \text{with } \frac{t}{m} \geq \frac{T}{M},$$

in the sense that $\{\omega_1 = 0\} = Z + b\Gamma_{a_0,2}$ (see (2.10) and (2.11)) for some integer $b > 0$. The construction proceeds as follows.

Let $u_1 = \pi_{2,1}^* \mathcal{O}_{X_{a_0,1}}(1)$, $u_2 = \mathcal{O}_{X_{2,a_0}}(1)$ and $h = \mathcal{O}_{X_{a_0}}(1)$. Then the zero divisor $Z_{\omega_0} = \{\omega_0 = 0\} \subset X_{a_0,2}$ is linearly equivalent to $Mu_2 - T\pi_{2,0}^*h$ in the Picard group

$$\text{Pic}(X_{a_0,2}) \cong \text{Pic}(X_{a_0}) \oplus \mathbb{Z}u_1 \oplus \mathbb{Z}u_2 \cong \mathbb{Z}^3.$$

Let $Z_{\omega_0} = \sum p_j Z_j$ be the irreducible decomposition of the divisor Z_{ω_0} , with each Z_j irreducible and reduced. Write the class of each component Z_j linearly in the Picard group $\text{Pic}(X_{a_0,2})$ as

$$Z_j \sim b_{1,j}u_1 + b_{2,j}u_2 - t_j\pi_{2,0}^*h, \quad b_{1,j}, b_{2,j}, t_j \in \mathbb{Z}.$$

By [18, Lemma 3.3], the effectiveness of Z_j implies that exactly one of the following mutually exclusive cases holds:

- $(b_{1,j}, b_{2,j}) = (0, 0)$ and $Z_j \in \pi_{2,0}^* \text{Pic}(X_{a_0})$, $-t_j > 0$;
- $(b_{1,j}, b_{2,j}) = (-1, 1)$, then Z_j contains $\Gamma_{a_0,2}$, where $\Gamma_{a_0,2}$ is an irreducible divisor on $X_{a_0,2}$. Thus $Z_j = \Gamma_{a_0,2}$ and $t_j = 0$;
- $b_{1,j} \geq 2b_{2,j} \geq 0$ and $m_j := b_{1,j} + b_{2,j} > 0$.

In the third case, the irreducible and reduced divisor Z_j corresponds to a section

$$\omega_j \in H^0(X_{a_0,2}, \pi_{2,1}^* \mathcal{O}_{X_{a_0,1}}(b_{1,j}) \otimes \mathcal{O}_{X_{a_0,2}}(b_{2,j}) \otimes \pi_{2,0}^* \mathcal{O}_{X_{a_0}}(-t_j)).$$

Noting that $\pi_{2,1}^* \mathcal{O}_{X_{a_0,1}}(b_1) \otimes \mathcal{O}_{X_{a_0,2}}(b_2) \cong \mathcal{O}_{X_{a_0,2}}(b_1 + b_2) \otimes \mathcal{O}_{X_{a_0,2}}(-b_1\Gamma_{a_0,2})$ (see (2.11)), we may regard

$$\omega_j \in H^0(X_{a_0,2}, \mathcal{O}_{X_{a_0,2}}(m_j) \otimes \pi_{2,0}^* \mathcal{O}_{X_{a_0}}(-t_j)),$$

whose zero divisor is $Z_j + b_{1,j}\Gamma_{a_0,2}$. Since both Z_j and $\Gamma_{a_0,2}$ are irreducible and reduced, the residual zero divisor of ω_j after removing its fixed Γ_2 -component is the irreducible reduced divisor Z_j . Thus ω_j is irreducible modulo the fixed vertical divisor Γ_2 .

Let $\max\{t_j/m_j\}$ denote the maximum ratio among those sections obtained from the third case. Since $M = \sum p_j m_j$ and $T = \sum p_j t_j$, we have $\max\{t_j/m_j\} \geq T/M$. Without loss of generality, we may assume that $t_1/m_1 = \max\{t_j/m_j\}$, and denote $m := m_1$ and $t := t_1$. By applying the Direct Image Formula (2.12) and abusing the notation for ω_1 , we thus obtain (5.3).

By our choice of $a_0 \notin \cup_{M,T \geq 1} Z_{M,T}$, automatically ω_1 extends to a holomorphic family

$$\{\omega_{1,a} \mid 0 \neq \omega_{1,a} \in H^0(X_a, E_{2,m}T_{X_a}^* \otimes \mathcal{O}_{X_a}(-t)), a \in U_{a_0}^{m,t}\},$$

which varies holomorphically with respect to the parameter a in some Zariski-open neighborhood $U_{a_0}^{m,t}$ of a_0 , with $\omega_{1,a_0} = \omega_1$. By shrinking to a smaller Zariski-open subset $U^{m,t} \subset U_{a_0}^{m,t}$ if necessary, we can ensure that each $a \in U^{m,t}$ defines a smooth surface of degree d in $\mathbb{P}^3$ while the zero locus $\{\omega_{1,a} = 0\}$ on the second level $X_{a,2}$ of the Demailly-Semple tower for X_a is irreducible and reduced modulo $\Gamma_{a,2}$. $\square$

By the same reasoning, the analogous statement holds in the logarithmic setting:

Proposition 5.5. *For some positive integer $m, t \geq 1$, there exists a Zariski-open subset $U^{m,t}$ in the parameter space $\mathbb{P}^{N_d^2}$ of algebraic curve $\mathcal{C}$ of degree $d \geq 11$ in $\mathbb{P}^2$, such that:*

- (1) *For each parameter $a \in U^{m,t}$, the corresponding $\mathcal{C}_a$ forms a smooth curve of degree d in $\mathbb{P}^2$;*
- (2) *There is a holomorphic family*

$$\{\omega_{1,a} \mid 0 \neq \omega_{1,a} \in H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}_a) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)), a \in U^{m,t}\},$$

which varies holomorphically with respect to the variable $a \in U^{m,t}$. Moreover, each zero locus $\{\omega_{1,a} = 0\}$ on the second level $X_{a,2}$ of the logarithmic Demailly-Semple tower for $(\mathbb{P}^2, \mathcal{C}_a)$ is irreducible and reduced modulo $\Gamma_{a,2}$ (see (2.10)). $\square$

5.4. Producing an Independent 2-Jet Differential via Siu's Slanted Vector Fields. Following Păun's argument [52, p. 889], we apply the slanted vector fields of pole order 7 in the compact case.

In both the compact and logarithmic universal families below, the initial jet differential is an equivariant polynomial on the vertical jet space. An invariant slanted vector field therefore acts on it naturally by Lie derivation, producing a genuine global invariant jet differential on each fibre; the pole order of the field is added to the base twist.

Proposition 5.6. *Given a holomorphic family $\{\omega_{1,a}\}_{a \in U}$ of negatively twisted invariant 2-jet differentials as in Proposition 5.4 with vanishing order $t \geq 8$, one can find some slanted vector field V of pole order 7 in Theorem 5.1, such that the derivative of $\{\omega_{1,a}\}$ along V gives a new family of holomorphic invariant 2-jet differentials $\omega_{2,a}$ with vanishing order $t - 7$. Furthermore, the common zero set $\{\omega_{1,a} = 0\} \cap \{\omega_{2,a} = 0\}$ in the second level $(X_a)_2$ of the Demailly-Semple tower of X_a has codimension at least 2 modulo $\Gamma_{a,2}$ for a generic parameter $a \in U$.* $\square$

Similarly, following Rousseau's argument [57, p. 38], we apply the slanted vector fields of pole order 7 in the logarithmic case.

Proposition 5.7. *Given a holomorphic family $\{\omega_{1,a}\}_{a \in U}$ of negatively twisted invariant logarithmic 2-jet differentials as in Proposition 5.5 with vanishing order $t \geq 8$, one can find some slanted vector field V of pole order 7 in Theorem 5.2, such that the derivative of $\{\omega_{1,a}\}$ along V gives a new family of holomorphic invariant logarithmic 2-jet differentials $\omega_{2,a}$ with vanishing order $t - 7$. Furthermore, the common zero set $\{\omega_{1,a} = 0\} \cap \{\omega_{2,a} = 0\}$ in the second level $(\mathbb{P}_a^2)_2$ of the logarithmic Demailly-Semple tower of $(\mathbb{P}^2, \mathcal{C}_a)$ has codimension at least 2 modulo $\Gamma_{a,2}$ for a generic parameter $a \in U$.* $\square$

6. Proof of the Main Results

6.1. Constructing a Second Independent Invariant 2-Jet Differential. We now give the detailed proof of Theorem 1.5. Let X be a generic surface of degree $d \geq 17$ in $\mathbb{P}^3$. By Proposition 2.1, for suitable positive integers M and T there exists a nonzero section

$$\omega_0 \in H^0(X, E_{2,M}T_X^* \otimes \mathcal{O}(K_X^{-T})) = H^0(X, E_{2,M}T_X^* \otimes \mathcal{O}_X(-T(d-4))).$$

Assume that there exists an entire curve $f: \mathbb{C} \rightarrow X$. Consider the Demailly–Semple tower

$$X_2 \longrightarrow X_1 \longrightarrow X_0 := X,$$

and let $f_{[2]}$ and $f_{[1]}$ denote the lifts of f to X_2 and X_1 , respectively.

Following the argument in the proof of Proposition 5.4, we select a very generic parameter a_0 in the moduli space $\mathbb{P}^{N_d^3}$ of degree- d surfaces in $\mathbb{P}^3$ such that:

- X_{a_0} is a smooth algebraic surface of general type;
- the Picard group satisfies $\text{Pic}(X_{a_0}) \cong \mathbb{Z}$ (by the Noether–Lefschetz theorem);
- for any positive integers M and T , every nonzero global section of $E_{2,M}T_{X_{a_0}}^* \otimes \mathcal{O}_{X_{a_0}}(-T)$ extends to a holomorphic family over a Zariski-open neighbourhood of a_0 .

Using Demailly–El Goul’s construction [18, p. 532] (see also the proof of Proposition 5.4), we obtain an irreducible invariant 2-jet differential

$$(6.1) \quad \omega_{1,a_0} \in H^0(X_{a_0}, E_{2,m}T_{X_{a_0}}^* \otimes \mathcal{O}_{X_{a_0}}(-t)),$$

whose zero divisor in X_{2,a_0} consists of an irreducible divisor Z_{a_0} modulo $\Gamma_{a_0,2}$.

The choice of a_0 guarantees that ω_{1,a_0} extends to a holomorphic family

$$(6.2) \quad \{\omega_{1,a} \mid 0 \neq \omega_{1,a} \in H^0(X_a, E_{2,m}T_{X_a}^* \otimes \mathcal{O}_{X_a}(-t)), a \in U_{a_0}^{m,t}\},$$

varying holomorphically with the parameter a in a Zariski-open neighbourhood $U_{a_0}^{m,t}$ of a_0 . By [29, Theorem 12.2.1], we may shrink to a smaller Zariski-open subset $U^{m,t} \subset U_{a_0}^{m,t}$ on which every $a \in U^{m,t}$ still defines a smooth surface X_a of general type and the zero locus satisfies

$$(6.3) \quad \{\omega_{1,a} = 0\} \text{ corresponds to an irreducible divisor } Z_a \text{ on } X_{a,2} \text{ modulo } \Gamma_{a,2}.$$

We now recall the following numerical criterion of Demailly–El Goul.

Proposition 6.1 ([18, Proposition 3.4]). *Let X be a smooth surface of degree- d of general type in $\mathbb{P}^3$ with $\text{Pic}(X) \cong \mathbb{Z}$. Suppose that there exists a nonzero irreducible negatively twisted 2-jet differential*

$$\omega_1 \in H^0(X, E_{2,m}T_X^* \otimes \mathcal{O}_X(-t))$$

satisfying

$$(6.4) \quad t < (d-4) \frac{13c_1^2 - 9c_2}{12c_1^2} m.$$

Then the line bundle $\mathcal{O}_{X_2}(1)$ restricted to the irreducible 3-dimensional component Z of the zero locus $\{\omega_1 = 0\}$ in X_2 is big. Here Z is the component that projects onto X_1 and is distinct from Γ_2 .

For the reader’s convenience, we provide a self-contained proof. Our argument follows a slightly different route from that of [18, Proposition 3.4], as we were unable to fully verify one of the technical steps in the original presentation.

The key ingredient is the following statement. A complete proof is given in our earlier work [31, Proposition 6.1].

Proposition 6.2. *Let $a_1, a_2 \in \mathbb{Z}$. Denote*

$$(6.5) \quad \mathcal{O}_{X_2}(a_1, a_2) := \pi_{2,1}^* \mathcal{O}_{X_1}(a_1) \otimes \mathcal{O}_{X_2}(a_2), \quad a_1, a_2 \in \mathbb{Z}.$$

(1) *If $a_2 = -1$, then*

$$R^q(\pi_{2,0})_* \mathcal{O}_{X_2}(a_1, -1) = 0 \quad \text{for every } q \geq 0.$$

(2) *If $a_2 \geq 0$, then*

$$R^q(\pi_{2,0})_* \mathcal{O}_{X_2}(a_1, a_2) = 0 \quad \text{for every } q \geq 2,$$

and

$$R^1(\pi_{2,0})_* \mathcal{O}_{X_2}(a_1, a_2) = 0 \iff a_1 - 2a_2 \geq -1.$$

In particular, for every $p \geq 0$,

$$R^q(\pi_{2,0})_* \mathcal{O}_{X_2}(2p, p) = 0 \quad \text{for every } q \geq 1.$$

□

Proof of Proposition 6.1. As in the proof of Proposition 5.4, one can show that the zero divisor of ω_1 on X_2 decomposes as

$$\{\omega_1 = 0\} = Z + b_1\Gamma_2,$$

where $Z \subset X_2$ is irreducible and reduced. Write $u_1 = \pi_{2,1}^*\mathcal{O}_{X_1}(1)$ and $u_2 = \mathcal{O}_{X_2}(1)$. Since $\text{Pic}(X_2)$ decomposes as $\text{Pic}(X) \oplus \mathbb{Z}u_1 \oplus \mathbb{Z}u_2$, the divisor Z is linearly equivalent to

$$(6.6) \quad Z \sim b_1u_1 + b_2u_2 - t\pi_{2,0}^*h, \quad b_1, b_2 \in \mathbb{Z}, \quad b_1 \geq 2b_2 > 0, \quad b_1 + b_2 = m, \quad t \in \mathbb{Z}_+,$$

where $h = \mathcal{O}_X(1)$.

We shall now prove that $\mathcal{O}_{X_2}(2, 1)|_Z$ is big whenever (6.4) holds, using a Riemann–Roch computation. With the intersection formulas on the Demailly–Sample tower given in [18, Lemma 3.3 (e)], a direct calculation yields

$$(2u_1 + u_2)^3 \cdot Z = m(13c_1^2 - 9c_2) + 12t(c_1 \cdot h) \quad [\text{cf. [18, p. 530 (†††)}],$$

where $c_1 = (4 - d)h$, $c_2 = (d^2 - 4d + 6)h^2$ and $h^2 = d$. Under condition (6.4) we therefore obtain

$$(2u_1 + u_2)^3 \cdot Z = m(13c_1^2 - 9c_2) - \frac{12}{d-4}tc_1^2 > 0.$$

Applying the Riemann–Roch theorem to the divisor Z , we have for sufficiently divisible $p \gg 1$

$$\begin{aligned} h^0(Z, \mathcal{O}_{X_2}(2p, p)|_Z) + h^2(Z, \mathcal{O}_{X_2}(2p, p)|_Z) \\ \geq \chi(Z, \mathcal{O}_{X_2}(2p, p)|_Z) \\ = \frac{(2u_1 + u_2)^3 \cdot Z}{3!} p^3 + O(p^2) \gg 1. \end{aligned}$$

We next show that

$$h^2(Z, \mathcal{O}_{X_2}(2p, p)|_Z) = 0 \quad \text{for all sufficiently divisible } p \gg 1.$$

From the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X_2}(-Z) \otimes \mathcal{O}_{X_2}(2p, p) \longrightarrow \mathcal{O}_{X_2}(2p, p) \longrightarrow \mathcal{O}_Z \otimes \mathcal{O}_{X_2}(2p, p) \longrightarrow 0,$$

the associated long exact cohomology sequence gives the estimate

$$h^2(Z, \mathcal{O}_{X_2}(2p, p)|_Z) \leq h^2(X_2, \mathcal{O}_{X_2}(2p, p)) + h^3(X_2, \mathcal{O}_{X_2}(-Z) \otimes \mathcal{O}_{X_2}(2p, p)).$$

Step 1. Vanishing of $h^2(X_2, \mathcal{O}_{X_2}(2p, p))$. Consider the direct image sheaf $(\pi_{2,0})_*\mathcal{O}_{X_2}(2p, p)$ on X . By Proposition 6.2 we have $R^q(\pi_{2,0})_*\mathcal{O}_{X_2}(2p, p) = 0$ for all $q \geq 1$. Hence Leray’s spectral sequence yields

$$H^2(X_2, \mathcal{O}_{X_2}(2p, p)) \cong H^2(X, (\pi_{2,0})_*\mathcal{O}_{X_2}(2p, p)).$$

Using the isomorphism (cf. [18, Lemma 3.3 (c)])

$$(\pi_{2,0})_*\mathcal{O}_{X_2}(2p, p) \cong E_{2,3p}T_X^*,$$

and Serre duality, we obtain

$$(6.7) \quad H^2(X, (\pi_{2,0})_*\mathcal{O}_{X_2}(2p, p)) \cong H^2(X, E_{2,3p}T_X^*) \cong H^0(X, E_{2,3p}T_X \otimes K_X)^*.$$

Here we use Serre duality $H^2(X, \mathcal{F}) \cong H^0(X, \mathcal{F}^* \otimes K_X)^*$, together with the fact that $(E_{2,3p}T_X^*)^* \cong E_{2,3p}T_X$.

The bundle $E_{2,3p}T_X \otimes K_X$ inherits a filtration from the dual of (2.3) twisted by K_X ; its graded pieces are

$$S^{3p-3j}T_X \otimes K_X^{1-j} \quad (j = 0, 1, \dots, p).$$

For all sufficiently divisible $p \gg 1$, one has $3p - 3j > 2(1 - j)$ for every $j = 0, 1, \dots, p$. By the Bogomolov vanishing theorem (Theorem 2.2) we therefore get

$$H^0(X, S^{3p-3j}T_X \otimes K_X^{1-j}) = 0 \quad (j = 0, 1, \dots, p),$$

which forces $H^0(X, E_{2,3p}T_X \otimes K_X) = 0$. Combining this with (6.7), we obtain

$$(6.8) \quad h^2(X_2, \mathcal{O}_{X_2}(2p, p)) = 0.$$

Step 2. Vanishing of $h^3(X_2, \mathcal{O}_{X_2}(-Z) \otimes \mathcal{O}_{X_2}(2p, p))$. By (6.6), set

$$\mathcal{M}_p := \mathcal{O}_{X_2}(2p, p) \otimes \mathcal{O}_{X_2}(-Z) = \mathcal{O}_{X_2}(2p - b_1, p - b_2) \otimes \pi_{2,0}^*\mathcal{O}_X(t).$$

Write

$$r := b_1 - 2b_2 \geq 0.$$

Choose p so large that $p - b_2 \geq 0$ and, when $r \geq 2$,

$$p - b_2 \geq \left\lfloor \frac{r-2}{3} \right\rfloor.$$

Let $\rho = \pi_{2,1}$ and $\pi = \pi_{1,0}$. Since $p - b_2 \geq 0$,

$$R^q \rho_* \mathcal{M}_p = 0 \quad (q \geq 1),$$

whereas $\rho_* \mathcal{M}_p$ has a filtration induced by

$$0 \longrightarrow \mathcal{O}_{X_1}(1) \longrightarrow V_1^* \longrightarrow \pi^* K_X \otimes \mathcal{O}_{X_1}(-2) \longrightarrow 0.$$

After reindexing by $j = p - b_2 - k$, the corresponding graded terms are

$$(6.9) \quad \mathcal{Q}_{p,k} = \pi^* \left(K_X^{p-b_2-k} \otimes \mathcal{O}_X(t) \right) \otimes \mathcal{O}_{X_1}(-r+3k), \quad 0 \leq k \leq p - b_2.$$

Set

$$A_{p,k} := K_X^{p-b_2-k} \otimes \mathcal{O}_{\mathbb{P}^2}(t), \quad d_k := -r + 3k,$$

so that

$$\mathcal{Q}_{p,k} = \pi^* A_{p,k} \otimes \mathcal{O}_{X_1}(d_k).$$

If $d_k \geq -1$, then the projective bundle formula and the projection formula give

$$R^1 \pi_* \mathcal{Q}_{p,k} \simeq A_{p,k} \otimes R^1 \pi_* \mathcal{O}_{X_1}(d_k) = 0.$$

Since π has one-dimensional fibers and $\dim X = 2$, the Leray spectral sequence yields

$$H^3(X_1, \mathcal{Q}_{p,k}) \simeq H^2(X, R^1 \pi_* \mathcal{Q}_{p,k}) = 0.$$

Consequently, only the indices satisfying $d_k \leq -2$ may contribute; equivalently,

$$0 \leq k \leq \left\lfloor \frac{r-2}{3} \right\rfloor.$$

The only possible nonvanishing terms are therefore indexed by the fixed finite set

$$(6.10) \quad 0 \leq k \leq \left\lfloor \frac{r-2}{3} \right\rfloor.$$

For such k , relative Serre duality for $\pi: X_1 = \mathbb{P}(T_X) \rightarrow \mathbb{P}^2$ gives

$$(6.11) \quad R^1 \pi_* \mathcal{Q}_{p,k} \simeq S^{r-3k-2} T_X \otimes K_X^{p-b_2-k-1} \otimes \mathcal{O}_X(t).$$

For each k in (6.10), the symmetric power in (6.11) is independent of p , whereas the power of the ample line bundle $K_X \simeq \mathcal{O}_X(d-4)$ tends to infinity with p . Serre vanishing therefore gives

$$H^2(X, R^1 \pi_* \mathcal{Q}_{p,k}) = 0 \quad (p \gg 1).$$

The Leray spectral sequence for π now yields $H^3(X_1, \mathcal{Q}_{p,k}) = 0$ for every graded term of the filtration. Induction along the filtration gives

$$H^3(X_1, \rho_* \mathcal{M}_p) = 0.$$

Using $R^q \rho_* \mathcal{M}_p = 0$ once more, we obtain

$$(6.12) \quad H^3(X_2, \mathcal{M}_p) = 0.$$

Combining (6.8) and (6.12) we obtain $h^2(Z, \mathcal{O}_{X_2}(2p, p)|_Z) = 0$ for all sufficiently divisible $p \gg 1$. Consequently,

$$h^0(Z, \mathcal{O}_{X_2}(2p, p)|_Z) \geq \frac{(2u_1 + u_2)^3 \cdot Z}{3!} p^3 + O(p^2) \longrightarrow \infty \quad \text{as } p \rightarrow \infty,$$

which proves that $\mathcal{O}_{X_2}(2, 1)|_Z$ is big.

Finally, by $\mathcal{O}_{X_2}(\Gamma_2) \simeq \mathcal{O}_{X_2}(-1, 1)$,

$$\mathcal{O}_{X_2}(3)|_Z \simeq \mathcal{O}_{X_2}(2, 1)|_Z \otimes \mathcal{O}_Z(2\Gamma_2).$$

Since Γ_2 is effective, $\mathcal{O}_{X_2}(3)|_Z$ is big, and therefore so is $\mathcal{O}_{X_2}(1)|_Z$. This completes the proof of the proposition. $\square$

6.2. Proof of Theorem 1.5 under Assumption (6.4). Assume that the pair (m, t) in (6.1) satisfies inequality (6.4). By Proposition 6.1, for every sufficiently large $\ell \gg 1$ the twisted bundle

$$(6.13) \quad \mathcal{O}_{X_{a_0,2}}(\ell) \otimes \pi_{2,0}^* \mathcal{O}_{X_{a_0}}(-1)|_{Z_{a_0}} \text{ admits a nonzero section } \omega_{2,a_0}.$$

We shall extend ω_{2,a_0} to a holomorphic family over a Zariski-open subset of $\mathbb{P}^{N_d^3}$. First we need a flat family whose fibre over a generic parameter a is the divisor Z_a described in (6.3).

Let

$$U := \{a \in U^{m,t} \mid X_a \text{ smooth, } \omega_{1,a} \not\equiv 0\},$$

and consider the total space $\mathcal{X}_{2,U} := \bigcup_{a \in U} X_{a,2}$. The family $\{\omega_{1,a}\}_{a \in U}$ gives a global section

$$\omega_1 \in H^0\left(\mathcal{X}_{2,U}, \bigcup_{a \in U} (\mathcal{O}_{X_{a,2}}(m) \otimes \pi_{2,0}^* \mathcal{O}_{X_a}(-t))\right),$$

restricting to $\omega_{1,a}$ on each fibre (see (6.2)). Let $\Gamma \subset \mathcal{X}_{2,U}$ be the relative divisor with fibres $\Gamma_{2,a}$ (see (2.10)).

On $\mathcal{X}_{2,U}$ we have the decomposition

$$\{\omega_1 = 0\} = \mathcal{Z} + b\Gamma,$$

where $\mathcal{Z}$ is effective and b is the vanishing order of ω_1 along Γ . After shrinking U we may assume that every $\omega_{1,a}$ vanishes along Γ with order b , then the restriction of $\mathcal{Z}$ to the fibre over a coincides with the irreducible divisor Z_a from (6.3).

The projection $\pi : \mathcal{Z} \rightarrow U$ is proper. After shrinking U , the residual divisor $\mathcal{Z} \subset X_{2,U}$ is a relative effective Cartier divisor. In particular, $\mathcal{Z} \rightarrow U$ is flat. Put

$$\mathcal{L}_\ell := (\mathcal{O}_{X_{2,U}}(\ell) \otimes \pi_{2,0}^* \mathcal{O}_{X_U}(-1))|_{\mathcal{Z}}.$$

By generic freeness and cohomology and base change, after shrinking to a nonempty Zariski-open subset $U' \subset U$, the sheaf

$$\mathcal{E}_\ell := \pi_* \mathcal{L}_\ell$$

is locally free and the natural maps

$$\mathcal{E}_\ell \otimes \mathbb{C}(a) \longrightarrow H^0(Z_a, \mathcal{L}_\ell|_{Z_a})$$

are isomorphisms for all $a \in U'$. Proposition 6.1 yields, after increasing ℓ if necessary, that these spaces are nonzero for a very generic $a_0 \in U'$ and admits a nonzero section ω_{2,a_0} in the fibre of $\mathcal{E}_\ell$ at a_0 . Since $\mathcal{E}_\ell$ is locally free, ω_{2,a_0} in (6.13) extends to an algebraic family of sections

$$(6.14) \quad \{\omega_{2,a} \mid 0 \not\equiv \omega_{2,a} \in H^0(X_{a,2}, \mathcal{O}_{X_{a,2}}(\ell) \otimes \pi_{2,0}^* \mathcal{O}_{X_a}(-1))|_{Z_a}, a \in U_{a_0}^\ell\},$$

defined on a Zariski-open neighbourhood $U_{a_0}^\ell \subset U'$ of a_0 .

Consequently, $\{(\omega_{2,a} = 0) \cap Z_a\}_{a \in U_{a_0}^\ell}$ is an algebraic family of 2-dimensional subvarieties. For any $a \in U_{a_0}^\ell$ and any entire curve $f : \mathbb{C} \rightarrow X_a$, we have $f^* \omega_{1,a} \equiv 0$. Hence the image of the lifting $f_{[2]}$ lies inside Z_a . The condition $f^* \omega_{2,a} \equiv 0$ further forces $f_{[2]}$ to be contained in

$$Z'_a := (\omega_{2,a} = 0) \cap Z_a,$$

a 2-dimensional subvariety of $X_{a,2}$. Therefore the image of $f_{[1]} = \pi_{2,1} \circ f_{[2]}$ lies in the proper subvariety

$$\pi_{2,1}(Z'_a) \subsetneq X_{a,1}.$$

This completes the proof. $\square$

6.3. Proof of Theorem 1.5 when (6.4) Fails. We now work on the complementary side of inequality (6.4), i.e. we assume

$$(6.15) \quad t \geq (d-4) \frac{13c_1^2 - 9c_2}{12c_1^2} m = \frac{4d^2 - 68d + 154}{12(d-4)} m =: C(d)m \quad [\text{see (2.13)}].$$

By the fundamental vanishing theorem for entire curves, it suffices to produce two algebraically independent negatively twisted 2-jet differentials ω_1 and ω_2 . Given an initial family ω_1 , if its vanishing order t satisfies $t \geq 8$, then differentiating ω_1 along a suitably chosen slanted vector field of pole order 7 (Proposition 5.6) yields a second independent family ω_2 .

For $d = 17$ we have $C(d) = 77/78$. By Key Vanishing Lemma 1.4, there is no non-zero negatively twisted holomorphic 2-jet differential satisfying condition (6.15) for $3 \leq m \leq 7$. Consequently, we must

have $m \geq 8$. In this range, condition (6.15) implies $t \geq 8$, so Proposition 5.6 supplies the required ω_2 . This completes the proof. $\square$

Remark 6.3. As an alternative to Păun’s approach using slanted vector fields of pole order 3 in [52, pp. 886–889] for the proof of Theorem 1.5 when $d \geq 18$, one may instead employ directly the slanted vector fields of pole order 7.

For $d \geq 21$, this alternative approach follows directly from a combination of the slanted vector fields technique of Siu and the zero-locus technique of Demailly–El Goul, as employed in this paper. For degrees $18 \leq d \leq 20$, the proof can be completed by establishing the following vanishing results, which can be verified using the supplementary Maple code mentioned in Section 4:

Fact.⁷ Let $X \subset \mathbb{P}^3$ be a generic smooth surface of degree d . Then:

- For $d = 20$, $H^0(X, E_{2,3}T_X^* \otimes \mathcal{O}(-7)) = 0$.
- For $d = 19$, $H^0(X, E_{2,m}T_X^* \otimes \mathcal{O}(-t)) = 0$ for all (m, t) in $\{(3, 6), (4, 7)\}$.
- For $d = 18$, $H^0(X, E_{2,m}T_X^* \otimes \mathcal{O}(-t)) = 0$ for all (m, t) in $\{(3, 5), (4, 6), (5, 7)\}$.

6.4. Proof of Theorem 1.8. Let $\mathcal{C}$ be a generic algebraic curve of degree $d \geq 12$ in $\mathbb{P}^2$. McQuillan’s theorem (Theorem 1.2) and the construction of slanted vector fields extend directly to the logarithmic setting; see [24, Section 2] and [57, Section 3].

First, by Proposition 2.4, there exist positive integers m, t and a nonzero section

$$\omega_1 \in H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)).$$

McQuillan’s result [44] shows that if the lift $f_{[1]}$ of an entire curve f to the first level X_1 of the Demailly–Semple tower of $(\mathbb{P}^2, \mathcal{C})$ is algebraically degenerate, then the image $f(\mathbb{C})$ is algebraically degenerate in $\mathbb{P}^2$. Let $C' \subset \mathbb{P}^2$ denote the Zariski closure of $f(\mathbb{C})$. For a generic curve $\mathcal{C}$ of degree $d \geq 11$, classical results in plane algebraic geometry guarantee that $C' \cap \mathcal{C}$ contains at least three distinct points. Consequently, the complement $C' \setminus \mathcal{C}$ is hyperbolic, contradicting the fact that it contains the image of the entire curve f . Hence, to derive a contradiction from the existence of such an f , it suffices to prove that $f_{[1]}$ is algebraically degenerate.

A numerical criterion analogous to Proposition 6.1 is provided by El Goul [24, Theorem 1.3.3]. It allows us to assume

$$(6.16) \quad t \geq (d-3) \frac{13\bar{c}_1^2 - 9\bar{c}_2}{12\bar{c}_1^2} m = \frac{4d^2 - 51d + 90}{12(d-3)} m =: \bar{C}(d) m \quad [\text{by (2.14)}].$$

Since $\text{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$, we can mimic and simplify the arguments of the compact case. For brevity, we work directly with a given ω_1 on a generic $\mathcal{C}$ and omit the step of extending it to a holomorphic family.

Under the condition (6.16) we now examine the remaining possibilities for (m, t) when $d = 13$ and $d = 12$.

For $d = 13$ we have $\bar{C}(d) = \frac{103}{120}$. By Key Vanishing Lemma 1.3, there is no non-zero negatively twisted logarithmic holomorphic 2-jet differential satisfying condition (6.16) for $3 \leq m \leq 8$. Consequently, we may assume $m \geq 9$. In this range, condition (6.16) implies $t \geq 8$, so Proposition 5.7 supplies the required second differential ω_2 .

For $d = 12$ we have $\bar{C}(d) = \frac{1}{2}$. Key Vanishing Lemma 1.3 ensures that no non-zero negatively twisted logarithmic 2-jet differential satisfying condition (6.16) exists for $3 \leq m \leq 14$. Hence, we may restrict to $m \geq 15$. For such m , condition (6.16) yields $t \geq 8$; therefore Proposition 5.7 again provides the desired ω_2 .

The preceding argument excludes every nonconstant entire curve in $\mathbb{P}^2 \setminus \mathcal{C}$, so the complement is Brody hyperbolic. Moreover, the smooth plane curve $\mathcal{C}$ has genus $(d-1)(d-2)/2 \geq 2$. Green’s criterion now applies: if a smooth projective variety X , the complement of a divisor D , and every open boundary stratum are Brody hyperbolic, then $X \setminus D$ is hyperbolically embedded in X ; see [26] and the precise restatement [38, Theorem 4.1]. For the present one-component divisor, the only boundary stratum is $\mathcal{C}$ itself. Hence $\mathbb{P}^2 \setminus \mathcal{C}$ is hyperbolically embedded and therefore Kobayashi hyperbolic. This completes the proof. $\square$

Remark 6.4. As an alternative to Rousseau’s approach using slanted vector fields of pole order 3 in [57, Proposition 17] for the proof of the Kobayashi hyperbolicity of the complement $\mathbb{P}^2 \setminus \mathcal{C}$ of a generic algebraic curve $\mathcal{C} \subset \mathbb{P}^2$ of degree $d \geq 14$, one may instead employ directly the slanted vector fields of pole order 7.

⁷The associated Maple exports are available at <https://github.com/LeiHou-code/Hyperbolicity-Dimension-2-Conventions-Facts-2026>.

For $d \geq 17$, this alternative approach follows directly from a combination of the slanted vector fields technique of Siu and the zero-locus technique of Demailly–El Goul, as employed in this paper. For degrees $14 \leq d \leq 16$, the proof can be completed by establishing the following vanishing results, which can be verified using the supplementary Maple code mentioned in Section 3:

Fact.⁸ *Let $\mathcal{C} \subset \mathbb{P}^2$ be a generic smooth curve of degree d . Then:*

- For $d = 17$, $H^0(\mathbb{P}^2, E_{2,3}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}(-7)) = 0$.
- For $d = 16$, $H^0(\mathbb{P}^2, E_{2,3}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}(-6)) = 0$.
- For $d = 15$, $H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}(-t)) = 0$ for all (m, t) in $\{(3, 5), (4, 7)\}$.
- For $d = 14$, $H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}(-t)) = 0$ for all (m, t) in $\{(3, 4), (4, 5), (5, 7)\}$.

6.5. Proof of Theorem 1.9. We now prove Theorem 1.9, following the strategy developed in our earlier work [31].

First, if the image of f is contained in an irreducible curve $\mathcal{D}$, this special case is already covered by stronger results in the literature; see e.g. [60, Chapter 4].

We now assume that f is algebraically nondegenerate. Propositions 2.4 and 5.5 guarantee the existence of an irreducible, negatively twisted invariant logarithmic 2-jet differential

$$(6.17) \quad \omega_1 \in H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)),$$

with weighted degree $m \geq 1$ and vanishing order $t \geq 1$.

If $f^*\omega_1 \not\equiv 0$, Theorem 2.6 immediately gives the desired Second Main Theorem with a finite constant C_d .

In particular, a Riemann–Roch computation based on the filtration (2.3) and a Bogomolov-type vanishing theorem show that for any rational number $\delta \in (0, \frac{1}{3})$ and sufficiently divisible $m \gg 1$,

$$h^0(E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \overline{K}_{\mathbb{P}^2}^{-\delta m}) \geq \frac{m^4}{648}((54\delta^2 - 48\delta + 13)\bar{c}_1^2 - 9\bar{c}_2) + O(m^3) \quad [\text{cf. [31, Proposition 4.2]}].$$

The leading coefficient (after removing the common factor $\frac{1}{648}$) is

$$\varphi(\delta) := (54\delta^2 - 48\delta + 13)\bar{c}_1^2 - 9\bar{c}_2 = 54\bar{c}_1^2\delta^2 - 48\bar{c}_1^2\delta + 13\bar{c}_1^2 - 9\bar{c}_2.$$

Note that $13\bar{c}_1^2 - 9\bar{c}_2 > 0$ for $d \geq 11$. The two roots of the equation $\varphi(\delta) = 0$ are

$$\delta_1 = \frac{8 - \sqrt{54\bar{c}_2/\bar{c}_1^2 - 14}}{18} > 0, \quad \delta_2 = \frac{8 + \sqrt{54\bar{c}_2/\bar{c}_1^2 - 14}}{18} > 0.$$

Hence $\varphi(\delta) > 0$ for all $\delta \in (0, \delta_1)$. Consequently, in the existence result for ω_1 stated in (6.17), the ratio t/m can be taken arbitrarily close to

$$(6.18) \quad (d-3)\delta_1 = \frac{8(d-3) - \sqrt{2(4d-3)(5d-6)}}{18} \quad [\text{use (2.14)}].$$

We may therefore assume that the ratio t/m of ω_1 in (6.17) satisfies

$$(d-3)\delta_1 - \varepsilon' < \frac{t}{m} \leq (d-3)\delta_1,$$

where $\varepsilon' > 0$ is a fixed arbitrarily small constant. If $f^*\omega_1 \not\equiv 0$, Theorem 2.6 yields the estimate (1.1) with constant $C_d = m/t$ satisfying

$$(6.19) \quad \frac{18}{8(d-3) - \sqrt{2(4d-3)(5d-6)}} = \frac{1}{(d-3)\delta_1} \leq C_d = \frac{m}{t} < \frac{1}{(d-3)\delta_1 - \varepsilon'}.$$

Thus we may assume $f^*\omega_1 \equiv 0$ in the remainder of the proof. The subsequent argument splits according to the values of (m, t) .

⁸The associated Maple exports are available at <https://github.com/LeiHou-code/Hyperbolicity-Dimension-2-Conventions-Facts-2026>.

Case 1: ω_1 has high vanishing order. When $m = 3, 4, 5$ and $t \geq 4$, we apply Siu's strategy of slanted vector fields [64], which in the two-dimensional logarithmic setting was carried out by Rousseau [57, Proposition 17]. Differentiating ω_1 by a slanted vector field twisted by $\mathcal{O}_{\mathbb{P}^2}(3)$ produces an “independent” nonzero invariant logarithmic 2-jet differential

$$\omega_2 \in H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t+3)).$$

When $m \geq 6$ and $t \geq 8$, we may instead differentiate ω_1 by a slanted vector field twisted by $\mathcal{O}_{\mathbb{P}^2}(7)$ (see Proposition 5.7), obtaining an “independent” nonzero invariant logarithmic 2-jet differential

$$\omega_2 \in H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t+7)).$$

We then distinguish two subcases:

- If $f^*\omega_2 \neq 0$, Theorem 2.6 directly gives a Second Main Theorem type estimate.
- If $f^*\omega_1 \equiv 0$ and $f^*\omega_2 \equiv 0$, the algebraic independence of ω_1 and ω_2 implies that the lift $f_{[1]}(\mathbb{C}) \subset X_1$ is algebraically degenerate. Hence McQuillan's Theorem 2.7 applies.

Case 2: ω_1 has low vanishing order. Assume

$$(6.20) \quad 0 < t < (d-3) \frac{13\bar{c}_1^2 - 9\bar{c}_2}{12\bar{c}_1^2} m = \frac{4d^2 - 51d + 90}{12(d-3)} m =: \bar{C}(d) m. \quad [\text{see (6.16)}]$$

Following a strategy of Demailly–El Goul [18, Proposition 3.4] (see also [24, Theorem 1.3.3]), a Riemann–Roch computation yields the existence of a nonzero invariant logarithmic 2-jet differential ω_2 with negative twist. The definition of ω_2 is subtle: it is not defined on the whole of $\mathbb{P}^2$, but only on the zero locus $\{\omega_1 = 0\}$ inside the second level X_2 of the Demailly–Semple tower for $(\mathbb{P}^2, \mathcal{C})$. We again distinguish two subcases:

- If $f^*\omega_2 \equiv 0$, the simultaneous vanishing of $f^*\omega_1$ and $f^*\omega_2$ forces the lift $f_{[1]}(\mathbb{C})$ to be algebraically degenerate in the first level X_1 of the logarithmic Demailly–Semple tower. McQuillan's Theorem 2.7 then applies.
- If $f^*\omega_2 \neq 0$, a Second Main Theorem type estimate can be obtained by an argument analogous to that in our earlier work [31, Section 6]. The reasoning proceeds as follows.

As in the proof of Proposition 6.1, one can show that the zero divisor of ω_1 on X_2 decomposes as

$$\{\omega_1 = 0\} = Z + b_1\Gamma_2,$$

where $Z \subset X_2$ is irreducible and reduced. We now determine an effective small rational number $\tau > 0$ such that the $\mathbb{Q}$ -line bundle

$$(\mathcal{O}_{X_2}(2, 1) \otimes \pi_{2,0}^*\mathcal{O}_{\mathbb{P}^2}(-\tau))|_Z$$

is big. (Here we use the same notation as in (6.5) for the logarithmic case). This is achieved by a similar Riemann–Roch computation and the use of Proposition 6.2 as in [31, Proposition 6.2].

Proposition 6.5. *If $0 < \frac{t}{m} < \bar{C}(d)$ and $0 \leq \tau < \tau_1(\frac{t}{m})$ with $\tau \in \mathbb{Q}$, where*

$$(6.21) \quad \begin{aligned} \tau_1\left(\frac{t}{m}\right) &= -2\bar{c}_1 - \frac{3}{2} \frac{t}{m} - \frac{\sqrt{3}}{6} \sqrt{27\left(\frac{t}{m}\right)^2 + 24\bar{c}_1 \frac{t}{m} + 36\bar{c}_2 - 4\bar{c}_1^2} \\ &= 2(d-3) - \frac{3}{2} \frac{t}{m} - \frac{\sqrt{3}}{6} \sqrt{27\left(\frac{t}{m}\right)^2 - 24(d-3)\frac{t}{m} + 4(8d^2 - 21d + 18)}, \end{aligned}$$

then the restricted $\mathbb{Q}$ -line bundle

$$(\mathcal{O}_{X_2}(2, 1) \otimes \pi_{2,0}^*\mathcal{O}_{\mathbb{P}^2}(-\tau))|_Z$$

is big on Z . □

We first make precise the evaluation of a section that is defined only on Z .

Lemma 6.6 (Tautological evaluation of a restricted section). *Let $Z \subset X_2$ be an effective Cartier divisor and*

$$\sigma \in H^0(Z, \mathcal{O}_{X_2}(2p, p) \otimes \pi_{2,0}^*\mathcal{O}_{\mathbb{P}^2}(-\tilde{t})|_Z), \quad p, \tilde{t} \in \mathbb{Z}_{>0}.$$

*Suppose that $g: \mathbb{C} \rightarrow \mathbb{P}^2$ is nonconstant, $g(\mathbb{C}) \not\subset \mathcal{C}$, and its logarithmic lift extends and satisfies $g_{[2]}(\mathbb{C}) \subset Z$. On $\mathbb{C} \setminus g^{-1}(\mathcal{C})$, regard $dg_{[1]}: T_{\mathbb{C}} \rightarrow g_{[1]}^*V_1$ as a differential in the directed structure. Wherever it is*

nonzero, its image is the tautological line $g_{[2]}^* \mathcal{O}_{X_2}(-1)$. It therefore defines, by meromorphic continuation across the remaining discrete set, the intrinsic tautological derivative

$$\mathcal{D}g_{[1]} \in H_{\text{mer}}^0(\mathbb{C}, K_{\mathbb{C}} \otimes g_{[2]}^* \mathcal{O}_{X_2}(-1)).$$

The inclusion

$$\iota_p: \mathcal{O}_{X_2}(2p, p) = \mathcal{O}_{X_2}(3p)(-2p\Gamma_2) \hookrightarrow \mathcal{O}_{X_2}(3p)$$

together with this derivative defines canonically

$$(6.22) \quad s_{\sigma, g} := \left\langle g_{[2]}^*(\iota_p \sigma), (\mathcal{D}g_{[1]})^{\otimes 3p} \right\rangle \in H_{\text{mer}}^0\left(\mathbb{C}, K_{\mathbb{C}}^{\otimes 3p} \otimes g^* \mathcal{O}_{\mathbb{P}^2}(-\tilde{t})\right).$$

It is independent of every ambient local lifting of σ . If it is not identically zero, then

$$(6.23) \quad \text{div}_{\infty}(s_{\sigma, g}) \leq 3p(g^* \mathcal{C})_{\text{red}}$$

and

$$(6.24) \quad \int_0^{2\pi} \log^+ \|s_{\sigma, g}(re^{\sqrt{-1}\theta})\| \frac{d\theta}{2\pi} = O(\log T_g(r) + \log r) \quad \parallel.$$

Proof. The factorization of $g_{[2]}$ through Z makes $g_{[2]}^*(\iota_p \sigma)$ intrinsic. Pairing it with the indicated power of the tautological derivative gives (6.22). If ambient local liftings are used to represent σ , their differences lie in the ideal of Z and therefore pull back to zero.

Equip the line bundles with smooth Hermitian metrics. Since Z is projective, the norm of the holomorphic section $\iota_p \sigma$ is bounded on Z . Hence

$$\|s_{\sigma, g}\| \leq C \|\mathcal{D}g_{[1]}\|^{3p}$$

for a uniform constant C .

Near $g^{-1}(\mathcal{C})$, take a logarithmic coordinate x with $\mathcal{C} = \{x = 0\}$. The boundary component of the derivative of the regular first lift is $(x \circ g)'/(x \circ g)$, while its vertical projective-coordinate components are holomorphic. Thus $\mathcal{D}g_{[1]}$ has at most a simple pole along $(g^* \mathcal{C})_{\text{red}}$, which proves (6.23) after taking the $3p$ -th tensor power. At an isolated zero of g' outside $g^{-1}(\mathcal{C})$, the regularized Semple lift gives only a removable singularity or a zero. If the critical point lies over $\mathcal{C}$, the logarithmic component $(x \circ g)'/(x \circ g)$ may still have a simple pole; that pole is already counted by (6.23), and no additional Semple-type pole occurs.

Choose a finite logarithmic base cover and, above it, finitely many relatively compact Semple charts. A partition of unity is used only to estimate the norm of the already-defined global section. On each chart, the components of $\mathcal{D}g_{[1]}$ are logarithmic derivatives of local coordinate functions. A projective Semple coordinate u is a rational function on X_1 and, in a logarithmic frame, a ratio of first logarithmic derivatives; therefore

$$T_{u \circ g_{[1]}}(r) = O(T_g(r) + \log r).$$

For a partition function χ supported in a relatively compact chart, choose $a \notin \overline{u(\text{supp } \chi)}$. On that support, $u - a$ is bounded above and away from zero, so

$$(u \circ g_{[1]})' = (u \circ g_{[1]} - a) \frac{(u \circ g_{[1]})'}{u \circ g_{[1]} - a}.$$

The logarithmic-derivative lemma applied to $u \circ g_{[1]} - a$ controls the vertical component by

$$O(\log^+ T_g(r) + \log r).$$

Summing over the finite cover, together with the analogous horizontal estimates, gives (6.24). This is precisely the partition-of-unity argument of [35, Thm. 3.1, (3.3)–(3.6)], with its Semple-coordinate adaptation made explicit. $\square$

The preceding lemma gives the following restricted-section form of the Second Main Theorem.

Theorem 6.7. *Let $g: \mathbb{C} \rightarrow \mathbb{P}^2$ be an entire curve whose lift $g_{[2]}: \mathbb{C} \rightarrow X_2$ satisfies $g_{[2]}(\mathbb{C}) \subset Z$. Suppose there exists a section*

$$\sigma \in H^0\left(Z, (\mathcal{O}_{X_2}(2, 1)^{\tilde{m}} \otimes \pi_{2,0}^* \mathcal{O}_X(-\tilde{t}))|_Z\right) \cong H^0\left(Z, (\mathcal{O}_{X_2}(3) \otimes \mathcal{O}(-2\Gamma_2))^{\tilde{m}} \otimes \pi_{2,0}^* \mathcal{O}_X(-\tilde{t})|_Z\right),$$

such that the canonical evaluation $s_{\sigma,g}$ of Lemma 6.6, with $p = \tilde{m}$, is not identically zero. Then the following Second Main Theorem type estimate holds:

$$T_g(r) \leq \frac{3\tilde{m}}{\tilde{t}} N_g^{(1)}(r, \mathcal{C}) + o(T_g(r)) \quad \parallel.$$

Proof. Give $K_{\mathbb{C}}$ the flat metric defined by dz and $\mathcal{O}_{\mathbb{P}^2}(-1)$ the metric dual to the Fubini–Study metric. Poincaré–Lelong and Jensen’s formula applied to the intrinsic section $s_{\sigma,g}$ give

$$\tilde{t}T_g(r) \leq N(r, \text{div}_{\infty}(s_{\sigma,g})) + \int_0^{2\pi} \log^+ \|s_{\sigma,g}(re^{\sqrt{-1}\theta})\| \frac{d\theta}{2\pi} + O(1).$$

Equations (6.23) and (6.24), with $p = \tilde{m}$, yield the stated coefficient $3\tilde{m}/\tilde{t}$. $\square$

Summarizing, Proposition 6.5 yields, for any rational $\tau \in (0, \tau_1(t/m))$, a nontrivial global section

$$\sigma \in H^0\left(Z, (\mathcal{O}_{X_2}(2, 1) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-\tau))^{\otimes \ell}\right)$$

for a sufficiently divisible integer $\ell \gg 1$ (depending on τ).

Recall that the assumption of the present subcase is $f^*\omega_1 \equiv 0$, while the canonical evaluation of σ along $f_{[2]}$ is nonzero. Thus Theorem 6.7 gives the estimate (1.1) with constant (see (6.21))

$$C_d = \frac{3}{\tau} > \frac{3}{\tau_1(t/m)}.$$

Since τ can be chosen arbitrarily close to $\tau_1(t/m)$, the estimate (1.1) holds for any constant $C_d > \frac{3}{\tau_1(t/m)}$.

Now we compute the constant C_d appearing in the estimate (1.1) for $d \geq 14$.

A direct computation gives

$$\overline{C}(d) = \frac{4d^2 - 51d + 90}{12(d-3)} = \frac{d-3}{3} - \frac{9}{4(d-3)} - \frac{9}{4} > \frac{7}{6},$$

where the last inequality follows from the assumption $d \geq 14$.

Let $\frac{7}{6} < \gamma < \overline{C}(d)$ be a parameter that will be chosen later.

If $t \leq \gamma m$ (which falls into **Case 2**), we obtain the estimate (1.1) with constant

$$C_d = \frac{3}{\tau_1(\gamma)} = \frac{3}{2\left(d-3-\frac{3}{4}\gamma-\frac{\sqrt{3}}{12}\sqrt{27\gamma^2-24(d-3)\gamma+4(8d^2-21d+18)}\right)} > \frac{1}{d-3},$$

where the lower bound $\frac{1}{d-3}$ is precisely the constant that would follow from Theorem 2.7.

If $t \geq \frac{6\gamma-7}{6}m + 7$ and $m \geq 6$ (or if $m = 3, 4, 5$ and $t \geq \frac{3\gamma-3}{3}m + 3$), then we are in **Case 1** and the estimate (1.1) holds with constant $C_d = \frac{6}{6\gamma-7}$ (respectively $C_d = \frac{3}{3\gamma-3}$). A straightforward computation shows that for $\frac{7}{6} < \gamma < \overline{C}(d)$ and $d \geq 14$,

$$(6.25) \quad \frac{6}{6\gamma-7} > \frac{3}{3\gamma-3} > \frac{1}{d-3}.$$

Observe the following:

- The pairs $(m, t) = (6, \lfloor 6\gamma \rfloor)$ and $(m, t) = (3, \lfloor 3\gamma \rfloor)$ satisfy $t \leq \gamma m$.
- The pairs $(m, t) = (6, \lceil 6\gamma \rceil)$ and $(m, t) = (3, \lceil 3\gamma \rceil)$ satisfy respectively $t \geq \frac{6\gamma-7}{6}m + 7$ and $t \geq \frac{3\gamma-3}{3}m + 3$.
- The lines $t = \gamma m$ and $t = \frac{6\gamma-7}{6}m + 7$ intersect at $(6, 6\gamma)$; the lines $t = \gamma m$ and $t = \frac{3\gamma-3}{3}m + 3$ intersect at $(3, 3\gamma)$.

Consequently, every possible value of (m, t) is covered by the above discussion. For a convenient uniform bound, let γ_0 be the unique solution of

$$\frac{6}{6\gamma-7} = \frac{3}{\tau_1(\gamma)}.$$

A straightforward calculation yields

$$(6.26) \quad \gamma_0(d) = \frac{32d - 33 - \sqrt{640d^2 - 1248d + 1017}}{72},$$

which lies in the interval $(\frac{7}{6}, \overline{C}(d))$. At this admissible split,

$$\frac{6}{6\gamma_0(d) - 7} = \frac{3}{\tau_1(\gamma_0(d))} = \frac{72}{32d - 117 - \sqrt{640d^2 - 1248d + 1017}}.$$

A direct comparison shows that this quantity is strictly larger, for every $d \geq 14$, than the other high-slope constant in (6.25) and the limiting initial-section constant in (6.19). Choosing the Riemann–Roch parameter sufficiently close to its endpoint therefore gives the uniform value

$$C_d = \frac{72}{32d - 117 - \sqrt{640d^2 - 1248d + 1017}} \quad (d \geq 14).$$

Case 3: Remaining possibilities. For $d = 12, 13$, there remain a few pairs (m, t) not covered by the previous two cases. Key Vanishing Lemma 1.3 excludes the existence of any nonzero ω_1 for those pairs. Note that vanishing for (m, t) implies vanishing for all (m, t') with $t' > t$. In enumerating the residual pairs we combine this monotonicity with the pole-order-three branch for $m = 3, 4, 5$; in particular, that branch, not the Key Vanishing list, handles $(m, t) = (5, 4)$ when $d = 13$.

We now compute the constant C_d in the estimate (1.1) for $d = 12$ and 13.

Case $d = 13$. A direct computation gives

$$\frac{1}{(d-3)\delta_1} = \frac{18}{80 - 7\sqrt{118}} \approx 4.545 \quad (d = 13).$$

Moreover, $\overline{C}(13) = \frac{103}{120} > \frac{7}{9}$. If $t \leq \frac{7}{9}m$ (which falls into **Case 2**), we obtain the estimate (1.1) with constant

$$C_{13} = \frac{3}{\tau_1(\frac{7}{9})} = \frac{3}{2\left(d - 3 - \frac{7}{12} - \frac{1}{12}\sqrt{96d^2 - 308d + 433}\right)} = \frac{18}{113 - \sqrt{12653}} > \frac{1}{d-3} \quad (d = 13).$$

If $t \geq \frac{1}{10}m + 7$ (which belongs to **Case 1**), the same estimate holds with constant $C_{13} = 10$.

Observe that

- The pairs $(m, t) = (10, 7)$ and $(11, 8)$ satisfy $t \leq \frac{7}{9}m$.
- The pair $(m, t) = (10, 8)$ satisfies $t \geq \frac{1}{10}m + 7$.

All other values of (m, t) are either covered by the pole-order-three subcase for $m \leq 5$, or ruled out by Key Vanishing Lemma 1.3 and twist monotonicity. Together with (6.19) applied to $d = 13$, we conclude that

$$C_{13} = \frac{18}{113 - \sqrt{12653}} \approx 34.989.$$

Case $d = 12$. A direct computation yields

$$\frac{1}{(d-3)\delta_1} = \frac{1}{4 - \sqrt{15}} \approx 7.873 \quad (d = 12).$$

Now $\overline{C}(12) = \frac{1}{2} > \frac{7}{15}$. If $t \leq \frac{7}{15}m$ (again **Case 2**), the estimate (1.1) holds with constant

$$C_{12} = \frac{3}{\tau_1(\frac{7}{15})} = \frac{3}{2\left(d - 3 - \frac{7}{20} - \frac{\sqrt{3}}{60}\sqrt{800d^2 - 2380d + 2787}\right)} = \frac{30}{173 - \sqrt{29809}} > \frac{1}{d-3} \quad (d = 12).$$

If $t \geq \frac{1}{17}m + 7$ (which belongs to **Case 1**), the constant becomes $C_{12} = 17$.

Notice that

- The pairs $(m, t) = (17, 7)$ and $(18, 8)$ satisfy $t \leq \frac{7}{15}m$.
- The pair $(m, t) = (17, 8)$ satisfies $t \geq \frac{1}{17}m + 7$.

All remaining values of (m, t) are either covered by the pole-order-three subcase for $m \leq 5$, or excluded by Key Vanishing Lemma 1.3 and twist monotonicity. Combined with (6.19) for $d = 12$, we obtain

$$C_{12} = \frac{30}{173 - \sqrt{29809}} \approx 86.413.$$

Thus we finish the proof of Theorem 1.9. □

7. Further Perspectives

7.1. A Computational Challenge. To reach the degree bounds 15 in the compact case and 11 in the logarithmic case — the current thresholds down to which the existence of nonzero negatively twisted invariant 2-jet differentials is guaranteed by Riemann–Roch computations — it would suffice to establish the following vanishing statements.

Question 7.1. Are the following vanishing statements true?

- (1) Let $\mathcal{C} \subset \mathbb{P}^2$ be a generic curve of degree 11. Then

$$H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}(-t)) = 0$$

for all pairs $(m, t) \in \mathbb{N}^2$ such that

$$t = \left\lceil \frac{13m}{96} \right\rceil, \quad 3 \leq m \leq 51.$$

- (2) Let $X \subset \mathbb{P}^3$ be a generic surface of degree d . Then:

- For $d = 16$, $H^0(X, E_{2,m}T_X^* \otimes \mathcal{O}_X(-t)) = 0$ for all (m, t) in

$$\{(3, 3), (4, 3), (5, 4), (6, 4), (7, 5), (8, 5), (9, 6), (10, 7), (11, 7)\}.$$

- For $d = 15$, $H^0(X, E_{2,m}T_X^* \otimes \mathcal{O}_X(-t)) = 0$ for all pairs $(m, t) \in \mathbb{N}^2$ such that

$$t = \left\lceil \frac{17m}{66} \right\rceil, \quad 3 \leq m \leq 27.$$

Remark 7.2. If the vanishing statements listed in Question 7.1–(1) for generic curves $\mathcal{C} \subset \mathbb{P}^2$ of degree 11 were to hold, then the argument presented in Subsection 6.5 would yield a Second Main Theorem type estimate valid for every entire curve $f: \mathbb{C} \rightarrow \mathbb{P}^2$:

$$T_f(r) \leq C_{11} N_f^{[1]}(r, \mathcal{C}) + o(T_f(r)) \quad \parallel,$$

After the proposed finite vanishing range is removed, the largest surviving ratio below $13/96$ is $5/37$. Hence the argument gives every constant

$$C_{11} > \frac{3}{\tau_1(5/37)} = \frac{3}{2} \left(3507 + \sqrt{12298605} \right) \approx 10520.905.$$

In particular, one may take the explicit integer constant $C_{11} = 10521$.

Remark 7.3. The hyperelliptic-type ansatz $y^d = P(x)$ is insufficient to prove the conjectured Key Vanishing Lemma for degree 11 (see Question 7.1–(1)): in all our Maple experiments, nontrivial sections invariably occur for $(m, t) = (3, 1)$. To settle degree 11 one must use more involved defining equations, which dramatically raise the computational cost — the resulting linear systems would involve far more than twenty million variables, well beyond Maple’s reach.

For surfaces in $\mathbb{P}^3$, the situation is mixed. With the one-term Fermat perturbation, degree 16 likely works (no obstruction at $(3, 1)$), but the linear system already exceeds two million variables. A C++ implementation with parallel computing could plausibly handle it; however, the main purpose of this paper is to introduce a usable algebraic reduction framework and test its viability on concrete non-trivial examples (degree 17 compact, degrees 12, 13 logarithmic), so we do not pursue degree 16 here.

Degree 15 is the hardest case: the same ansatz *fails*, since our Maple experiments already exhibit nontrivial sections for $(m, t) = (3, 1)$ on $X^{15} + Y^{15} + Z^{15} + T^{15} + X^7T^8 = 0$. Resolving degree 15 would demand more sophisticated equations and lead to linear systems with well over sixty million variables — an immense computational challenge that will require new algorithmic ideas and a substantial engineering effort.

Question 7.4. Is there a natural algebro-geometric construction that explains some of the nonvanishing sections for $(m, t) = (3, 1)$ reported in Remark 7.3?

7.2. Relaxing “Very Generic” to “Generic” in the Compact Case. The issue of replacing “very generic” by “generic” in the compact case (for low degrees) was noted earlier by Demailly–El Goul [18, Remark 7.2] and remains unresolved. The obstacle lies in the hypothesis of Theorem 1.1, which itself demands that the surface be very generic.

Question 7.5. For every integer $d \geq 17$, does a *generic* surface $X \subset \mathbb{P}^3$ of degree d contain no rational or elliptic curves? Note that Theorem 1.5 gives a strong constraint: if a rational or elliptic curve C were to exist on a generic surface X_a of degree $d \geq 17$, then C must necessarily be an *invariant curve* of some multi-foliation $\mathcal{F}_a$.

Remark 7.6. If the answer to Question 7.5 is affirmative, then Theorem 1.7 can be strengthened to hold for *generic* surfaces.

Remark 7.7. Question 7.5 is related to the classical Poincaré problem [53], which seeks to bound the degree of an algebraic solution of a differential equation in terms of the degree of the equation itself. Lins Neto [41] showed that no such universal bound exists in general. Thus, to answer Question 7.5 affirmatively, one must exploit additional geometric structure. Theorem 1.5 provides such structure by forcing any rational or elliptic curve to be invariant under a multi-foliation arising from jet differentials.

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Author contributions and AI-assisted preparation Building on the exponent-bound method introduced in the preceding three-conics paper [31], Lei Hou and Song-Yan Xie recognized that the present one-component problem should admit a finite exact treatment. They developed the theoretical framework for the Key Vanishing Lemmas and selected the defining equations used in their finite verification. Starting from this framework, Joël Merker implemented the verification in Maple and carried out the computations using high-performance computing resources in Orsay. Dinh Tuan Huynh carried out the Riemann–Roch computations. Hou and Xie assembled these ingredients into the final mathematical argument and prepared the manuscript.

Generative AI was used only for language polishing. The mathematical ideas, arguments, and computational specifications are due to the authors.

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ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA
Email address: houlei@amss.ac.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF EDUCATION, HUE UNIVERSITY, 34 LE LOI ST., HUE CITY, VIETNAM
Email address: dinhtuanhuynh@hueuni.edu.vn

LABORATOIRE DE MATHÉMATIQUES D’ORSAY, UNIVERSITÉ PARIS-SACLAY, 91405 ORSAY CEDEX, FRANCE
Email address: joel.merker@universite-paris-saclay.fr

STATE KEY LABORATORY OF MATHEMATICAL SCIENCES, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA; SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF CHINESE ACADEMY OF SCIENCES, BEIJING 100049, CHINA
Email address: xiesongyan@amss.ac.cn