

A SECOND MAIN THEOREM FOR ENTIRE CURVES INTERSECTING THREE CONICS

LEI HOU, DINH TUAN HUYNH, JOËL MERKER, AND SONG-YAN XIE

ABSTRACT. We establish a Second Main Theorem for algebraically nondegenerate entire holomorphic curves $f : \mathbb{C} \rightarrow \mathbb{P}^2$ intersecting a generic configuration of three conics $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ in the complex projective plane $\mathbb{P}^2$. Using invariant logarithmic 2-jet differentials with negative twists, we prove the estimate

$$T_f(r) \leq 5 \sum_{i=1}^3 N_f^{[1]}(r, \mathcal{C}_i) + o(T_f(r)) \quad \parallel,$$

where $T_f(r)$ is the Nevanlinna characteristic function, and $N_f^{[1]}(r, \mathcal{C}_i)$ is the 1-truncated counting function.

We also prove that the logarithmic cotangent bundle of three generic conics is nef and that all of its negatively twisted symmetric powers have no nonzero global sections. Consequently, within the negatively twisted jet-differential approach, order two is the first level at which an effective argument can operate.

The key innovation of our approach is establishing new vanishing lemmas of the form

$$H^0(\mathbb{P}^2, E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)) = 0$$

for specific pairs (m, t) , achieved by combining algebro-geometric arguments with computer-assisted computations through a mod- p reduction technique. This provides a systematic method for proving vanishing results for negatively twisted jet differentials — a key component in complex hyperbolic geometry — thereby filling a missing piece in two-dimensional complex hyperbolicity theory.

1. Introduction

1.1. Motivation. In the complex projective plane $\mathbb{P}^2$, the *logarithmic Kobayashi conjecture* asserts that if $\mathcal{C} \subset \mathbb{P}^2$ is a *generic* algebraic curve of degree $d \geq 5$, then the complement $\mathbb{P}^2 \setminus \mathcal{C}$ should be hyperbolic [39]. Various notions of hyperbolicity exist; here we use the most straightforward one: $\mathbb{P}^2 \setminus \mathcal{C}$ is said to be *hyperbolic* if there is no nonconstant holomorphic map $f : \mathbb{C} \rightarrow \mathbb{P}^2 \setminus \mathcal{C}$. This conjecture generalizes the classical Little Picard Theorem on the hyperbolicity of $\mathbb{P}^1 \setminus \{3 \text{ points}\}$, and is supported by explicit examples constructed by Zaidenberg [74]. For generic curves $\mathcal{C}$ of sufficiently large degree in $\mathbb{P}^2$, the conjecture was confirmed by Siu and Yeung [67] by constructing explicit logarithmic 2-jet differentials with negative twists. Later, building on the work [18], El Goul [25] improved the degree bound to 15 for *very generic* $\mathcal{C}$. Combining this with Siu's strategy of using slanted vector fields [65] and a technique of Păun [54], Rousseau [58] further reduced the degree bound to 14.

These results for high-degree curves naturally extend to configurations with multiple components. In the case where the curve $\mathcal{C} = \cup_{i=1}^q \mathcal{C}_i$ decomposes into the union of $q \geq 2$ generic algebraic curves $\mathcal{C}_i$ of degree d_i , the hyperbolicity of the complement $\mathbb{P}^2 \setminus \mathcal{C}$ has been established with smaller degree bounds $d = \sum_{i=1}^q d_i$ in various works. For instance, a result in [1] settles the case where $q \geq 5$. The case $(q = 4, \sum_{i=1}^4 d_i \geq 5)$ was treated in [21]. The configuration of three conics was studied in [29, 21, 2] by different methods. Dethloff, Schumacher, and Wong treated the case of two conics and one cubic using symmetric differential 1-forms [22], and settled the case of three generic conics by a clever application of Nevanlinna theory [21]. Rousseau [58] addressed the case $q = 2$ under the assumptions $(d_1 = 1, d_2 \geq 11)$, $(d_1 = 2, d_2 \geq 8)$, $(d_1 = 3, d_2 \geq 5)$, or $(d_1 = 4, d_2 \geq 4)$. Notably, for $q \geq 3$, Noguchi, Winkelmann, and Yamanoi [53] established the *algebraic degeneracy* of every entire curve in $\mathbb{P}^2 \setminus \mathcal{C}$ under the mild condition $\sum_{i=1}^q d_i \geq 4$. Their result also extends to the corresponding higher-dimensional setting.

Thus, in the aforementioned settings, every algebraically nondegenerate entire curve $f : \mathbb{C} \rightarrow \mathbb{P}^2$ must intersect the curve $\mathcal{C}$. In this article, we aim to quantify these geometric results by means of Nevanlinna

Date: August 25, 2026.

2020 *Mathematics Subject Classification.* 32H30, 32Q45.

Key words and phrases. Nevanlinna theory, Second Main Theorem, entire curves, invariant logarithmic 2-jet differentials, Siu's slanted vector fields, Demailly-Semple tower.

Appendix C was authored by Pengchao Wang.

theory. Specifically, we aim to establish a Second Main Theorem type estimate:

$$(1.1) \quad T_f(r) \leq c \sum_{i=1}^q N_f^{[k]}(r, \mathcal{C}_i) + o(T_f(r)) \quad \parallel,$$

where

$$N_f^{[k]}(r, \mathcal{C}_i) := \int_1^r \sum_{|z| < t} \min\{k, \text{ord}_z f^* \mathcal{C}_i\} \frac{dt}{t} \quad (i = 1, \dots, q; r > 1)$$

are the k -truncated counting functions, measuring the frequency of intersections $f(\mathbb{C}) \cap \mathcal{C}_i$; and

$$T_f(r) := \int_1^r \frac{dt}{t} \int_{|z| < t} f^* \omega_{\text{FS}} \quad (r > 1)$$

is the *order function*, capturing the area growth rate of the holomorphic discs $f(\{|z| < t\})$ with respect to the Fubini-Study metric ω_{FS} on $\mathbb{P}^2$. The constant c on the right-hand side of (1.1) is independent of r , and the notation “ $\parallel$ ” indicates that the inequality (1.1) holds for all $r > 1$ outside a set of finite Lebesgue measure.

When $\mathcal{C}$ has at least 4 irreducible components, such a Second Main Theorem type estimate with truncation at level 2 was established by H. Cartan [7] in the linear case where $\mathcal{C}_i$ are lines:

$$(q-3)T_f(r) \leq \sum_{i=1}^q N_f^{[2]}(r, \mathcal{C}_i) + o(T_f(r)) \quad \parallel.$$

In the nonlinear case, analogous results (without truncation) were obtained by Eremenko and Sodin [27] via pluripotential theory, and by Ru [60] through a synthesis of Cartan’s method and Diophantine geometry. These results also extend to the corresponding higher-dimensional setting (see [51, 61]).

The above estimate becomes vacuous when $q = 3$, as the factor $(q-3)$ vanishes; thus the case of three curves lies precisely on the borderline where classical methods break down. From the viewpoint of the logarithmic Bloch–Ochiai theorem, the condition $q > 3$ was originally required to guarantee algebraic degeneracy of every entire curve $f : \mathbb{C} \rightarrow \mathbb{P}^2 \setminus \bigcup_{i=1}^q \mathcal{C}_i$ (see the series of works by Noguchi [47, 48, 49]).

In the prior work [10], the authors established a Second Main Theorem for $n+1$ hypersurfaces $\{D_i\}_{i=1}^{n+1}$ in $\mathbb{P}^n$ in general position and intersecting transversally, proving the defect relation

$$(1.2) \quad \sum_{i=1}^{n+1} \delta_f(D_i) < n+1$$

for any algebraically nondegenerate entire curve $f : \mathbb{C} \rightarrow \mathbb{P}^n$ under the optimal total degree assumption $\sum_{i=1}^{n+1} \deg D_i \geq n+2$. Here, each defect $\delta_f(D_i)$ is defined by

$$\delta_f(D_i) := 1 - \limsup_{r \rightarrow \infty} \frac{N_f(r, D_i)}{\deg(D_i) T_f(r)}.$$

The proof combined parabolic Nevanlinna theory [55] with the Second Main Theorems for semi-Abelian varieties [52] and for $n+2$ hypersurfaces [59]. Subsequently, Ru and Wang [62] extended the result (1.2) to moving targets with truncation level 1, while preserving the upper bound $n+1$ on the right-hand side (see also [63]).

In the case of $n = 2$, the defect relation (1.2) might be viewed as the weakest, albeit nontrivial, form of a Second Main Theorem in the vein of (1.1) with noneffective $c < \infty$. The primary objective of this article is to establish an effective Second Main Theorem, with truncation level 1, for three generic conics with an explicit constant $c = 5$.

1.2. Two Key Theorems. The pursuit of a Second Main Theorem encounters a fundamental obstacle: the very nature of entire curves exhibits an inherent and wild flexibility in large-scale geometry, as evidenced by the exotic examples in [38, 73, 71, 9]. Our strategy constrains this flexibility by leveraging additional algebro-geometric structures to extract the underlying rigidity of entire curves, building on the following two key results.

Theorem 1.1 (McQuillan [41]). *Let $\mathcal{C} = \sum_{i=1}^q \mathcal{C}_i$ be a normal crossings divisor in $\mathbb{P}^2$, where each $\mathcal{C}_i$ is a smooth algebraic curve of degree d_i with $d = \sum_{i=1}^q d_i \geq 4$. Let $f : \mathbb{C} \rightarrow \mathbb{P}^2$ be an algebraically nondegenerate entire holomorphic curve. If the lifting $f_{[1]} : \mathbb{C} \rightarrow X_1 = \mathbb{P} T_{\mathbb{P}^2}(-\log \mathcal{C})$ is algebraically degenerate, then the following Second Main Theorem type estimate holds:*

$$(d-3)T_f(r) \leq N_f^{[1]}(r, \mathcal{C}) + o(T_f(r)) \quad \parallel.$$

Theorem 1.2 ([37, Theorem 3.1]). *Let $\mathcal{C} = \sum_{i=1}^q \mathcal{C}_i$ be a normal crossings divisor in $\mathbb{P}^2$. For any logarithmic jet differential*

$$\omega \in H^0(\mathbb{P}^2, E_{k,m}^{GG} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)) \quad (k, m, t \geq 1),$$

and any entire holomorphic curve $f: \mathbb{C} \rightarrow \mathbb{P}^2$ such that $f^ \omega \not\equiv 0$, the following Second Main Theorem type estimate holds:*

$$(1.3) \quad T_f(r) \leq \frac{m}{t} \sum_{i=1}^q N_f^{[1]}(r, \mathcal{C}_i) + O(\log T_f(r) + \log r) \quad \parallel.$$

Here, $E_{k,m}^{GG} T_{\mathbb{P}^2}^*(\log \mathcal{C})$ denotes the logarithmic Green–Griffiths k -jet bundle of weighted order m (see Section 2). The use of such higher-order jet differentials in the study of complex hyperbolicity goes back to André Bloch [3]. In the present work, we restrict attention to the “invariant” subbundle $E_{k,m} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \subset E_{k,m}^{GG} T_{\mathbb{P}^2}^*(\log \mathcal{C})$, originally introduced by Demailly [17]. This subbundle possesses stronger positivity properties, making it particularly suitable for constructing sections; further details can be found in Section 2.

Theorem 1.2 can be seen as a quantitative refinement of the Fundamental Vanishing Theorem for entire curves (cf. [30, pp. 55-59], [66, Lemma 1.2] and [17, Corollary 7.9]). Although the original statement in [37] is formulated for hypersurfaces in $\mathbb{P}^n$, the proof extends directly to the case of simple normal crossing divisors in projective manifolds (cf. [61, Theorem A10.7.2]).

1.3. Our First Result. The first result of this article is an application of Theorem 1.2 in the case $k = 1$.

Theorem 1.3. *Let $\mathcal{C} = \cup_{i=1}^3 \mathcal{C}_i$ be a union of three smooth curves $\mathcal{C}_i$ of degrees d_i with simple normal crossings. Assuming that $d_1 \geq d_2 \geq d_3 \geq 2$ and $d_1 \geq 3$, so that the total degree $d = d_1 + d_2 + d_3 \geq 7$. For a nonconstant algebraically nondegenerate entire holomorphic curve $f: \mathbb{C} \rightarrow \mathbb{P}^2$ whose image is not contained in any $\mathcal{C}_i$, one has the estimate*

$$T_f(r) \leq c \sum_{i=1}^3 N_f^{[1]}(r, \mathcal{C}_i) + o(T_f(r)) \quad \parallel,$$

for any positive constant c satisfying

$$c > \frac{1}{(d-3)\delta_1}, \quad \text{where} \quad \delta_1 = \frac{1}{2} - \frac{\sqrt{9(d-1)^2 - 12(d_1 d_2 + d_2 d_3 + d_3 d_1)}}{6(d-3)}.$$

Here, the numerical conditions on d_1, d_2, d_3 are imposed to ensure the positivity of the Euler characteristic of the logarithmic symmetric differential bundles $S^m T_{\mathbb{P}^2}^*(\log \mathcal{C})$. Specifically, we have:

$$\chi(S^m T_{\mathbb{P}^2}^*(\log \mathcal{C})) = L_{d_1 d_2 d_3} \cdot m^3 + O(m^2) > 0, \quad \forall m \gg 1,$$

where $L_{d_1 d_2 d_3}$ is a positive leading coefficient.

Remark 1.4. In the case where $\mathcal{C}$ consists of two conics and one cubic, a direct computation shows that δ_1 takes the value $\frac{6-\sqrt{33}}{12}$. This leads to a Second Main Theorem of the form

$$T_f(r) \leq c \sum_{i=1}^3 N_f^{[1]}(r, \mathcal{C}_i) + o(T_f(r)) \quad \parallel,$$

valid for any positive constant c satisfying $c > \frac{3}{6-\sqrt{33}} \approx 11.74 \dots$. This quantifies the hyperbolicity result of $\mathbb{P}^2 \setminus \mathcal{C}$ due to Dethloff–Schumacher–Wong [22].

1.4. Proof of Theorem 1.3. Let f be algebraically nondegenerate, as in the statement of the theorem. The key step is to find a nonzero global negatively twisted logarithmic symmetric differential form

$$(1.4) \quad \omega \in H^0(\mathbb{P}^2, S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)), \quad m \geq 1, t \geq 1, m/t \leq c, \omega \not\equiv 0.$$

Proof of Theorem 1.3 assuming (1.4). If $f^* \omega \not\equiv 0$, then Theorem 1.2 yields the desired estimate. On the other hand, if $f^* \omega \equiv 0$, then $f_{[1]}(\mathbb{C})$ lies in the proper zero divisor of ω on X_1 . We can therefore apply McQuillan’s Theorem 1.1 to obtain the stronger inequality

$$(d-3)T_f(r) \leq N_f^{[1]}(r, \mathcal{C}) + o(T_f(r)) \quad \parallel.$$

In either case, the conclusion of the theorem holds. $\square$

The proof of (1.4) will be given in Section 3, based on Riemann–Roch computations and a Bogomolov-type vanishing theorem (Theorem 2.1).

1.5. Our Second, Main Result: Second Main Theorem for Three Conics. Having established an effective Second Main Theorem for configurations of large degree, we now turn to the more challenging case of three generic conics $\mathcal{C}$ in $\mathbb{P}^2$. Put

$$E := T_{\mathbb{P}^2}^*(\log \mathcal{C}).$$

The residue sequence gives $c_1(E) = 3H$ and $c_2(E) = 9H^2$. Consequently, Hirzebruch–Riemann–Roch yields the exact formula

$$(1.5) \quad \chi(S^m E(-t)) = \frac{m+1}{2}((t-1)(t-2) - 3mt).$$

In particular, $\chi(S^m E) = m+1$; thus the untwisted Euler characteristic does not have a negative cubic leading term. The authors formulated the corresponding one-jet vanishing problem, which is settled completely by the following theorem.

Theorem 1.5 (One-jet vanishing for three conics). *For a generic simple-normal-crossing configuration of three smooth conics $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ in $\mathbb{P}^2$, the logarithmic cotangent bundle*

$$E = T_{\mathbb{P}^2}^*(\log \mathcal{C})$$

is nef. Moreover, for every pair of integers $m, t \geq 1$,

$$(1.6) \quad H^0(\mathbb{P}^2, S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)) = 0.$$

The proof is given in Appendix B. It is instructive to compare the theorem with known borderline cases. For certain special configurations of six lines, negatively twisted logarithmic symmetric differentials do exist; see [34, p. 133] and [69, Theorem 1.4]. For a generic sextic curve, they do not exist [25, Lemma 1.4.1]. Theorem 1.5 shows that three generic conics satisfy the stronger vanishing for every negative twist. Thus the one-jet route is genuinely unavailable in this low-degree setting. Invariant logarithmic 2-jets are therefore not merely a technical convenience: they are the first possible jet order for the negatively twisted jet-differential strategy. This necessity gives the two-jet construction in the present paper additional geometric significance. This is a precise instance of the principle that “the lower the degree, the harder the hyperbolicity” [23, p. 3]. Our main result is as follows.

Theorem 1.6 (Second Main Theorem for Three Conics). *For a generic configuration of three conics $\{\mathcal{C}_i\}_{i=1}^3$ in $\mathbb{P}^2$ and a nonconstant algebraically nondegenerate entire holomorphic curve $f: \mathbb{C} \rightarrow \mathbb{P}^2$ whose image is not contained in any $\mathcal{C}_i$, the following inequality holds:*

$$(1.7) \quad T_f(r) \leq 5 \sum_{i=1}^3 N_f^{[1]}(r, \mathcal{C}_i) + o(T_f(r)) \quad \parallel.$$

Remark 1.7. The constant $c = 5$ in this theorem is smaller than the constant ≈ 11.74 appearing in Theorem 1.3 for the case of two conics and one cubic. This may seem counterintuitive, since the smaller total degree $2 + 2 + 2 < 2 + 2 + 3$ would normally suggest a larger constant. Theorem 1.5 shows that negatively twisted 1-jets do not exist for three generic conics; the improvement therefore comes from the more refined invariant 2-jet method. A detailed 2-jet analysis is likewise expected to sharpen the constants for the two-conics-and-one-cubic case and for the other configurations covered by Theorem 1.3.

Remark 1.8. An unexpected computational byproduct appears for the three Fermat-type conics $\mathcal{C}$ defined by the quadratic polynomials

$$2Z_0^2 + Z_1^2 + Z_2^2, \quad Z_0^2 + 2Z_1^2 + Z_2^2, \quad Z_0^2 + Z_1^2 + 2Z_2^2.$$

For this explicit configuration, the finite calculation detects a 9-dimensional space of negatively twisted invariant logarithmic 2-jets of weighted degree 5 and twist -1 ; see Remark 7.5. Together with the preceding vanishings in weighted degrees 1, 2, 3, 4, this forces the space to contain two algebraically independent members and hence yields a Second Main Theorem with constant 5 for this Fermat-type configuration. This byproduct is not used in the proof of the Main Theorem, whose constant is instead governed by the finite set of vanishings in Key Vanishing Lemma II.

We now outline the main ideas underlying the proof of Theorem 1.6 for a finite constant $c < \infty$, before refining the argument to achieve the sharper goal $c = 5$. Let f be algebraically nondegenerate, as in Theorem 1.6.

The core idea is to construct two independent negatively twisted invariant logarithmic 2-jet differentials ω_1 and ω_2 :

- If $f^*\omega_1 \neq 0$ or $f^*\omega_2 \neq 0$, then Theorem 1.2 directly gives the desired Second Main Theorem type estimate.
- If both pullbacks vanish, then their independence forces the lifting $f_{[1]}(\mathbb{C}) \subset X_1$ to be algebraically degenerate. Hence McQuillan's Theorem 1.1 applies.

In either case, we obtain the desired Second Main Theorem type estimate.

In the numerical range where differentiation preserves a negative twist, Siu's slanted vector fields realize this idea directly: the natural action of an invariant vector field on the universal logarithmic jet space sends ω_1 to a genuine global invariant logarithmic 2-jet differential ω_2 . When this direct construction is not numerically available, we retain Miyaoka's zero-locus argument (see also Lu–Yau), in the jet-differential form used by Demailly–El Goul [45, 40, 18], as a logically necessary fallback. That fallback is not the conceptual origin of the method; sufficiently many additional Key Vanishing Lemmas would eliminate it.

The central difficulty is to control the common base locus of the resulting jet differentials. Such constructions are available in high degree, as in Siu–Yeung [67], but are far more difficult in low degree because of the algebraic complexity of jet differentials and the ensuing elimination problem; compare [42]. For three generic conics, Theorem 1.5 rules out the negatively twisted 1-jet alternative altogether.

A notable illustration of these difficulties arises even in the 1-jet case. Debarre [16] conjectured that for a generic choice of high-degree hypersurfaces $H_1, \dots, H_c \subset \mathbb{P}^N$ with $c \geq N/2$, the complete intersection $X := H_1 \cap \dots \cap H_c$ should have an ample cotangent bundle T_X^* , which is equivalent to the existence of sufficiently many negatively twisted 1-jet differentials on X with empty common base locus. The first proof of Debarre's ampleness conjecture was established by Xie [72], using explicit 1-jet differentials obtained by Brotbek [5] for Fermat-type hypersurfaces. A key step in Xie's approach is a core lemma [72, p. 969] that overcomes substantial difficulties in the elimination process. Shortly thereafter, Brotbek and Darondeau [6] found an alternative proof, which instead applies Nakamaye's theorem [46] to tackle the elimination.

1.6. Roadmap for Theorem 1.6: Towards an Effective Finite Constant $c < \infty$. We now outline the proof of our main result, focusing first on establishing an explicit finite constant c in the inequality (1.7). The argument proceeds in several steps.

First, for generic three conics $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$, standard Riemann–Roch computations (detailed in Section 4) guarantee the existence of an irreducible, negatively twisted invariant logarithmic 2-jet differential

$$(1.8) \quad \omega_1 \in H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)),$$

with weighted degree $m \geq 1$, vanishing order $t \geq 1$.

If $f^*\omega_1 \neq 0$, then Theorem 1.2 immediately yields the desired Second Main Theorem with a finite constant c . We may therefore assume $f^*\omega_1 \equiv 0$ in what follows.

The subsequent strategy depends on the values of (m, t) :

Case 1: $t > 7$. We apply Siu's strategy [65] of slanted vector fields, as realized in dimension two by Păun [54] (compact case) and Rousseau [58] (logarithmic case). On the universal logarithmic jet space, an invariant pole-order-7 field acts naturally on the equivariant polynomial representing ω_1 and produces a genuine global invariant logarithmic 2-jet differential

$$\omega_2 \in H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(7-t)).$$

The field can be chosen so that ω_1 and ω_2 are independent modulo Γ_2 . Theorem 1.2 applied to ω_2 gives the coefficient $m/(t-7)$; if both pullbacks vanish, their independence forces $f_{[1]}$ to be algebraically degenerate. See Section 5.

Case 2: $t < m$. Following Miyaoka's zero-locus method (see also Lu–Yau), in the form developed by Demailly–El Goul [45, 40, 18] (see Section 6), Riemann–Roch computations give a nonzero negatively twisted section on the residual zero divisor of ω_1 in the second level X_2 of the Demailly–Semple tower; see Proposition 6.2. We then distinguish two subcases:

- If the canonical evaluation of the restricted section along $f_{[2]}$ is nonzero, we derive a Second Main Theorem-type estimate from the intrinsic tautological evaluation and the partition-of-unity argument of [37, Theorem 3.1]; see Theorem 6.5.
- If the canonical evaluation is identically zero, Lemma 6.6 shows that the lifting $f_{[1]}(\mathbb{C})$ is algebraically degenerate in the first level X_1 of the logarithmic Demailly–Semple tower for $(\mathbb{P}^2, \mathcal{C})$, and hence McQuillan's Theorem 1.1 applies.

Case 3: The Remaining Possibilities. For the remaining pairs (m, t) not covered by the previous two cases, we establish the following Key Vanishing Lemma I to rule out the existence of any nonzero ω_1 in these settings. Note that vanishing for (m, t) implies vanishing for all (m, t') with $t' > t$. Thus we complete the proof of a Second Main Theorem for three generic conics with a finite constant c . Moreover, careful numerical computation shows that we can take $c = 19$; see Subsection 6.3.

1.7. New Vanishing Results and Proof Strategy.

Key Vanishing Lemma I. *For a generic configuration of three conics $\mathcal{C} = \cup_{i=1}^3 \mathcal{C}_i$ in $\mathbb{P}^2$, one has:*

$$H^0(\mathbb{P}^2, E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)) = 0$$

for all (m, t) in the set $\{(3, 3), (4, 4), (5, 5), (6, 6), (7, 7)\}$.

While such vanishing results have long been recognized as fundamental in complex hyperbolicity theory, a proof has remained out of reach for lack of a viable method. The decisive innovation is a new exponent bound that converts the a priori infinite extension problem into finitely many exact linear-algebra calculations.

The project began with the question of whether three conics admit an effective Second Main Theorem and with an attempt to use slanted vector fields of pole order 3. After substantial work, that route did not yield a complete argument; moreover, lowering the pole order would not by itself remove the Key Vanishing obstruction. The present proof therefore uses the classical pole-order-7 package and resolves the remaining finite obstruction through the new exponent bound.

We later observed, with the benefit of the pole-lowering ideas developed in the subsequent papers [14, 36], that a pole-order-4 version can also be made to close. We do not pursue that variant here: the pole-order-7 argument keeps the exposition focused on the genuinely new ingredient of the present paper, namely the exponent bound and the resulting finite Key Vanishing problem. Lowering the pole order changes the finite list, but does not eliminate the need to prove its members.

Each case in Key Vanishing Lemma I follows from the corresponding stronger case in Key Vanishing Lemma II by twist monotonicity. The substantive finite verification is therefore the proof of Key Vanishing Lemma II, presented in Subsection 7.2. It has three steps:

- **All-layer finite shape:** The exponent bound is established uniformly in the Wronskian layer q . It gives a finite homogeneous polynomial ansatz for every coefficient, including all layers required when m is large.
- **Exact C++ verification:** For explicit Fermat-type conics, a C++ program constructs the complete coupled transition matrix over $\mathbb{F}_5$ and proves full column rank by exact sparse elimination.
- **Geometric extension:** Upper semicontinuity globalizes the verified vanishing from this explicit configuration to a nonempty Zariski-open set of triples of conics.

Remark 1.9. An alternative approach to establishing such vanishing results is through cohomological diagram chasing based on the filtration structure (2.3). This method enables systematic construction of long exact sequences in cohomology, thereby facilitating stepwise computation of $H^0(\mathbb{P}^2, E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}(-t))$. When $m = t$ is not divisible by 3, the vanishing reduces to negatively twisted logarithmic 1-jet differentials. This observation is due to Pengchao Wang; see Appendix C. Theorem 1.5 supplies all the required symmetric-power vanishings. Consequently, Proposition C.2 gives, in particular, the cases $(m, t) = (4, 4), (5, 5), (7, 7)$ in Key Vanishing Lemma I.

However, in more general cases—such as when $m > t$ or when 3 divides $m = t$ —this diagram-chasing approach becomes substantially more complicated and its feasibility remains uncertain. Determining the essential information required for the diagram chasing presents a significant challenge. In particular, computing H^i for $i = 0, 1, 2$ for each term in the long exact sequence requires either analyzing the connecting homomorphisms from H^i to H^{i+1} , or computing other H^i groups that typically do not vanish, thereby introducing considerable additional complexity.

Remark 1.10. The significance of the present method extends well beyond the numerical constant in the Second Main Theorem. To the best of our knowledge, this is the first effective Second Main Theorem in higher-dimensional Nevanlinna theory whose decisive input is a pair of algebraically independent, negatively twisted invariant logarithmic 2-jet differentials, rather than a Cartan-type Wronskian construction. Here the distinction concerns the proof mechanism: the local algebra of invariant 2-jets still contains its

intrinsic Wronskian generator, but the Second Main Theorem is not produced by the classical Cartan–Wronskian method. This introduces a new algebraic-geometric mechanism for effectiveness. Equally importantly, the exponent bound introduced here reduces the extension problem to finite exact algebra and became a foundational input in the subsequent papers [35, 14]. In those works, the compact threshold case $d = 15$ and the logarithmic threshold case $d = 11$ are reduced to explicit finite rank-verification problems. The corresponding computer calculations are now in progress; no further conceptual ingredient is required, so reaching the two threshold degrees predicted by the 2-jet method is now a matter of computation time.

1.8. Proof Blueprint of Theorem 1.6 with $c = 5$. We first show that in the existence result for ω_1 in (1.8), the ratio t/m can be made close to $\frac{4-\sqrt{10}}{3}$; see Proposition 4.2. If $f^*\omega_1 \neq 0$, then Theorem 1.2 yields the estimate (1.7) with constant c around $\frac{3}{4-\sqrt{10}} = 3.58\dots$. This value represents the theoretical limit of our current approach. Our goal of achieving $c = 5$ is strategically set near this limit.

In **Case 1** ($t > 7$), the original differential ω_1 gives the coefficient m/t , while the global differential ω_2 produced by a pole-order-7 slanted vector field gives the coefficient $m/(t-7)$. Requiring both to be at most 5 is equivalent to $t \geq m/5 + 7$.

In **Case 2** ($t < m$), we refine Demailly–El Goul’s approach [18] to estimate the ratio of the vanishing order of ω_2 to its weighted degree. Through the analysis in Section 6, we show that the restricted-section argument gives any constant $c > 3/\tau_1(\frac{t}{m})$, where

$$\tau_1(\frac{t}{m}) = \frac{3}{2} \left(4 - \frac{t}{m}\right) - \frac{\sqrt{3}}{2} \sqrt{32 - 8\frac{t}{m} + 3\left(\frac{t}{m}\right)^2}.$$

A direct computation shows that $3/\tau_1(\frac{t}{m}) < 5$ is equivalent to $t/m < 43/85$.

Although most positive integer pairs (m, t) satisfy either $t/m < 43/85$ or $t \geq m/5 + 7$, exceptional cases remain where neither condition holds. To address these, we need the following strengthened vanishing results:

Key Vanishing Lemma II. *For a generic configuration of three conics $\mathcal{C} = \cup_{i=1}^3 \mathcal{C}_i$ in $\mathbb{P}^2$, one has:*

$$(1.9) \quad H^0(\mathbb{P}^2, E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)) = 0$$

for all

$$(m, t) \in \{(3, 2), (4, 3), (5, 3), (6, 4), (7, 4), (8, 5), (9, 5), \\ (10, 6), (11, 6), (12, 7), (13, 7), (14, 8), (15, 8), (16, 9), \\ (17, 9), (18, 10), (19, 10), (21, 11)\}.$$

The proof in Subsection 7.2 first establishes the uniform all-layer finite shape. It then specializes to Fermat-type conics and proves, by exact full-coupled sparse elimination over $\mathbb{F}_5$, that the resulting integral matrix has trivial kernel.

1.9. Structure of the article. Section 2 provides necessary background. Section 3 presents detailed Riemann–Roch computations that form the core of the proof of Theorem 1.3. Section 4 advances to more sophisticated Riemann–Roch computations, specifically designed to establish the existence of negatively twisted invariant logarithmic 2-jet differentials for three generic conics. Section 5 details the application of Siu’s slanted vector fields in **Case 1**. Section 6 then introduces a refinement of the Demailly–El Goul argument, adapted to handle **Case 2**. The main innovation of the article is contained in Section 7, where we overcome fundamental obstructions in our low-degree setting by establishing Key Vanishing Lemmas. Appendix A records the explicit quotient polynomials in the four factorization identities, Appendix B proves the one-jet vanishing theorem, and Appendix C presents Pengchao Wang’s cohomological observation.

2. Preliminaries

2.1. Logarithmic Jet Bundles and Invariant Differential Forms. Let X be an n -dimensional complex manifold. The k -th jet bundle $J_k(X)$ is defined by considering holomorphic germs $f: (\mathbb{C}, 0) \rightarrow (X, x)$, where two germs are considered equivalent if their Taylor expansions coincide up to order k in some local coordinate system. The equivalence class of a germ f , denoted by $j_k(f)$, is said to be regular if $f'(0) \neq 0$.

The total space $J_k(X) = \cup_{x \in X} J_k(X)_x$ carries a natural topology and, together with the natural projection $\pi_k: J_k(X) \rightarrow X$, forms a holomorphic fiber bundle. Note that $J_1(X)$ coincides with the tangent bundle T_X .

Now let $D \subset X$ be a normal crossing divisor. The logarithmic cotangent bundle $T_X^*(\log D)$ is a locally free sheaf generated by meromorphic 1-forms with at most logarithmic poles along D . Precisely, for $x \in X \setminus D$, the stalk $T_{X,x}^*(\log D)$ coincides with the stalk $T_{X,x}^*$ of the usual cotangent sheaf T_X^* at x . For $x \in D$, take a neighborhood U of x and irreducible holomorphic functions $\eta_1, \dots, \eta_\ell$ on U such that $D \cap U = \{\eta_1 \cdots \eta_\ell = 0\}$; then the stalk is given by $T_{X,x}^*(\log D) = \sum_{j=1}^\ell \mathcal{O}_{X,x} \frac{d\eta_j}{\eta_j} + T_{X,x}^*$. Its dual, the logarithmic tangent bundle $T_X(-\log D)$, consists of holomorphic vector fields tangent to each component of D (cf. [50, 25]).

Consider an open subset $U \subset X$, a section $\omega \in H^0(U, T_X^*(\log D))$, and a jet $j_k(f)$ in $J_k(X)|_U$. The pullback $f^*\omega$ takes the form $A(z)dz$ for some meromorphic function A , and the derivatives $A^{(j)}(0)$ for $0 \leq j \leq k-1$ depend only on the jet $j_k(f)$. This construction induces a well-defined map:

$$(2.1) \quad \tilde{\omega}: J_k(X)|_U \rightarrow (\mathbb{C} \cup \{\infty\})^k, \quad j_k(f) \mapsto (A(0), A'(0), \dots, A^{(k-1)}(0)).$$

A holomorphic section $s \in H^0(U, J_k(X))$ is called a logarithmic k -jet field on U if for every open subset $U' \subset U$ and every $\omega \in H^0(U', T_X^*(\log D))$, the composition $\tilde{\omega} \circ s$ is holomorphic on U' , where $\tilde{\omega}$ is defined as in (2.1). These logarithmic k -jet fields form a subsheaf of $J_k(X)$, which corresponds to the sections of a holomorphic fiber bundle over X known as the logarithmic k -jet bundle along D , denoted $J_k(X, -\log D)$ (cf. [50, 25, 58]).

Furthermore, given a local basis $\omega_1, \dots, \omega_n$ of $T_X^*(\log D)$ on U , we obtain a trivialization:

$$H^0(U, J_k(X, -\log D)) \rightarrow U \times (\mathbb{C}^n)^k, \quad \sigma \mapsto (\pi_k \circ \sigma; \tilde{\omega}_1 \circ \sigma, \dots, \tilde{\omega}_n \circ \sigma).$$

A logarithmic jet differential of order k and weighted degree m at a point $x \in X$ is a polynomial $Q(f^{(1)}, \dots, f^{(k)})$ on the fiber of $J_k(X, -\log D)$ over x , satisfying the weighted homogeneity condition:

$$Q(\lambda f', \lambda^2 f'', \dots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \dots, f^{(k)}),$$

for all $\lambda \in \mathbb{C}^*$ and all $(f', f'', \dots, f^{(k)}) \in J_k(X, -\log D)_x$.

The vector space $E_{k,m}^{GG} T_X^*(\log D)_x$ consists of all such jet differential polynomials defined at the point $x \in X$. Then the total space

$$E_{k,m}^{GG} T_X^*(\log D) := \cup_{x \in X} E_{k,m}^{GG} T_X^*(\log D)_x,$$

acquires the structure of a holomorphic vector bundle over X by Faà di Bruno's formula (cf. e.g. [12, 44]), referred to as the logarithmic Green–Griffiths bundle of jet differentials of order k and weighted degree m .

Let $\mathbb{G}_k$ denote the group of germs of k -jets of biholomorphisms of $(\mathbb{C}, 0)$:

$$\mathbb{G}_k := \left\{ \varphi: t \mapsto \sum_{j=1}^k c_j t^j \mid c_1 \in \mathbb{C}^*, c_j \in \mathbb{C} \text{ for } j \geq 2 \right\},$$

where the group operation is defined by composition modulo t^{k+1} . This group acts naturally on $J_k(X)$ via the right action: $\varphi \cdot j_k(f) := j_k(f \circ \varphi)$.

A logarithmic jet differential operator $P \in \Gamma(U, E_{k,m}^{GG} T_X^*(\log D))$ over an open set $U \subset X$ is said to be invariant under the reparametrization group $\mathbb{G}_k$ if for every $\varphi \in \mathbb{G}_k$ the following identity holds:

$$(2.2) \quad P((f \circ \varphi)', (f \circ \varphi)'', \dots, (f \circ \varphi)^{(k)}) = (\varphi')^m \cdot P(f', f'', \dots, f^{(k)}).$$

The subbundle $E_{k,m} T_X^*(\log D) \subset E_{k,m}^{GG} T_X^*(\log D)$ is then defined as the holomorphic vector bundle whose fibers consist of all such invariant logarithmic jet differential polynomials.

This notion was initially introduced by Demailly [17] in the compact case, and subsequently generalized to the logarithmic case by Dethloff–Lu [20].

Throughout this article, we only concern the situation $k = 1$ or 2 . In the case that $\dim_{\mathbb{C}} X = 2$, the bundle of invariant logarithmic jet differentials of order 2 admits a filtration structure with the grading [17]:

$$(2.3) \quad \text{Gr}^\bullet E_{2,m} T_X^*(\log D) = \bigoplus_{0 \leq j \leq \lfloor m/3 \rfloor} S^{m-3j} T_X^*(\log D) \otimes \bar{K}_X^j,$$

where $\bar{K}_X$ is the logarithmic canonical line bundle $K_X \otimes \mathcal{O}_X(D)$. This filtration corresponds to the expression of an invariant logarithmic 2-jet differential $P(f', f'')$ outside D as:

$$P(f', f'') = \sum_{0 \leq j \leq \lfloor m/3 \rfloor} \sum_{\substack{\alpha_1 + \alpha_2 = m-3j \\ \alpha_1, \alpha_2 \geq 0}} R_{\alpha_1, \alpha_2, j}(f_1, f_2) \cdot (f_1')^{\alpha_1} (f_2')^{\alpha_2} (f_1' f_2'' - f_2' f_1'')^j,$$

where $f = (f_1, f_2)$ represents the holomorphic germs in local coordinates, and where $R_\bullet$ are holomorphic functions.

2.2. Logarithmic Demailly-Semple Tower. The main reference of this subsection is [17, 20].

For simplicity, consider the log-direct manifold (X_0, D_0, V_0) associated with three generic conics $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ in the complex projective plane $\mathbb{P}^2$, where $X_0 = X := \mathbb{P}^2$, $D_0 := \mathcal{C} = \sum_{i=1}^3 \mathcal{C}_i$ is a simple normal crossing divisors on X whose irreducible components $\mathcal{C}_i$ are smooth conics and $V_0 := T_X(-\log D_0)$. Define $X_1 := \mathbb{P}(V_0)$ with the natural projection $\pi_{1,0} : X_1 \rightarrow X_0$. Set $D_1 := \pi_{1,0}^* D_0$. The subbundle $V_1 \subset T_{X_1}(-\log D_1)$ is defined fiberwise by:

$$(2.4) \quad V_{1,(x,[v])} := \{\xi \in T_{X_1,(x,[v])}(-\log D_1) \mid (\pi_{1,0})_* \xi \in \mathbb{C} \cdot v\},$$

with $\mathbb{C} \cdot v \subset V_{0,x} = T_{X_0,x}(-\log D_0)$. Thus we obtain the first level (X_1, D_1, V_1) of the logarithmic Demailly-Semple tower for $(\mathbb{P}^2, \mathcal{C})$.

Proceeding likewise, we can construct the second level of logarithmic Demailly-Semple tower:

$$(X_2, D_2, V_2) \rightarrow (X_1, D_1, V_1) \rightarrow (X_0, D_0, V_0),$$

together with the projections $\pi_{2,1} : X_2 \rightarrow X_1$ and $\pi_{2,0} : X_2 \rightarrow X_0$.

Let $\mathcal{O}_{X_1}(-1)$ denote the tautological line bundle of $X_1 = \mathbb{P}(V_0)$ such that $\mathcal{O}_{X_1}(-1)_{(x,[v])} = \mathbb{C} \cdot v$. By definition, there is a short exact sequence

$$(2.5) \quad 0 \longrightarrow T_{X_1/X_0} \longrightarrow V_1 \xrightarrow{(\pi_{1,0})_*} \mathcal{O}_{X_1}(-1) \longrightarrow 0,$$

where T_{X_1/X_0} is the relative tangent bundle of the fibration $\pi_{1,0} : X_1 \rightarrow X_0$. Moreover, there is a relative version of the Euler sequence of the tangent bundle of the fiber $\mathbb{P}(V_{0,x})$:

$$(2.6) \quad 0 \longrightarrow \mathcal{O}_{X_1} \longrightarrow \pi_{1,0}^* V_0 \otimes \mathcal{O}_{X_1}(1) \longrightarrow T_{X_1/X_0} \longrightarrow 0.$$

Furthermore, there exists a canonical injection $\mathcal{O}_{X_2}(-1) \hookrightarrow \pi_{2,1}^* V_1$. Combining this with the above sequence yields a natural morphism of line bundles:

$$\mathcal{O}_{X_2}(-1) \hookrightarrow \pi_{2,1}^* V_1 \xrightarrow{\pi_{2,1}(\pi_{1,0})_*} \pi_{2,1}^* \mathcal{O}_{X_1}(-1),$$

whose zero divisor is the hyperplane subbundle:

$$(2.7) \quad \Gamma_2 := \mathbb{P}(T_{X_1/X_0}) \subset \mathbb{P}(V_1) = X_2.$$

Consequently, we can obtain the isomorphism:

$$(2.8) \quad \mathcal{O}_{X_2}(\Gamma_2) \cong \pi_{2,1}^* \mathcal{O}_{X_1}(-1) \otimes \mathcal{O}_{X_2}(1).$$

Let $f : \mathbb{D}_R \rightarrow X_0$ be a nonconstant holomorphic map from the disc $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ of radius $R \in \mathbb{R}_+ \cup \{\infty\}$. To define its liftings to the first level X_1 of the logarithmic Demailly-Semple tower, we consider the following two cases for $f'(t) \in T_{X_0,f(t)}$:

When $0 \neq f'(t) \in T_{X_0,f(t)}(-\log D_0)$, the point $[f'(t)] \in \mathbb{P}(T_{X_0,f(t)}(-\log D_0))$ is well defined;

When $f'(t_0) = 0$ or $f'(t_0) \in T_{X_0,f(t_0)} \setminus T_{X_0,f(t_0)}(-\log D_0)$, we can define $[f'(t_0)] \in \mathbb{P}(T_{X_0,f(t_0)}(-\log D_0))$ as following: near $t_0 \in \mathbb{D}_R$, choose a local frame (e_1, e_2) for $T_{X_0}(-\log D_0)$ in a neighborhood of $f(t_0)$. In this frame, the derivative $f'(t)$ has a local expression $f'(t) = u_1(t)e_1 + u_2(t)e_2$, where $u_1(t), u_2(t)$ are meromorphic functions with at most finite-order poles. Let $\ell \in \mathbb{Z}$ be the minimum vanishing order (equivalently, the negative of the maximal pole order) of $u_1(t)$ and $u_2(t)$ at t_0 . Factoring out $(t - t_0)^\ell$, we can write

$$f'(t) = (t - t_0)^\ell (v_1(t)e_1 + v_2(t)e_2),$$

where $v_1(t)$ and $v_2(t)$ are holomorphic near t_0 and not simultaneously vanishing at t_0 . We define the projective class:

$$[f'(t_0)] := [v_1(t_0) : v_2(t_0)] \in \mathbb{P}(T_{X_0,f(t_0)}(-\log D_0)).$$

This definition is independent of the choice of local frame, so $[f'(t_0)] \in \mathbb{P}(T_{X_0,f(t_0)}(-\log D_0))$ is well defined.

With this convention, the first lifting $f_{[1]} = (f; [f'])$ of f is a well-defined holomorphic map from $\mathbb{D}_R$ to X_1 . Repeating the process for $f_{[1]}$, the second lifting $f_{[2]} = (f_{[1]}; [f'_{[1]}])$ of f is also well-defined and holomorphic from $\mathbb{D}_R$ to X_2 .

Moreover, from the definition of Γ_2 and the identity $\pi_{1,0} \circ f_{[1]} = f$, it follows that $f_{[2]}(t) \in \Gamma_2$ if and only if $(\pi_{1,0})_* f'_{[1]}(t) = f'(t) = 0$.

Demailly [17, Corollary 5.12] observed that, for any $w_0 \in X_2$, there exists an open neighborhood $U_{w_0} \subset X_2$ such that for all the $w \in U_{w_0}$, one can find a holomorphic family of germs $f_w: (\mathbb{C}, 0) \rightarrow U$ such that $(f_w)_{[2]}(0) = w$ and $(f_w)'_{[1]}(0) \neq 0$. By the construction of the Demailly–Sample tower, we have $f'_w(t) \in \mathcal{O}_{X_1}(-1)_{(f_w)_{[1]}(t)}$ and $(f_w)'_{[1]}(t) \in \mathcal{O}_{X_2}(-1)_{(f_w)_{[2]}(t)}$. Then for any $P \in E_{2,m}T_X^*(\log D)_x$ with $x \in \pi_{2,0}(U_{w_0}) \subset X_0$, this induces a holomorphic section $\sigma_P(w)$ of $\mathcal{O}_{X_2}(m)$ over the fiber $X_{2,x}$ defined by:

$$\sigma_P(w) := P(f'_w, f''_w)(0) \left((f_w)'_{[1]}(0) \right)^{-m}.$$

More generally, the following fundamental isomorphism holds:

Direct image formula (cf. [17, Theorem 6.8], [20, Proposition 3.9]). *The direct image sheaf*

$$(2.9) \quad (\pi_{2,0})_* \mathcal{O}_{X_2}(m) \cong \mathcal{O}(E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}))$$

is isomorphic to the sheaf of invariant logarithmic 2-jet differentials of weighted degree m .

2.3. A Bogomolov-type Vanishing Theorem. In the subsequent Sections 3 and 4, the existence of negatively twisted logarithmic 1-jet and 2-jet differentials will be established via Riemann–Roch calculations. This reduces to verifying the vanishing of the corresponding second cohomology groups:

$$H^2(\mathbb{P}^2, S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)) \quad \text{and} \quad H^2(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)),$$

respectively. A key tool in proving these vanishing results is the following Bogomolov-type vanishing theorem.

Theorem 2.1 (cf. [8, Theorem 4.4]). *Let (X, D) be a log pair where X is a smooth projective surface and D is a reduced simple normal crossing divisor. Assume that (X, D) is of general type (i.e., $K_X + D$ is big) and $K_X + D$ is nef. Then*

$$H^0(X, S^p T_X(-\log D) \otimes \mathcal{O}_X(q(K_X + D))) = 0,$$

for all integers p, q satisfying $p > 2q$.

In the compact case, Bogomolov [4] established a seminal vanishing theorem under the assumption that tangent bundle T_X is semi-stable. Later, Enoki [26] showed that T_X is K_X -semi-stable for minimal Kähler space X with big K_X , and Tsuji [68] proved that T_X is K_X -semi-stable for smooth minimal algebraic variety (i.e., K_X is nef). Combining these results, the Bogomolov-type vanishing theorem holds for projective variety X with big and nef K_X ; see [17, Theorem 14.1].

3. Proof of Theorem 1.3

Let $\mathcal{C} = \sum_{i=1}^3 \mathcal{C}_i$ be a simple normal crossing divisor of total degree $d = d_1 + d_2 + d_3$, where $d_i = \deg \mathcal{C}_i$ satisfy $d_1 \geq d_2 \geq d_3 \geq 2$ and $d_1 \geq 3$. To derive an effective Second Main Theorem, we seek nontrivial global sections of the twisted sheaf

$$S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(1)^{-(d-3)\delta m} \cong S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{-\delta m},$$

where $\delta > 0$ is a small rational number to be chosen, and m is taken sufficiently divisible.

Definition 3.1 (cf. [18, p. 517], [25, Definition 1.3.1]). The 1-jet log threshold of $(\mathbb{P}^2, \mathcal{C})$ is defined by

$$\bar{\theta}_1(\mathbb{P}^2, \mathcal{C}) := \sup \left\{ \frac{t}{m} \mid m, t > 0, H^0(\mathbb{P}^2, S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)) \neq 0 \right\}.$$

If no such pair (m, t) exists, the threshold is defined to be zero.

For a generic configuration of three conics, Theorem 1.5 gives $\bar{\theta}_1(\mathbb{P}^2, \mathcal{C}) = 0$.

Proposition 3.2 (Proof of (1.4)). *Let $\mathcal{C} = \cup_{i=1}^3 \mathcal{C}_i$ be a union of three smooth curves $\mathcal{C}_i$ of degrees d_i with simple normal crossings. Assume that $d_1 \geq d_2 \geq d_3 \geq 2$ and $d_1 \geq 3$. Then one has the estimate*

$$0 < \frac{1}{\bar{\theta}_1(\mathbb{P}^2, \mathcal{C})} \leq \frac{1}{(d-3)\delta_1},$$

where

$$(3.1) \quad \delta_1 = \frac{1}{2} - \frac{\sqrt{9(d-1)^2 - 12(d_1 d_2 + d_2 d_3 + d_3 d_1)}}{6(d-3)} > 0.$$

Proof. We follow the standard Riemann-Roch argument (cf. [18, 25]).

By Serre duality, the dual of

$$(3.2) \quad H^2(\mathbb{P}^2, S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{-\delta m})$$

is

$$H^0(\mathbb{P}^2, K_{\mathbb{P}^2} \otimes S^m T_{\mathbb{P}^2}(-\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{\delta m}).$$

Take m sufficiently divisible that δm is an integer. Multiplication by a nonzero section of $\mathcal{O}_{\mathbb{P}^2}(3)$ injects the last sheaf into

$$S^m T_{\mathbb{P}^2}(-\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{\delta m}.$$

For $0 < \delta < 1/2$ one has $m > 2\delta m$. The Bogomolov-type Vanishing Theorem 2.1 therefore annihilates the target, and hence also the Serre-dual group. We obtain

$$H^2(\mathbb{P}^2, S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{-\delta m}) = 0$$

which yields

$$\dim H^0(\mathbb{P}^2, S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{-\delta m}) \geq \chi(S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{-\delta m}),$$

for large m . A standard Riemann-Roch computation gives

$$\begin{aligned} \chi(S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{-\delta m}) &= \frac{1}{6}((m - \delta m)^3 - (-\delta m)^3) \bar{c}_1^2 - \frac{1}{6} m^3 \bar{c}_2 + O(m^2) \\ &= \frac{1}{2} m^3 \left((d-3)^2 \delta^2 - (d-3)^2 \delta + \frac{1}{3} (d_1 d_2 + d_2 d_3 + d_3 d_1) - d + 2 \right) + O(m^2). \end{aligned}$$

The leading coefficient $\varphi(\delta) = (d-3)^2 \delta^2 - (d-3)^2 \delta + \frac{1}{3} (d_1 d_2 + d_2 d_3 + d_3 d_1) - d + 2$ has two positive roots given by

$$\delta_1 = \frac{1}{2} - \frac{\sqrt{9(d-1)^2 - 12(d_1 d_2 + d_2 d_3 + d_3 d_1)}}{6(d-3)}, \quad \delta_2 = \frac{1}{2} + \frac{\sqrt{9(d-1)^2 - 12(d_1 d_2 + d_2 d_3 + d_3 d_1)}}{6(d-3)} > 0.$$

We can check the positivity of δ_1 directly. Hence $\varphi(\delta) > 0$ for all $\delta \in (0, \delta_1)$. Thus we conclude the proof. $\square$

4. Twisted Invariant Logarithmic 2-Jet Differentials and Threshold

To obtain a constant c as small as possible in the Second Main Theorem for three conics $\mathcal{C} \subset \mathbb{P}^2$, by Theorem (1.2), we seek negatively twisted invariant logarithmic 2-jet differentials with relatively high vanishing order t compared to the degree m . This motivates the study of the following threshold:

Definition 4.1 (cf. [18, p. 517], [25, Definition 1.3.1]). The 2-jet log threshold of $(\mathbb{P}^2, \mathcal{C})$ is defined by

$$\bar{\theta}_2(\mathbb{P}^2, \mathcal{C}) := \sup \left\{ \frac{t}{m} \mid m, t > 0, H^0(X_2, \mathcal{O}_{X_2}(m) \otimes \pi_{2,0}^* \mathcal{O}_X(-t)) \neq 0 \right\}.$$

If no such pair (m, t) exists, the threshold is defined to be zero.

If $\bar{\theta}_2 > 0$, then for any sufficiently small $\varepsilon > 0$, there exists a negatively twisted logarithmic 2-jet differential

$$\omega \in H^0(\mathbb{P}^2, E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t))$$

with ratio $\bar{\theta}_2 - \varepsilon < \frac{t}{m} \leq \bar{\theta}_2$. Furthermore, if $f^* \omega \neq 0$ also holds, then Theorem 1.2 yields the Second Main Theorem with constant

$$\frac{1}{\bar{\theta}_2} \leq c = \frac{m}{t} < \frac{1}{\bar{\theta}_2 - \varepsilon}.$$

The significance of the 2-jet log threshold stems from its role as the fundamental barrier that dictates the possibility of constructing jet differentials with a prescribed ratio of vanishing orders to weighted degrees.

The following result establishes an explicit upper bound for $\frac{1}{\bar{\theta}_2}$ in the case of three smooth conics with simple normal crossings:

Proposition 4.2. *Let $\mathcal{C} = \cup_{i=1}^3 \mathcal{C}_i$ be a union of three smooth conics $\mathcal{C}_i$ with simple normal crossings. Then one has the estimate*

$$0 < \frac{1}{\bar{\theta}_2(\mathbb{P}^2, \mathcal{C})} \leq \frac{3}{4 - \sqrt{10}} = 3.58 \dots$$

The proof employs a Riemann–Roch calculation, which is standard (cf. [18]).

Proof. We seek nonzero invariant logarithmic 2-jet differential forms with high vanishing order, which correspond to global sections of the twisted vector bundle

$$E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(1)^{-3\delta m} \cong E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{-\delta m},$$

where the logarithmic canonical line bundle

$$\bar{K}_{\mathbb{P}^2} = K_{\mathbb{P}^2} \otimes \mathcal{O}(\mathcal{C}) \cong \mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(6) \cong \mathcal{O}_{\mathbb{P}^2}(3),$$

with $\delta > 0$ a rational number to be chosen and m sufficiently divisible.

The standard approach begins with Serre duality:

$$(4.1) \quad H^2(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{-\delta m})^\vee = H^0(\mathbb{P}^2, K_{\mathbb{P}^2} \otimes (E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}))^\vee \otimes \bar{K}_{\mathbb{P}^2}^{\delta m}).$$

The bundle on the right admits the filtration induced by the dual of (2.3), with graded pieces

$$S^{m-3j}T_{\mathbb{P}^2}(-\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{-j} \otimes K_{\mathbb{P}^2} \otimes \bar{K}_{\mathbb{P}^2}^{\delta m} = S^{m-3j}T_{\mathbb{P}^2}(-\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{\delta m-j-1}$$

for $0 \leq j \leq \frac{m}{3}$.

For any rational number $\delta \in (0, \frac{1}{3})$ and sufficiently divisible $m \gg 1$, the inequality

$$m - 3j > 2(\delta m - j - 1)$$

automatically holds for all $0 \leq j \leq \frac{m}{3}$. By applying the Bogomolov-type Vanishing Theorem 2.1, we deduce:

$$H^0(\mathbb{P}^2, S^{m-3j}T_{\mathbb{P}^2}(-\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{\delta m-j-1}) = 0 \quad \text{for all } 0 \leq j \leq \frac{m}{3}.$$

This establishes the vanishing of the right-hand side in (4.1), which consequently implies the vanishing of the left-hand side as well. We thus obtain the lower bound

$$\dim H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{\delta m}) \geq \chi(E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{\delta m}),$$

since the Euler characteristic decomposes as

$$\chi(\cdots) = \dim H^0(\cdots) - \dim H^1(\cdots) + \dim H^2(\cdots) = \dim H^0(\cdots) - \dim H^1(\cdots).$$

To estimate the asymptotic growth of the Euler characteristic, we employ the Schur bundle filtration (see (2.3)):

$$(4.2) \quad \begin{aligned} \text{Gr}^\bullet E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) &= \bigoplus_{a+3b=m} \Gamma^{(a+b,b)}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \quad (\text{cf. [17, p. 67], [25, p. 474]}) \\ &= \bigoplus_{0 \leq b \leq \frac{m}{3}} \Gamma^{(m-2b,b)}T_{\mathbb{P}^2}^*(\log \mathcal{C}). \end{aligned}$$

Together with the Pieri formula (cf. [28, p. 455], [17, p. 66]):

$$\Gamma^{(\ell_1, \ell_2)}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{\delta m} = \Gamma^{(\ell_1-\delta m, \ell_2-\delta m)}T_{\mathbb{P}^2}^*(\log \mathcal{C}),$$

we derive the following computation (cf. [44, p. 77]):

$$\begin{aligned} &\chi(E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{\delta m}) \\ &= \sum_{b=0}^{\frac{m}{3}} \chi(\Gamma^{(m-2b-\delta m, b-\delta m)}T_{\mathbb{P}^2}^*(\log \mathcal{C})) \\ &= \left(\frac{1}{6} \int_0^{\frac{m}{3}} [(m-2b-\delta m)^3 - (b-\delta m)^3] db \right) \bar{c}_1^2 - \left(\frac{1}{6} \int_0^{\frac{m}{3}} (m-3b)^3 db \right) \bar{c}_2 + O(m^3) \\ &= \frac{m^2(m+3)}{108} (54\delta^2 m - 48\delta m + 13m - 18\delta + 15) \bar{c}_1^2 - \frac{m^4}{72} \bar{c}_2 + O(m^3) \\ &= \frac{m^4}{648} ((54\delta^2 - 48\delta + 13) \bar{c}_1^2 - 9\bar{c}_2) + O(m^3). \end{aligned}$$

Here the logarithmic Chern numbers are determined by the formula $\bar{c}_1^2 = (d-3)^2 = 9$ and $\bar{c}_2 = d^2 - 3d + 3 - \sum_{i < j} d_i d_j = 9$, where $d_1 = d_2 = d_3 = 2$ and $d = d_1 + d_2 + d_3 = 6$.

The quadratic leading coefficient (after removing the common factor $\frac{9}{648}$)

$$\varphi(\delta) := (54\delta^2 - 48\delta + 13) - 9 = 54\delta^2 - 48\delta + 4$$

has two positive roots given by $\delta_1 = \frac{4-\sqrt{10}}{9}$ and $\delta_2 = \frac{4+\sqrt{10}}{9}$.

Hence, $\varphi(\delta) > 0$ for all $\delta \in (0, \delta_1)$. It follows that $\bar{\theta}_2 \geq 3\delta_1 = 0.279\dots$, which completes the proof of Proposition 4.2. $\square$

To apply Siu's strategy of slanted vector fields, we follow the argument of Siu [65, pp. 558–559] to extend a negatively twisted invariant logarithmic 2-jet differential as a holomorphic family, invoking the following standard result in algebraic geometry:

Theorem 4.3 ([33, p. 288]). *Let $\tau: \mathcal{L} \rightarrow S$ be a flat holomorphic family of compact complex spaces and let $\mathcal{L} \rightarrow \mathcal{L}$ be a holomorphic vector bundle. Then there exists a proper subvariety $Z \subset S$ such that for each $s_0 \in S \setminus Z$, the restriction map $H^0(\tau^{-1}(U_{s_0}), \mathcal{L}) \rightarrow H^0(\tau^{-1}(s_0), \mathcal{L}|_{\tau^{-1}(s_0)})$ is onto, for some Zariski-dense open set $U_{s_0} \subset S$ containing s_0 .*

Now we apply Demailly–El Goul's argument (cf. [18, p. 532], [25, Lemma 1.3.2]) to show the existence of “irreducible” negatively twisted invariant logarithmic 2-jet differentials.

Proposition 4.4. *For some positive integer $m, t \geq 1$ with $m < 5t$, there exists a Zariski-open subset $U^{m,t}$ in the parameter space $\mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^5$ (see (5.1)) of three conics $\mathcal{C}$ in $\mathbb{P}^2$, such that:*

- (1) *For each parameter $a \in U^{m,t}$, the corresponding $\mathcal{C}_a$ forms a simple normal crossing divisor.*
- (2) *There is a holomorphic family*

$$\{\omega_{1,a} \mid 0 \neq \omega_{1,a} \in H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}_a) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)), a \in U^{m,t}\},$$

which varies holomorphically with respect to the variable $a \in U^{m,t}$. Moreover, each zero locus $\{\omega_{1,a} = 0\}$ on the second level $X_{2,a}$ of the logarithmic Demailly-Semple tower for $(\mathbb{P}^2, \mathcal{C}_a)$ is irreducible and reduced modulo $\Gamma_{2,a}$ (see (2.7)).

Proof. First, for any integers $M, T \geq 1$, Theorem 4.3 guarantees that there exists a proper Zariski-closed subset $Z_{M,T}$ in the parameter space $\mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^5$ of three conics, such that, for any parameter a outside $Z_{M,T}$, every nonzero global section of $E_{2,M}T_{\mathbb{P}^2}^*(\log \mathcal{C}_a) \otimes \mathcal{O}_{\mathbb{P}^2}(-T)$ extends to a holomorphic family parameterized by some Zariski open neighborhood of a . Thus, to ensure the local extendability of any nonzero negatively twisted logarithmic 2-jet differential on $(\mathbb{P}^2, \mathcal{C}_{a_0})$, we choose a point a_0 in $(\mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^5) \setminus \cup_{M,T \geq 1} Z_{M,T}$ such that $\mathcal{C}_{a_0}$ is a simple normal crossing divisor. Such a point exists because the parameter space is irreducible over the uncountable field $\mathbb{C}$, so it cannot be covered by countably many proper closed subsets; Bertini supplies the additional simple-normal-crossing open condition.

By Proposition 4.2, noting that $5 > \frac{3}{4-\sqrt{10}} = 3.58\dots$, for some integer $M, T \geq 1$ with ratio $\frac{1}{5} < \frac{T}{M} \leq \bar{\theta}_2(\mathbb{P}^2, \mathcal{C}_{a_0})$, there exists a nonzero negatively twisted invariant logarithmic 2-jet differential

$$\omega_0 \in H^0(\mathbb{P}^2, E_{2,M}T_{\mathbb{P}^2}^*(\log \mathcal{C}_{a_0}) \otimes \mathcal{O}_{\mathbb{P}^2}(-T)).$$

Next, we decompose the zero locus $\{\omega_0 = 0\}$ in the second level X_{2,a_0} of the logarithmic Demailly-Semple tower of $(\mathbb{P}^2, \mathcal{C}_{a_0})$ into irreducible components. By the same argument as Demailly and El Goul [18, p. 532], [25, Lemma 1.3.2], we can extract an irreducible and reduced component Z , which in fact “corresponds” to some irreducible negatively twisted invariant logarithmic 2-jet differential

$$(4.3) \quad \omega_1 \in H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}_{a_0}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)), \quad \text{with } \frac{t}{m} \geq \frac{T}{M} > \frac{1}{5},$$

in the sense that $\{\omega_1 = 0\} = Z + b\Gamma_{2,a_0}$ (see (2.7) and (2.8)) for some integer $b > 0$. The argument is as follows.

Let $u_1 = \pi_{2,1}^* \mathcal{O}_{X_{1,a_0}}(1)$, $u_2 = \mathcal{O}_{X_{2,a_0}}(1)$ and $h = \mathcal{O}_{\mathbb{P}^2}(1)$. Then the zero divisor $Z_{\omega_0} = \{\omega_0 = 0\} \subset X_{2,a_0}$ is linearly equivalent to $Mu_2 - T\pi_{2,0}^*h$ in the Picard group

$$\text{Pic}(X_{2,a_0}) \cong \text{Pic}(\mathbb{P}^2) \oplus \mathbb{Z}u_1 \oplus \mathbb{Z}u_2 \cong \mathbb{Z}^3.$$

Let $Z_{\omega_0} = \sum p_j Z_j$ be the irreducible decomposition of the divisor Z_{ω_0} , with each Z_j irreducible and reduced. Decompose each component Z_j linearly equivalently in the Picard group $\text{Pic}(X_{2,a_0})$ as

$$Z_j \sim b_{1,j}u_1 + b_{2,j}u_2 - t_j\pi_{2,0}^*h, \quad b_{1,j}, b_{2,j}, t_j \in \mathbb{Z}.$$

By [18, Lemma 3.3], the effectiveness of Z_j implies that exactly one of the following mutually exclusive cases holds:

- $(b_{1,j}, b_{2,j}) = (0, 0)$ and $Z_j \in \pi_{2,0}^* \text{Pic}(\mathbb{P}^2)$, $-t_j > 0$;
- $(b_{1,j}, b_{2,j}) = (-1, 1)$, then Z_j contains Γ_{2,a_0} , where Γ_{2,a_0} is an irreducible divisor on X_{2,a_0} . Thus $Z_j = \Gamma_{2,a_0}$ and $t_j = 0$;
- $b_{1,j} \geq 2b_{2,j} \geq 0$ and $m_j := b_{1,j} + b_{2,j} > 0$.

In the third case, the irreducible and reduced divisor Z_j corresponds a section

$$\omega_j \in H^0(X_{2,a_0}, \pi_{2,1}^* \mathcal{O}_{X_{1,a_0}}(b_{1,j}) \otimes \mathcal{O}_{X_{2,a_0}}(b_{2,j}) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-t_j)).$$

Noting that $\pi_{2,1}^* \mathcal{O}_{X_{1,a_0}}(b_1) \otimes \mathcal{O}_{X_{2,a_0}}(b_2) \cong \mathcal{O}_{X_{2,a_0}}(b_1 + b_2) \otimes \mathcal{O}_{X_{2,a_0}}(-b_1 \Gamma_{2,a_0})$ (see (2.8)), we may regard

$$\omega_j \in H^0(X_{2,a_0}, \mathcal{O}_{X_{2,a_0}}(m_j) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-t_j)),$$

whose zero divisor is $Z_j + b_{1,j} \Gamma_{2,a_0}$. Since both Z_j and Γ_{2,a_0} are irreducible and reduced, ω_j can not be factorized into a product of other logarithmic 2-jet differentials.

Let $\max\{t_j/m_j\}$ denote the maximum ratio among those sections obtained from the third case. Since $M = \sum p_j m_j$ and $T = \sum p_j t_j$, we have $\max\{t_j/m_j\} \geq T/M > 1/5$. Without loss of generality, we may assume that $t_1/m_1 = \max\{t_j/m_j\}$, and denote $m := m_1$ and $t := t_1$. By applying the direct image formula (2.9) and abusing the notation for ω_1 , we thus obtain (4.3).

By our choice of $a_0 \notin \cup_{M,T \geq 1} Z_{M,T}$, automatically ω_1 extends to a holomorphic family

$$\{\omega_{1,a} \mid 0 \neq \omega_{1,a} \in H^0(\mathbb{P}^2, E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C}_a) \otimes \mathcal{O}_{\mathbb{P}^2}(-t)), a \in U_{a_0}^{m,t}\},$$

which varies holomorphically with respect to the parameter a in some Zariski-open neighborhood $U_{a_0}^{m,t}$ of a_0 , with $\omega_{1,a_0} = \omega_1$. By shrinking to a smaller Zariski-open subset $U^{m,t} \subset U_{a_0}^{m,t}$ if necessary, we may ensure that each $a \in U^{m,t}$ defines three conics $\mathcal{C}_a$ with simple normal crossings while the zero locus $\{\omega_{1,a} = 0\}$ on the second level $X_{2,a}$ of the logarithmic Demailly-Semple tower for $(\mathbb{P}^2, \mathcal{C}_a)$ is irreducible and reduced modulo $\Gamma_{2,a}$. $\square$

5. Generating Independent Twisted Jet Differentials via Slanted Vector Fields

To construct an additional family of independent negatively twisted jet differentials, we employ Siu's method of slanted vector fields [65]. This approach generalizes to k -jet spaces the original technique developed for 1-jet spaces by Clemens [11], Ein [24], and Voisin [70] in their study of rational curves.

The method was first implemented by Păun [54] for compact surfaces in $\mathbb{P}^3$. Rousseau [58] subsequently established the logarithmic theory for pairs $(\mathbb{P}^2, D)$ with curve configurations D , while in [57, 56] he further extended both Păun's setting and the logarithmic case to threefolds. For hypersurfaces $H \subset \mathbb{P}^n$ of arbitrary dimension, Merker [43] established the complete generalization, thereby realizing Siu's original vision concerning generic global generation. The logarithmic counterpart for pairs $(\mathbb{P}^n, H)$ was subsequently obtained by Darondeau [15].

This section adapts the standard technique of Siu's slanted vector fields to the case of three generic conics in $\mathbb{P}^2$. The content is entirely classical and no originality is claimed.

5.1. Logarithmic Slanted Vector Fields for Three Conics in $\mathbb{P}^2$. The framework presented in this subsection specializes Rousseau's logarithmic setting in $\mathbb{P}^2$ to configurations of three conics. We emphasize that this subsection contains no original contributions and is included solely for completeness and the reader's convenience.

Let

$$(5.1) \quad \mathcal{X} = \cup_{\lambda=1}^3 \mathcal{X}_\lambda \subset \mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}$$

denote the universal family of algebraic curves in $\mathbb{P}^2$ consisting of three components of degrees d_1, d_2 , and d_3 , respectively. Here, each $\mathcal{X}_\lambda$ is the universal curve of degree d_λ , defined by the equation $\sum_{|\alpha|=d_\lambda} A_\alpha^\lambda Z^\alpha = 0$, where $[A^\lambda] = [A_\alpha^\lambda]_{|\alpha|=d_\lambda} \in \mathbb{P}^{N_{d_\lambda}}$ are the homogeneous coefficients and $[Z] = [Z_0 : Z_1 : Z_2] \in \mathbb{P}^2$ are the projective coordinates. For a multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}^3$ with length $|\alpha| := \alpha_0 + \alpha_1 + \alpha_2$, we employ the notation $Z^\alpha := Z_0^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2}$. The dimension of the parameter space for each component is given by $N_{d_\lambda} = \binom{d_\lambda+2}{2} - 1$.

For any integer $k \geq 1$, the *vertical logarithmic k -jet space* is then defined as

$$\overline{J}_k^v \left(\mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}} \right) := \ker(\text{pr}_2)_* \subset J_k \left(\mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}, -\log \mathcal{X} \right),$$

where $\text{pr}_2: \mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}} \longrightarrow \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}$ denotes the natural projection onto the parameter space and $J_k(\mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}, -\log \mathcal{X})$ is the logarithmic k -jet bundle along the universal family $\mathcal{X}$.

We define the universal intersection family

$$\mathcal{Y} = \cap_{\lambda=1}^3 \mathcal{Y}_\lambda \subset \mathbb{P}^5 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}+1},$$

where for each $1 \leq \lambda \leq 3$, the hypersurface $\mathcal{Y}_\lambda$ is the universal family of degree d_λ hypersurfaces in $\mathbb{P}^5$, given explicitly by $\mathcal{Y}_\lambda := \{A_0^\lambda Z_{2+\lambda}^\lambda + \sum_{|\alpha|=d_\lambda} A_\alpha^\lambda Z^\alpha = 0\} \subset \mathbb{P}^5 \times \mathbb{P}^{N_{d_\lambda}+1}$. Here, the space $\mathbb{P}^5$ has homogeneous coordinates $[Z : Z_3 : Z_4 : Z_5] = [Z_0 : \dots : Z_5]$, and the parameter space $\mathbb{P}^{N_{d_\lambda}+1}$ is equipped with homogeneous coordinates $[A^\lambda : A_0^\lambda] = [\dots : A_\alpha^\lambda : \dots : A_0^\lambda]_{|\alpha|=d_\lambda}$.

For notational convenience, we denote the exceptional locus $\{Z = \mathbf{0}\} \cup \cup_{\lambda=1}^3 \{A^\lambda = \mathbf{0}\}$, where $\{Z = \mathbf{0}\}$ denotes the subset of $\mathbb{P}^5$ defined by $Z_0 = Z_1 = Z_2 = 0$ and $\{A^\lambda = \mathbf{0}\}$ refers to the locus in $\mathbb{P}^{N_{d_\lambda}+1}$ where all coordinates vanish except A_0^λ . There exists a natural forgetful map

$$\pi: \mathbb{P}^5 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}+1} \setminus \left(\{Z = \mathbf{0}\} \cup \cup_{\lambda=1}^3 \{A^\lambda = \mathbf{0}\} \right) \longrightarrow \mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}$$

that forgets the auxiliary coordinates. Explicitly, π is given by

$$([Z : Z_3 : Z_4 : Z_5], [A^1 : A_0^1], [A^2 : A_0^2], [A^3 : A_0^3]) \longmapsto ([Z], [A^1], [A^2], [A^3]).$$

We observe the containment relation

$$(5.2) \quad \mathcal{Y} \cap \left(\{Z = \mathbf{0}\} \cup \cup_{\lambda=1}^3 \{A^\lambda = \mathbf{0}\} \right) \subset \mathcal{Y} \cap \left(\cup_{\lambda=1}^3 \left(\{A_0^\lambda = 0\} \cup \{A^\lambda = \mathbf{0}\} \right) \right),$$

from the inclusion $\mathcal{Y} \cap \{Z = \mathbf{0}\} \subset \mathcal{Y} \cap \cup_{\lambda=1}^3 \{A_0^\lambda = 0\}$, which is deduced by the definition of $\mathcal{Y}$.

Now we consider $\mathcal{Y}^* := \mathcal{Y} \setminus \left(\cup_{\lambda=1}^3 \left(\{A_0^\lambda = 0\} \cup \{A^\lambda = \mathbf{0}\} \right) \right)$ on which the restricted forgetful map

$$\pi|_{\mathcal{Y}^*}: \mathcal{Y}^* \rightarrow \mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}$$

is well-defined according to the containment relation (5.2). Furthermore, we obtain the identification of the preimage $(\pi|_{\mathcal{Y}^*})^{-1}(\mathcal{X}) = \{Z_3 Z_4 Z_5 = 0\} =: \mathcal{D}$. This shows that π induces a logarithmic morphism from $(\mathcal{Y}^*, \mathcal{D})$ to $(\mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}, \mathcal{X})$, which consequently gives rise to a dominant map between jet spaces:

$$(5.3) \quad \pi_{[2]}: \overline{J}_2^v(\mathcal{Y}^*) \longrightarrow \overline{J}_2^v\left(\mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}\right).$$

Here, $\overline{J}_2^v(\mathcal{Y}^*)$ represents the space of vertical logarithmic 2-jets with respect to the normal crossing divisor $\mathcal{D} = \{Z_3 Z_4 Z_5 = 0\}$. This construction allows us to perform all local computations on the more tractable space $\mathcal{Y}^*$ through pullback via π , where we have access to explicit coordinate expressions.

On the affine chart $\{Z_0 \neq 0\} \times \prod_{\lambda=1}^3 \{A_0^\lambda \neq 0\}$, we employ the affine coordinates $z_i := Z_i/Z_0$ for $i = 1, \dots, 5$ and $a_\alpha^\lambda := A_{\alpha_0 \alpha_1 \alpha_2}^\lambda / A_0^\lambda$ with $\alpha_0 = d_\lambda - \alpha_1 - \alpha_2$. To streamline notation, for each jet of order $j = 1, 2$ we define:

$$\xi_i^{(j)} := \begin{cases} z_i^{(j)} & \text{for } i = 1, 2, \\ (\log z_{2+\lambda})^{(j)} & \text{for } i = 3, 4, 5 \ (\lambda = i - 2). \end{cases}$$

Then local coordinates for $\overline{J}_2^v(\{Z_0 \neq 0\} \times \prod_{\lambda=1}^3 \{A_0^\lambda \neq 0\}) \subset J_2(\mathbb{P}^5 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}+1}, -\log \mathcal{D})$ are given by the coordinate tuple $(z_i, a_\alpha^\lambda, \xi_{i_1}'^{\lambda}, \xi_{i_2}''^{\lambda})$.

Following the strategy of Siu [65] and the realizations of Păun [54] and Rousseau [58], we consider the following family of Siu's slanted vector fields with pole order 7:

Theorem 5.1. *Let Σ_0 be the zero locus of the 2×2 Wronskian $\det(\xi_i^{(j)})_{1 \leq i, j \leq 2}$:*

$$\Sigma_0 = \left\{ \left(z_i, a_\alpha^\lambda, \xi_{i_1}'^{\lambda}, \xi_{i_2}''^{\lambda} \right) \mid \det \left(\xi_i^{(j)} \right)_{1 \leq i, j \leq 2} = 0 \right\} \subset \overline{J}_2^v \left(\mathcal{Y} \cap \left(\{Z_0 \neq 0\} \times \prod_{\lambda=1}^3 \{A_0^\lambda \neq 0\} \right) \right),$$

and Σ be the closure of Σ_0 in $\overline{J}_2^v(\mathcal{Y}^*)$. Then the twisted tangent bundle

$$(5.4) \quad T_{\overline{J}_2^v(\mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}})} \otimes p^* \text{pr}_1^* \mathcal{O}_{\mathbb{P}^2}(7) \otimes p^* \text{pr}_2^* \mathcal{O}_{\prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}}(1, 1, 1),$$

is generated by its global sections at every point of $\overline{J}_2^v(\mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}) \setminus (\pi_{[2]}(\Sigma) \cup p^{-1}(\mathcal{X}))$, where $\pi_{[2]}$ is introduced in (5.3), p is the natural projection

$$p: \overline{J}_2^v(\mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}) \longrightarrow \mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}},$$

and pr_1 (resp. pr_2) is the projection from $\mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}$ to $\mathbb{P}^2$ (resp. $\prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}$). $\square$

5.2. Producing an Independent 2-Jet via Siu's Slanted Vector Fields of Pole Order 7.

Proposition 5.2 (The second global jet differential). *Let $\{\omega_{1,a}\}_{a \in U}$ be the holomorphic family supplied by Proposition 4.4, with weighted degree m and negative base twist $t \geq 8$. For a generic parameter a , one can choose a pole-order-7 slanted vector field V such that*

$$\omega_{2,a} := \mathcal{L}_V P|_a \in H^0(\mathbb{P}^2, E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C}_a) \otimes \mathcal{O}_{\mathbb{P}^2}(7-t))$$

is nonzero and the common zero locus of $\omega_{1,a}$ and $\omega_{2,a}$ in $X_{2,a}$ has codimension at least 2 modulo $\Gamma_{2,a}$. Consequently, if $f^* \omega_{1,a} \equiv 0$, then either

$$T_f(r) \leq \frac{m}{t-7} \sum_{i=1}^3 N_f^{[1]}(r, \mathcal{C}_i) + O(\log T_f(r) + \log r) \quad \parallel,$$

or $f_{[1]}(\mathbb{C})$ is algebraically degenerate in $X_{1,a}$.

This is Siu's global differentiation mechanism [65], in the logarithmic form developed by Rousseau [58]; see also the explicit invariant formulation in [19, pp. 171–173]. The point is that P is not being differentiated as an abstract line-bundle-valued section after an arbitrary choice of frame. On the universal vertical jet space it is an equivariant polynomial function of the jet variables, and an invariant slanted vector field acts on it naturally by derivation. Equivariance makes the result independent of the homogeneous chart and preserves the weighted degree and reparametrization invariance.

Proof. Regard the family $\{\omega_{1,a}\}_{a \in U}$ as the equivariant holomorphic map

$$P: \overline{J}_2^v(\mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}) \big|_{\text{pr}_2^{-1}(U)} \longrightarrow p^* \text{pr}_1^* \mathcal{O}_{\mathbb{P}^2}(-t).$$

If V is a global invariant section of the twisted tangent bundle (5.4), its natural action on equivariant functions gives

$$\mathcal{L}_V P: \overline{J}_2^v(\mathbb{P}^2 \times \prod_{\lambda=1}^3 \mathbb{P}^{N_{d_\lambda}}) \big|_{\text{pr}_2^{-1}(U)} \longrightarrow p^* \text{pr}_1^* \mathcal{O}_{\mathbb{P}^2}(7-t),$$

where the parameter-space twist becomes a constant one-dimensional factor after restriction to a fixed fibre. Since V is invariant, $\mathcal{L}_V P$ is again an invariant polynomial of weighted degree m in the jet variables. Its restriction to the fibre over a is therefore the asserted global differential $\omega_{2,a}$.

By Proposition 4.4, the residual zero divisor of $\omega_{1,a}$ is irreducible and reduced modulo $\Gamma_{2,a}$. The global generation in Theorem 5.1 allows us to choose V non-tangent to that divisor at a general smooth point. Hence $\mathcal{L}_V P$ does not vanish identically there, and the common zero locus has codimension at least 2 modulo $\Gamma_{2,a}$.

If $f^* \omega_{2,a} \not\equiv 0$, apply Theorem 1.2 to $\omega_{2,a}$, whose negative twist is $t-7$. If both pullbacks vanish, then $f_{[2]}(\mathbb{C})$ lies in their common zero locus. Its image in $X_{1,a}$ has dimension at most 2, strictly smaller than $\dim X_{1,a} = 3$; hence $f_{[1]}$ is algebraically degenerate. $\square$

6. Refinement of Demailly–El Goul's Arguments

6.1. More Riemann-Roch Computations. In Proposition 4.4, we show the existence of a nonzero negatively twisted invariant logarithmic 2-jet differential

$$\omega_1 \in H^0(X_2, \mathcal{O}_{X_2}(m) \otimes \pi_{2,0}^* \mathcal{O}_X(-t)),$$

for some positive integers $m, t \geq 1$ with $t/m \geq 1/5$, whose zero divisor on the second level of the Demailly-Semple tower X_2 is of the form $Z + b_1 \Gamma_2$, where $Z \subset X_2$ is irreducible and reduced. Moreover, we have the linear equivalence

$$(6.1) \quad Z \sim b_1 u_1 + b_2 u_2 - t \pi_{2,0}^* h, \quad (b_1, b_2) \in \mathbb{Z}^2, \quad b_1 \geq 2b_2 > 0, \quad b_1 + b_2 = m, \quad t \in \mathbb{Z}_+,$$

where $u_1 = \pi_{2,1}^* \mathcal{O}_{X_1}(1)$, $u_2 = \mathcal{O}_{X_2}(1)$, and $h = \mathcal{O}_{\mathbb{P}^2}(1)$.

Hereafter, we use the standard notation:

$$\mathcal{O}_{X_2}(b_1, b_2) := \pi_{2,1}^* \mathcal{O}_{X_1}(b_1) \otimes \mathcal{O}_{X_2}(b_2), \quad b_1, b_2 \in \mathbb{Z}.$$

Using the same argument as in Demailly–El Goul [18, Proposition 3.4] — Miyaoka’s method [45]; see also Lu–Yau [40] — one can show that $\mathcal{O}_{X_2}(2, 1)|_Z$ is big, provided that $t < m$. Now, our goal is to determine an effective small rational number $\tau > 0$ such that the $\mathbb{Q}$ -line bundle $(\mathcal{O}_{X_2}(2, 1) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-\tau))|_Z$ remains big. This will be achieved through Riemann-Roch computations and the following proposition observed by Demailly [17, p. 73].

Proposition 6.1. *Let $a_1, a_2 \in \mathbb{Z}$.*

(1) *If $a_2 = -1$, then*

$$R^q(\pi_{2,0})_* \mathcal{O}_{X_2}(a_1, -1) = 0 \quad \text{for every } q \geq 0.$$

(2) *If $a_2 \geq 0$, then*

$$R^q(\pi_{2,0})_* \mathcal{O}_{X_2}(a_1, a_2) = 0 \quad \text{for every } q \geq 2,$$

and

$$R^1(\pi_{2,0})_* \mathcal{O}_{X_2}(a_1, a_2) = 0 \iff a_1 - 2a_2 \geq -1.$$

In particular, for every $p \geq 0$,

$$R^q(\pi_{2,0})_* \mathcal{O}_{X_2}(2p, p) = 0 \quad \text{for every } q \geq 1.$$

Proof. Set

$$\rho := \pi_{2,1}: X_2 = \mathbb{P}(V_1) \longrightarrow X_1, \quad \pi := \pi_{1,0}: X_1 \longrightarrow X_0.$$

Recall that V_1 is the rank-two directed bundle defined by (2.5), and that V_1^* denotes its dual. We use the convention that $\mathbb{P}(V_1)$ parametrizes lines in the fibers of V_1 , so that $\mathcal{O}_{X_2}(-1) \subset \rho^* V_1$ is the tautological line subbundle.

Suppose first that $a_2 = -1$. The restriction of $\mathcal{O}_{X_2}(a_1, -1)$ to every fiber of ρ is $\mathcal{O}_{\mathbb{P}^1}(-1)$, whose cohomology vanishes in every degree. Cohomology and base change therefore gives

$$R^q \rho_* \mathcal{O}_{X_2}(a_1, -1) = 0 \quad (q \geq 0),$$

and the first assertion follows from $\pi_{2,0} = \pi \circ \rho$.

Assume now that $a_2 \geq 0$. By the projective bundle formula and the projection formula,

$$(6.2) \quad R^q \rho_* \mathcal{O}_{X_2}(a_1, a_2) = 0 \quad (q \geq 1), \quad \rho_* \mathcal{O}_{X_2}(a_1, a_2) = \mathcal{O}_{X_1}(a_1) \otimes S^{a_2} V_1^*.$$

Hence the Leray spectral sequence yields

$$R^q(\pi_{2,0})_* \mathcal{O}_{X_2}(a_1, a_2) \simeq R^q \pi_* (\mathcal{O}_{X_1}(a_1) \otimes S^{a_2} V_1^*).$$

Fix $x \in X_0$ and set

$$F_x := \pi^{-1}(x) \simeq \mathbb{P}^1.$$

Restricting (2.5) to F_x gives

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow V_1|_{F_x} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0.$$

This sequence splits, since

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(2)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)) = 0.$$

Thus

$$V_1^*|_{F_x} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(1),$$

and, since $\mathcal{O}_{X_1}(1)|_{F_x} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$,

$$(6.3) \quad (\mathcal{O}_{X_1}(a_1) \otimes S^{a_2} V_1^*)|_{F_x} \simeq \bigoplus_{j=0}^{a_2} \mathcal{O}_{\mathbb{P}^1}(a_1 + a_2 - 3j).$$

The smallest degree in (6.3) is $a_1 - 2a_2$. Since

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = 0 \iff d \geq -1,$$

the fiberwise first cohomology vanishes precisely when $a_1 - 2a_2 \geq -1$. The splitting type is independent of x , so Grauert’s theorem gives

$$R^1(\pi_{2,0})_* \mathcal{O}_{X_2}(a_1, a_2) = 0 \iff a_1 - 2a_2 \geq -1.$$

Finally, π has one-dimensional fibers; hence

$$R^q \pi_* (\mathcal{O}_{X_1}(a_1) \otimes S^{a_2} V_1^*) = 0 \quad (q \geq 2).$$

Together with (6.2), this proves the remaining assertion. Taking $(a_1, a_2) = (2p, p)$ gives $a_1 - 2a_2 = 0$, and therefore all higher direct images vanish. $\square$

Proposition 6.2. *If $0 < \frac{t}{m} < 1$ and $0 \leq \tau < \tau_1(\frac{t}{m})$, $\tau \in \mathbb{Q}$, where*

$$\tau_1\left(\frac{t}{m}\right) = \frac{3}{2} \left(4 - \frac{t}{m}\right) - \frac{\sqrt{3}}{2} \sqrt{32 - 8\frac{t}{m} + 3\left(\frac{t}{m}\right)^2},$$

then the restricted $\mathbb{Q}$ -line bundle $(\mathcal{O}_{X_2}(2, 1) \otimes \pi_{2,0}^ \mathcal{O}_{\mathbb{P}^2}(-\tau))|_Z$ is big on Z .*

Proof. We begin by recalling some key intersection formulas on the Demailly–Semple tower. By [25, p. 478], we have the identities:

$$u_1^2 + \bar{c}_1 u_1 + \bar{c}_2 = 0 \quad \text{and} \quad u_2^2 + \bar{c}_1(V_1) u_2 + \bar{c}_2(V_1) = 0,$$

where $\bar{c}_1(V_1) = \bar{c}_1 + u_1$ (see (2.4)) and $\bar{c}_2(V_1) = \bar{c}_2 - u_1^2 = 2\bar{c}_2 + \bar{c}_1 u_1$.

On the second level X_2 of the Demailly–Semple tower, the following identities hold:

$$u_1^4 = 0, \quad u_1^3 u_2 = \bar{c}_1^2 - \bar{c}_2, \quad u_1^2 u_2^2 = \bar{c}_2, \quad u_1 u_2^3 = \bar{c}_1^2 - 3\bar{c}_2, \quad u_2^4 = 5\bar{c}_2 - \bar{c}_1^2.$$

Furthermore, for any pullback $F \in \pi_{2,0}^* \text{Pic}(\mathbb{P}^2)$, we have:

$$u_1^3 \cdot F = 0, \quad u_1^2 u_2 \cdot F = -\bar{c}_1 \cdot F, \quad u_1 u_2^2 \cdot F = 0, \quad u_2^3 \cdot F = 0.$$

A crucial part of our approach relies on the following additional identities:

$$u_1^2 \cdot F_1 \cdot F_2 = 0, \quad u_1 u_2 \cdot F_1 \cdot F_2 = F_1 \cdot F_2, \quad u_2^2 \cdot F_1 \cdot F_2 = -F_1 \cdot F_2, \quad \forall F_1, F_2 \in \pi_{2,0}^* \text{Pic}(\mathbb{P}^2).$$

Then by direct computation, we can obtain

$$(6.4) \quad (2u_1 + u_2 - \tau \pi_{2,0}^* h)^3 \cdot Z = m(13\bar{c}_1^2 - 9\bar{c}_2) + 12t(\bar{c}_1 \cdot h) + 12m(\bar{c}_1 \cdot h)\tau + 3m(h^2)\tau^2 + 9(h^2)t\tau.$$

In our setting of three generic conics, we have $\bar{c}_1 = -3h$ and $\bar{c}_2 = 9h^2$. Substituting these into (6.4), together with $h^2 = 1$, yields:

$$(6.5) \quad (2u_1 + u_2 - \tau \pi_{2,0}^* h)^3 \cdot Z = 3(m\tau^2 - 3(4m - t)\tau + 12(m - t)).$$

This quadratic polynomial has two real roots:

$$(6.6) \quad \tau_1\left(\frac{t}{m}\right) = \frac{3}{2} \left(4 - \frac{t}{m}\right) - \frac{\sqrt{3}}{2} \sqrt{32 - 8\frac{t}{m} + 3\left(\frac{t}{m}\right)^2} < \tau_2\left(\frac{t}{m}\right) = \frac{3}{2} \left(4 - \frac{t}{m}\right) + \frac{\sqrt{3}}{2} \sqrt{32 - 8\frac{t}{m} + 3\left(\frac{t}{m}\right)^2},$$

because $32 - 8\frac{t}{m} + 3\left(\frac{t}{m}\right)^2 > 0$. Moreover, $\tau_1(\frac{t}{m}) > 0$ if and only if $\frac{t}{m} < 1$.

Thus when $0 < \frac{t}{m} < 1$ and $0 \leq \tau < \tau_1(\frac{t}{m})$, we have $(2u_1 + u_2 - \tau \pi_{2,0}^* h)^3 \cdot Z > 0$. Riemann–Roch Theorem then gives, for sufficiently divisible integer $p \gg 1$, that:

$$\begin{aligned} & h^0(Z, (\mathcal{O}_{X_2}(2, 1) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-\tau))^p|_Z) + h^2(Z, (\mathcal{O}_{X_2}(2, 1) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-\tau))^p|_Z) \\ & \geq \chi(Z, (\mathcal{O}_{X_2}(2, 1) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-\tau))^p|_Z) = \frac{(2u_1 + u_2 - \tau \pi_{2,0}^* h)^3 \cdot Z}{3!} p^3 + O(p^2) \gg 1. \end{aligned}$$

Now we prove that

$$(6.7) \quad H^2(Z, (\mathcal{O}_{X_2}(2, 1) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-\tau))^p|_Z) = 0$$

for all sufficiently divisible $p \gg 1$. Put

$$\mathcal{L}_p = \mathcal{O}_{X_2}(2p, p) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-\tau p).$$

Tensoring $0 \rightarrow \mathcal{O}_{X_2}(-Z) \rightarrow \mathcal{O}_{X_2} \rightarrow \mathcal{O}_Z \rightarrow 0$ with $\mathcal{L}_p$ gives

$$(6.8) \quad h^2(Z, \mathcal{L}_p|_Z) \leq h^2(X_2, \mathcal{L}_p) + h^3(X_2, \mathcal{L}_p \otimes \mathcal{O}_{X_2}(-Z)).$$

For the first term, Proposition 6.1 applies because $2p - 2p = 0 \geq -1$. Hence

$$R^q(\pi_{2,0})_* \mathcal{O}_{X_2}(2p, p) = 0 \quad (q \geq 1)$$

and

$$(\pi_{2,0})_* \mathcal{O}_{X_2}(2p, p) \simeq E_{2,3p} T_{\mathbb{P}^2}^*(\log \mathcal{C}).$$

Indeed, in filtration layer q a weighted-degree $3p$ invariant monomial has $3p - 3q$ first-derivative factors and q Wronskian factors. On X_2 its class is

$$\mathcal{O}_{X_2}(3p - q, q) = \mathcal{O}_{X_2}(3p)(-(3p - q)\Gamma_2).$$

Since $0 \leq q \leq p$, every section of $\mathcal{O}_{X_2}(3p)$ coming from an invariant differential vanishes along Γ_2 at least to order $2p$. Hence

$$(\pi_{2,0})_* \mathcal{O}_{X_2}(2p, p) \simeq (\pi_{2,0})_* \mathcal{O}_{X_2}(3p) \simeq E_{2,3p} T_{\mathbb{P}^2}^*(\log \mathcal{C}),$$

which justifies the displayed zeroth-direct-image identification. By Leray and the projection formula,

$$H^2(X_2, \mathcal{L}_p) \simeq H^2(\mathbb{P}^2, E_{2,3p} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-\tau p)).$$

The argument of Proposition 4.2, with jet degree $3p$ and $\delta = \tau/9$, gives the vanishing of this group. Indeed,

$$0 \leq \tau < \tau_1(t/m) \leq 6 - 2\sqrt{6} < 3,$$

so that $0 \leq \delta < 1/3$. Therefore

$$(6.9) \quad H^2(X_2, \mathcal{L}_p) = 0.$$

It remains to treat the second term in (6.8). By (6.1), set

$$\mathcal{M}_p := \mathcal{L}_p \otimes \mathcal{O}_{X_2}(-Z) = \mathcal{O}_{X_2}(2p - b_1, p - b_2) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(t - \tau p).$$

Write

$$r := b_1 - 2b_2 \geq 0.$$

Choose p so large that $p - b_2 \geq 0$ and, when $r \geq 2$,

$$p - b_2 \geq \left\lfloor \frac{r-2}{3} \right\rfloor.$$

Let $\rho = \pi_{2,1}$ and $\pi = \pi_{1,0}$. Since $p - b_2 \geq 0$,

$$R^q \rho_* \mathcal{M}_p = 0 \quad (q \geq 1),$$

whereas $\rho_* \mathcal{M}_p$ has a filtration induced by

$$0 \longrightarrow \mathcal{O}_{X_1}(1) \longrightarrow V_1^* \longrightarrow \pi^* \bar{K}_{\mathbb{P}^2} \otimes \mathcal{O}_{X_1}(-2) \longrightarrow 0.$$

After reindexing by $j = p - b_2 - k$, the corresponding graded terms are

$$(6.10) \quad \mathcal{Q}_{p,k} = \pi^* \left(\bar{K}_{\mathbb{P}^2}^{p-b_2-k} \otimes \mathcal{O}_{\mathbb{P}^2}(t - \tau p) \right) \otimes \mathcal{O}_{X_1}(-r + 3k), \quad 0 \leq k \leq p - b_2.$$

Set

$$A_{p,k} := \bar{K}_{\mathbb{P}^2}^{p-b_2-k} \otimes \mathcal{O}_{\mathbb{P}^2}(t - \tau p), \quad d_k := -r + 3k,$$

so that

$$\mathcal{Q}_{p,k} = \pi^* A_{p,k} \otimes \mathcal{O}_{X_1}(d_k).$$

If $d_k \geq -1$, then the projective bundle formula and the projection formula give

$$R^1 \pi_* \mathcal{Q}_{p,k} \simeq A_{p,k} \otimes R^1 \pi_* \mathcal{O}_{X_1}(d_k) = 0.$$

Since π has one-dimensional fibers and $\dim \mathbb{P}^2 = 2$, the Leray spectral sequence yields

$$H^3(X_1, \mathcal{Q}_{p,k}) \simeq H^2(\mathbb{P}^2, R^1 \pi_* \mathcal{Q}_{p,k}) = 0.$$

Consequently, only the indices satisfying $d_k \leq -2$ may contribute; equivalently,

$$0 \leq k \leq \left\lfloor \frac{r-2}{3} \right\rfloor.$$

The only possible nonvanishing terms are therefore indexed by the fixed finite set

$$(6.11) \quad 0 \leq k \leq \left\lfloor \frac{r-2}{3} \right\rfloor.$$

For such k , relative Serre duality for $\pi: X_1 = \mathbb{P}(T_{\mathbb{P}^2}(-\log \mathcal{C})) \rightarrow \mathbb{P}^2$ gives

$$(6.12) \quad R^1 \pi_* \mathcal{Q}_{p,k} \simeq S^{r-3k-2} T_{\mathbb{P}^2}(-\log \mathcal{C})$$

$$(6.13) \quad \otimes \mathcal{O}_{\mathbb{P}^2}((3 - \tau)p + t - 3(b_2 + k + 1)),$$

because $\overline{K}_{\mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^2}(3)$. For each k in (6.11), the symmetric power in (6.12) is independent of p , while the twist tends to $+\infty$ since $\tau < 3$. Serre vanishing therefore gives

$$H^2(\mathbb{P}^2, R^1\pi_*\mathcal{D}_{p,k}) = 0 \quad (p \gg 1).$$

The Leray spectral sequence for π now yields $H^3(X_1, \mathcal{D}_{p,k}) = 0$ for every graded term of the filtration. Induction along the filtration gives

$$H^3(X_1, \rho_*\mathcal{M}_p) = 0.$$

Using $R^q\rho_*\mathcal{M}_p = 0$ once more, we obtain

$$(6.14) \quad H^3(X_2, \mathcal{M}_p) = 0.$$

Combining (6.8), (6.9), and (6.14) proves (6.7). Thus Riemann–Roch gives

$$h^0(Z, \mathcal{L}_p|_Z) \geq \frac{(2u_1 + u_2 - \tau\pi_{2,0}^*h)^3 \cdot Z}{3!} p^3 + O(p^2),$$

and the positive leading coefficient proves that the restricted $\mathbb{Q}$ -line bundle is big. $\square$

6.2. A Refined Estimate. Here is a key step in proving our main theorem.

Lemma 6.3 (Tautological evaluation of a restricted section). *Let $Z \subset X_2$ be an effective Cartier divisor, let $p, \tilde{t} \in \mathbb{Z}_{>0}$, and let*

$$\sigma \in H^0(Z, \mathcal{O}_{X_2}(2p, p) \otimes \pi_{2,0}^*\mathcal{O}_{\mathbb{P}^2}(-\tilde{t})|_Z).$$

*Suppose that $g: \mathbb{C} \rightarrow \mathbb{P}^2$ is nonconstant, $g(\mathbb{C}) \not\subset \mathcal{C}$, and that its logarithmic Semple lift extends across its isolated indeterminacy points and satisfies $g_{[2]}(\mathbb{C}) \subset Z$. On $\mathbb{C} \setminus g^{-1}(\mathcal{C})$, regard $dg_{[1]}: T_{\mathbb{C}} \rightarrow g_{[1]}^*V_1$ as a differential in the directed structure. Wherever it is nonzero, its image is the tautological line $g_{[2]}^*\mathcal{O}_{X_2}(-1)$. It therefore defines, by meromorphic continuation across the remaining discrete set, the intrinsic tautological derivative*

$$\mathcal{D}g_{[1]} \in H_{\text{mer}}^0(\mathbb{C}, K_{\mathbb{C}} \otimes g_{[2]}^*\mathcal{O}_{X_2}(-1)).$$

Here $H_{\text{mer}}^0(Y, L)$ denotes the vector space of global meromorphic sections of a line bundle L on a complex manifold Y . The canonical inclusion

$$\iota_p: \mathcal{O}_{X_2}(2p, p) = \mathcal{O}_{X_2}(3p)(-2p\Gamma_2) \hookrightarrow \mathcal{O}_{X_2}(3p)$$

defines an intrinsic meromorphic section

$$(6.15) \quad s_{\sigma,g} := \left\langle g_{[2]}^*(\iota_p\sigma), (\mathcal{D}g_{[1]})^{\otimes 3p} \right\rangle \in H_{\text{mer}}^0(\mathbb{C}, K_{\mathbb{C}}^{\otimes 3p} \otimes g^*\mathcal{O}_{\mathbb{P}^2}(-\tilde{t})),$$

This construction is independent of every choice of an ambient local lifting of σ . If $s_{\sigma,g} \not\equiv 0$, then its pole divisor satisfies

$$(6.16) \quad \text{div}_{\infty}(s_{\sigma,g}) \leq 3p \sum_{i=1}^3 (g^*\mathcal{C}_i)_{\text{red}},$$

and, outside a set of finite Lebesgue measure,

$$(6.17) \quad \int_0^{2\pi} \log^+ \|s_{\sigma,g}(re^{\sqrt{-1}\theta})\| \frac{d\theta}{2\pi} = O(\log T_g(r) + \log r).$$

Proof. Since the map $g_{[2]}$ factors through Z , one may pull back $\iota_p\sigma$ directly from Z , without extending σ to a neighborhood. Pairing the two line bundles gives (6.15). If one nevertheless represents σ by ambient local liftings, two such liftings differ by a section in the ideal of Z ; their pullbacks by $g_{[2]}$ therefore agree.

Equip the line bundles on X_2 and $\mathbb{P}^2$ with smooth Hermitian metrics. Since Z is projective, the norm of the holomorphic section $\iota_p\sigma$ is bounded on Z . Consequently, for a uniform constant $C > 0$,

$$\|s_{\sigma,g}\| \leq C \|\mathcal{D}g_{[1]}\|^{3p}.$$

This estimate uses only the intrinsic section on Z and does not extend it to a saturated inverse image in the Semple tower.

We next analyze the tautological derivative. Near a point of $g^{-1}(\mathcal{C})$, choose logarithmic coordinates with $\mathcal{C} = \{x_1 \cdots x_q = 0\}$, $q \leq 2$. The horizontal components of the derivative of the regular first lift are $(x_i \circ g)' / (x_i \circ g)$ in the boundary directions; all vertical projective-coordinate components are holomorphic because the first lift extends. Thus $\mathcal{D}g_{[1]}$ has at most a simple pole along each $(g^*\mathcal{C}_i)_{\text{red}}$. Its $3p$ -th tensor

power gives (6.16). At an isolated zero of g' outside $g^{-1}(\mathcal{C})$, the regular Semple lift extends and the evaluation has only a removable singularity or a zero. If the critical point lies over $\mathcal{C}$, a logarithmic component such as $(x_i \circ g)'/(x_i \circ g)$ may still have a simple pole; this is precisely one of the boundary poles already counted in (6.16), and no additional Semple-type pole occurs.

Finally choose a finite logarithmic coordinate cover and, above it, a finite cover by relatively compact Semple charts. A partition of unity is used only to estimate the norm of the already-defined global section. On each chart, the components of $\mathcal{D}g_{[1]}$ are logarithmic derivatives of local coordinate functions. A projective Semple coordinate u is a rational function on X_1 and, in a logarithmic frame, a ratio of two first logarithmic derivatives. Hence

$$T_{u \circ g_{[1]}}(r) = O(T_g(r) + \log r).$$

For a partition function χ supported in a relatively compact Semple chart, choose $a \notin \overline{u(\text{supp } \chi)}$. On this support, $u - a$ is bounded above and away from zero, and

$$(u \circ g_{[1]})' = (u \circ g_{[1]} - a) \frac{(u \circ g_{[1]})'}{u \circ g_{[1]} - a}.$$

The logarithmic-derivative lemma applied to $u \circ g_{[1]} - a$ controls the vertical component by

$$O(\log^+ T_g(r) + \log r).$$

Summing this estimate and the analogous horizontal estimates over the finite cover gives (6.17); this is the partition-of-unity argument of [37, Thm. 3.1, (3.3)–(3.6)] with the preceding Semple-coordinate adaptation made explicit. $\square$

Lemma 6.4 (Tautological evaluation of a restricted $\mathcal{O}_{X_2}(m)$ -section). *Let $Z \subset X_2$ be a projective subvariety and let*

$$\sigma \in H^0(Z, \mathcal{O}_{X_2}(m) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-s)|_Z), \quad m, s \geq 1.$$

If $g_{[2]}(\mathbb{C}) \subset Z$, the tautological pairing defines an intrinsic meromorphic section

$$\langle g_{[2]}^* \sigma, (\mathcal{D}g_{[1]})^{\otimes m} \rangle \in H_{\text{mer}}^0(\mathbb{C}, K_{\mathbb{C}}^{\otimes m} \otimes g^* \mathcal{O}_{\mathbb{P}^2}(-s)).$$

If it is not identically zero, then

$$T_g(r) \leq \frac{m}{s} \sum_{i=1}^3 N_g^{[1]}(r, \mathcal{C}_i) + O(\log T_g(r) + \log r) \quad \parallel.$$

Proof. The proof is the same intrinsic pairing and logarithmic/Semple-coordinate estimate as in Lemma 6.3, with $\mathcal{O}_{X_2}(m)$ paired directly against $\mathcal{O}_{X_2}(-m)$. Thus the reduced boundary pole divisor has coefficient at most m ; Poincaré–Lelong and Jensen’s formula give the stated inequality. $\square$

Theorem 6.5. *Let $g: \mathbb{C} \rightarrow \mathbb{P}^2$ be an entire curve whose lift $g_{[2]}: \mathbb{C} \rightarrow X_2$ satisfies $g_{[2]}(\mathbb{C}) \subset Z$. Suppose there exists a section*

$$(6.18) \quad \sigma \in H^0\left(Z, (\mathcal{O}_{X_2}(2, 1)^{\tilde{m}} \otimes \pi_{2,0}^* \mathcal{O}_X(-\tilde{t}))|_Z\right) \cong H^0\left(Z, (\mathcal{O}_{X_2}(3) \otimes \mathcal{O}(-2\Gamma_2))^{\tilde{m}} \otimes \pi_{2,0}^* \mathcal{O}_X(-\tilde{t})|_Z\right),$$

such that the canonical evaluation $s_{\sigma, g}$ of Lemma 6.3, with $p = \tilde{m}$, is not identically zero. Then the following Second Main Theorem type estimate holds:

$$T_g(r) \leq \frac{3\tilde{m}}{\tilde{t}} \sum_{i=1}^3 N_g^{[1]}(r, \mathcal{C}_i) + o(T_g(r)) \quad \parallel.$$

The fixed double vertical factor is retained through the canonical inclusion in Lemma 6.3. The lemma, rather than an identification of an arbitrary local lifting with a jet polynomial, supplies the global meromorphic evaluation and the coefficient $3\tilde{m}/\tilde{t}$.

Proof. Equip $\mathcal{O}_X(-1)$ with the standard norm, whose Chern curvature is $-\omega_{\text{FS}}$. By (6.18), the canonical evaluation $s_{\sigma, g}$ is a nonzero meromorphic section of $K_{\mathbb{C}}^{\otimes 3\tilde{m}} \otimes g^* \mathcal{O}_{\mathbb{P}^2}(-\tilde{t})$. Give $K_{\mathbb{C}}$ the flat metric defined by dz . Poincaré–Lelong and Jensen’s formula give

$$\tilde{t} T_g(r) \leq N(r, \text{div}_{\infty}(s_{\sigma, g})) + \int_0^{2\pi} \log^+ \|s_{\sigma, g}(re^{\sqrt{-1}\theta})\| \frac{d\theta}{2\pi} + O(1).$$

Applying (6.16) and (6.17), with $p = \tilde{m}$, yields

$$\tilde{t}T_g(r) \leq 3\tilde{m} \sum_{i=1}^3 N_g^{[1]}(r, \mathcal{C}_i) + O(\log T_g(r) + \log r) \quad ||,$$

which is the asserted estimate. $\square$

Lemma 6.6 (The zero-evaluation branch). *In the setting of Theorem 6.5, assume that Z is irreducible and that $\sigma \neq 0$. If the canonical evaluation $s_{\sigma,g}$ is identically zero, then*

$$g_{[2]}(\mathbb{C}) \subset Z(\sigma) \subsetneq Z,$$

and $g_{[1]}(\mathbb{C})$ is contained in a proper algebraic subset of X_1 .

Proof. The tautological derivative $\mathcal{D}g_{[1]}$ is not identically zero, since otherwise $g_{[1]}$, and hence g , would be constant. On the dense open set where this derivative is nonzero and $g_{[2]}$ is outside Γ_2 , the canonical inclusion t_p is nonvanishing. The equality $s_{\sigma,g} = 0$ therefore implies $\sigma(g_{[2]}(z)) = 0$. Analytic continuation gives the first inclusion. Since σ is a nonzero section on the irreducible threefold Z , $Z(\sigma)$ is proper and has dimension at most 2. Its image under the proper morphism $\pi_{2,1}$ is consequently a proper algebraic subset of the threefold X_1 containing $g_{[1]}(\mathbb{C})$. $\square$

6.3. An Effective Constant c in the Second Main Theorem for Three Conics. Suppose there exists a nonzero negatively twisted invariant logarithmic 2-jet differential

$$\omega_1 \in H^0(\mathbb{P}^2, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t))$$

satisfying $0 < t < m$ and such that the divisor Z of ω_1 on the second level X_2 of the logarithmic Demailly-Semple tower for $(\mathbb{P}^2, \mathcal{C})$ is irreducible and reduced modulo Γ_2 . Then, by the argument in Section 6.1, for any rational number $\tau \in (0, \tau_1(t/m))$, we obtain a nontrivial global section

$$\sigma \in H^0\left(Z, (\mathcal{O}_{X_2}(2, 1) \otimes \pi_{2,0}^* \mathcal{O}_{\mathbb{P}^2}(-\tau))^{\otimes \ell}\right)$$

for sufficiently divisible $\ell \gg 1$ (depending on τ).

Furthermore, if $f^*\omega_1 \equiv 0$ and the canonical evaluation of σ along $f_{[2]}$ is nonzero, then Theorem 6.5 yields the estimate (1.7) with constant (see (6.6)) $c = \frac{3}{\tau} > \frac{3}{\tau_1(t/m)}$. If the canonical evaluation is zero, Lemma 6.6 shows that $f_{[1]}$ is algebraically degenerate, and Theorem 1.1 gives the stronger estimate $3T_f \leq \sum_i N_f^{[1]} + o(T_f)$. Thus, in either branch and by the arbitrariness of τ , the estimate (1.7) holds for any constant $c > 3/\tau_1(\frac{t}{m})$.

Direct computation shows that $\inf_{0 < x < 1} \frac{3}{\tau_1(x)} = \frac{3+\sqrt{6}}{2}$. Thus $\frac{3}{\tau_1(t/m)} < c$ is equivalent to $c > \frac{3+\sqrt{6}}{2}$ and $0 < \frac{t}{m} < \frac{4c^2-12c+3}{c(4c-3)}$.

Meanwhile, in **Case 1**, Proposition 5.2 gives the coefficient $m/(t-7)$. For this coefficient to be at most c , the positive integers m and t associated to the first family ω_1 must satisfy $t \geq \frac{m}{c} + 7$.

Taking $c = 19$, we obtain the desired estimate (1.7) for positive integer pairs (m, t) satisfying $0 < t < \frac{1219}{1387}m$ or $t \geq \frac{m}{19} + 7$. Note that $8 \cdot \frac{1219}{1387} > 7$, whereas $7 \cdot \frac{1219}{1387} < 7$. To exclude the exceptional cases of (m, t) , it therefore suffices to prove Key Vanishing Lemma I. In fact, the exact real boundary is $c > (75 + \sqrt{5241})/8$, so 19 is the smallest integral constant supplied by this argument.

Taking $c = 5$, we obtain the desired estimate (1.7) for positive integer pairs (m, t) satisfying $0 < t < \frac{43}{85}m$ or $t \geq \frac{m}{5} + 7$. For $m \geq 23$ these two regions overlap, since $43m/85 \geq m/5 + 7$. For $3 \leq m \leq 22$, direct integer enumeration shows that, after using monotonicity in the negative twist, the only minimal uncovered pairs are precisely the eighteen pairs in Key Vanishing Lemma II. (For $m = 20$ and $m = 22$ there is no gap.)

Proof of Theorem 1.6. Work on the common nonempty Zariski-open locus on which Proposition 4.4, the slanted-vector-field construction, and Key Vanishing Lemma II hold. Let f be algebraically nondegenerate. The threshold and irreducible-component argument give a section ω_1 with $m < 5t$ and residual irreducible divisor Z modulo Γ_2 . If $f^*\omega_1 \not\equiv 0$, Theorem 1.2 gives the result because $m/t < 5$.

Assume $f^*\omega_1 \equiv 0$. A nonconstant lifted curve is not contained in Γ_2 , so analytic continuation gives $f_{[2]}(\mathbb{C}) \subset Z$. If $t \geq m/5 + 7$, Proposition 5.2 either gives the constant-5 estimate or makes $f_{[1]}$ algebraically degenerate, in which case Theorem 1.1 is stronger. If $t/m < 43/85$, then $\tau_1(t/m) > 3/5$; choose $3/5 < \tau <$

$\tau_1(t/m)$. The restricted-section construction and Theorem 6.5 give coefficient $3/\tau < 5$ when the canonical evaluation is nonzero, while Lemma 6.6 and Theorem 1.1 handle the zero-evaluation branch.

It remains only to see that one of these alternatives occurs. For $m = 1, 2$ this follows from Theorem 1.5. For $m \geq 23$ the two numerical regions overlap. For $3 \leq m \leq 22$, the eighteen minimal uncovered pairs are exactly those excluded by Key Vanishing Lemma II; twist monotonicity excludes every larger twist with the same m . Finally, algebraic nondegeneracy implies $O(\log T_f(r) + \log r) = o(T_f(r))$ outside the usual exceptional set. This proves (1.7). $\square$

7. Proof of the Key Vanishing Lemmas

7.1. Basic Calculations. Let $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ be a divisor of three smooth conics in $\mathbb{P}^2$, defined by homogeneous quadratic polynomials:

$$\begin{aligned} A(Z) &= A_{200}Z_0^2 + A_{020}Z_1^2 + A_{002}Z_2^2 + A_{110}Z_0Z_1 + A_{101}Z_0Z_2 + A_{011}Z_1Z_2, \\ B(Z) &= B_{200}Z_0^2 + B_{020}Z_1^2 + B_{002}Z_2^2 + B_{110}Z_0Z_1 + B_{101}Z_0Z_2 + B_{011}Z_1Z_2, \\ C(Z) &= C_{200}Z_0^2 + C_{020}Z_1^2 + C_{002}Z_2^2 + C_{110}Z_0Z_1 + C_{101}Z_0Z_2 + C_{011}Z_1Z_2, \end{aligned}$$

where $Z = (Z_0, Z_1, Z_2)$ and all coefficients $A_\bullet, B_\bullet, C_\bullet \in \mathbb{C}$ are generic.

On the affine chart $U_0 = \{Z_0 \neq 0\}$ with coordinates $(x_1, x_2) = (Z_1/Z_0, Z_2/Z_0)$, the dehomogenized polynomials are:

$$\begin{aligned} a(x_1, x_2) &= A_{200} + A_{020}x_1^2 + A_{002}x_2^2 + A_{110}x_1 + A_{101}x_2 + A_{011}x_1x_2, \\ b(x_1, x_2) &= B_{200} + B_{020}x_1^2 + B_{002}x_2^2 + B_{110}x_1 + B_{101}x_2 + B_{011}x_1x_2, \\ c(x_1, x_2) &= C_{200} + C_{020}x_1^2 + C_{002}x_2^2 + C_{110}x_1 + C_{101}x_2 + C_{011}x_1x_2. \end{aligned}$$

Their partial derivatives are given by:

$$\begin{aligned} a_1 &:= \frac{\partial a}{\partial x_1} = 2A_{020}x_1 + A_{110} + A_{011}x_2, & a_2 &:= \frac{\partial a}{\partial x_2} = 2A_{002}x_2 + A_{101} + A_{011}x_1, \\ b_1 &:= \frac{\partial b}{\partial x_1} = 2B_{020}x_1 + B_{110} + B_{011}x_2, & b_2 &:= \frac{\partial b}{\partial x_2} = 2B_{002}x_2 + B_{101} + B_{011}x_1, \\ c_1 &:= \frac{\partial c}{\partial x_1} = 2C_{020}x_1 + C_{110} + C_{011}x_2, & c_2 &:= \frac{\partial c}{\partial x_2} = 2C_{002}x_2 + C_{101} + C_{011}x_1. \end{aligned}$$

The Jacobian determinant is the homogeneous cubic polynomial:

$$J(Z) := \begin{vmatrix} \frac{\partial}{\partial Z_0} A & \frac{\partial}{\partial Z_0} B & \frac{\partial}{\partial Z_0} C \\ \frac{\partial}{\partial Z_1} A & \frac{\partial}{\partial Z_1} B & \frac{\partial}{\partial Z_1} C \\ \frac{\partial}{\partial Z_2} A & \frac{\partial}{\partial Z_2} B & \frac{\partial}{\partial Z_2} C \end{vmatrix}.$$

Its vanishing locus $\{J = 0\}$ is the ramification divisor of the endomorphism:

$$(7.1) \quad \sigma: \mathbb{P}^2 \rightarrow \mathbb{P}^2, \quad [Z_0 : Z_1 : Z_2] \mapsto [A : B : C].$$

The utility of such Jacobian curves has been well-documented in the literature, cf. e.g. [2, 13, 31, 32, 10].

Under generic coefficient assumptions, the Jacobian curve $\{J = 0\}$ satisfies the following transversality properties (cf., e.g., [32, 10]):

(TP1) Transverse intersection with each conic $\mathcal{C}_i$ for $i = 1, 2, 3$.

(TP2) Transverse intersection with each coordinate hyperplane $\{Z_i = 0\}$ for $i = 0, 1, 2$.

For the Key Vanishing Lemma I (but not for II), we will work with parameters satisfying these transversality conditions.

We begin with the following two results, which follow from direct computation on the affine charts $\{Z_i \neq 0\} \subset \mathbb{P}^2$, $i = 0, 1, 2$.

Proposition 7.1. *Let $\mathcal{H} := \{Z_0 = 0\} + \{Z_1 = 0\} + \{Z_2 = 0\}$ be a divisor consisting of the three coordinate hyperplanes in $\mathbb{P}^2$. Then the logarithmic cotangent bundle $T_{\mathbb{P}^2}^*(\log \mathcal{H})$ is globally generated by the sections $(\log \frac{Z_0}{Z_2})'$ and $(\log \frac{Z_1}{Z_2})'$.* $\square$

Proposition 7.2. *The logarithmic 2-jet differential bundle $E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{H})$ of weighted degree m is globally generated by the following sections:*

$$(7.2) \quad \left(\left(\log \frac{Z_0}{Z_2} \right)' \right)^k \left(\left(\log \frac{Z_1}{Z_2} \right)' \right)^{m-3j-k} \left| \begin{array}{cc} \left(\log \frac{Z_0}{Z_2} \right)' & \left(\log \frac{Z_1}{Z_2} \right)' \\ \left(\log \frac{Z_0}{Z_2} \right)'' & \left(\log \frac{Z_1}{Z_2} \right)'' \end{array} \right|^j \quad (0 \leq j \leq \lfloor m/3 \rfloor, 0 \leq k \leq m-3j).$$

□

The endomorphism σ defined in (7.1) is in fact a morphism $\sigma : (\mathbb{P}^2, \mathcal{C}) \rightarrow (\mathbb{P}^2, \mathcal{H})$ between two log pairs. The functoriality immediately yields the relations between logarithmic differential 1-forms

$$\left(\log \frac{A}{C} \right)' = \sigma^* \left(\log \frac{Z_0}{Z_2} \right)' \quad \text{and} \quad \left(\log \frac{B}{C} \right)' = \sigma^* \left(\log \frac{Z_1}{Z_2} \right)'.$$

Combining this with Proposition 7.1, we obtain:

Proposition 7.3. *On the Zariski-open set $\{J \neq 0\} \subset \mathbb{P}^2$, the logarithmic 1-forms $(\log \frac{A}{C})'$ and $(\log \frac{B}{C})'$ generate the sheaf $T_{\mathbb{P}^2}^*(\log \mathcal{C})$.*

We complement this conceptual proof with explicit computations. While the functorial approach is elegant, the detailed coordinate computations reveal specific algebraic structures that will play an essential role in our later analysis.

Proof. The sections $(\log \frac{A}{C})'$ and $(\log \frac{B}{C})'$ are clearly holomorphic on $\{J \neq 0\}$. We prove generation by working on affine charts.

On $U_0 = \{Z_0 \neq 0\}$ with coordinates (x_1, x_2) , consider the transformation matrix between differentials:

$$\begin{pmatrix} \left(\log \frac{a}{c} \right)' \\ \left(\log \frac{b}{c} \right)' \end{pmatrix} = M \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \quad M = \begin{pmatrix} \frac{\partial}{\partial x_1} \log \frac{a}{c} & \frac{\partial}{\partial x_2} \log \frac{a}{c} \\ \frac{\partial}{\partial x_1} \log \frac{b}{c} & \frac{\partial}{\partial x_2} \log \frac{b}{c} \end{pmatrix}.$$

A direct computation gives the determinant of M :

$$(7.3) \quad \det M = \frac{D}{abc}, \quad \text{where} \quad D = \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \frac{J}{2Z_0^3}.$$

Thus the matrix M is invertible on the Zariski open set $U_0 \cap \{J \neq 0\} \cap \{ABC \neq 0\}$, with its inverse given explicitly by:

$$M^{-1} = \frac{1}{D} \begin{pmatrix} a(cb_2 - bc_2) & b(ac_2 - ca_2) \\ a(bc_1 - cb_1) & b(ca_1 - ac_1) \end{pmatrix}.$$

Hence inverting the relation (7.3) yields the explicit coordinate transformation:

$$(7.4) \quad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \frac{1}{D} \begin{pmatrix} a(cb_2 - bc_2) & b(ac_2 - ca_2) \\ a(bc_1 - cb_1) & b(ca_1 - ac_1) \end{pmatrix} \begin{pmatrix} \left(\log \frac{a}{c} \right)' \\ \left(\log \frac{b}{c} \right)' \end{pmatrix}.$$

This establishes that the logarithmic differential forms $(\log \frac{A}{C})'$ and $(\log \frac{B}{C})'$ generate the sheaf of logarithmic cotangent bundle $T_{\mathbb{P}^2}^*(\log \mathcal{C})$ over the Zariski-open set $\{J \neq 0\} \cap \{ABC \neq 0\}$, as the transformation matrix consists of regular functions on this domain.

Furthermore, the differential $(\log c)'$ can be expressed in terms of our generators:

$$\begin{aligned} (\log c)' &= \frac{c'}{c} = \begin{pmatrix} c_1 & c_2 \\ c & c \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \\ \text{[use (7.4)]} &= \frac{1}{D} \left(a \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \right) \begin{pmatrix} \left(\log \frac{a}{c} \right)' \\ \left(\log \frac{b}{c} \right)' \end{pmatrix}. \end{aligned}$$

Thus the sections $(\log \frac{A}{C})'$ and $(\log \frac{B}{C})'$ continue to generate the logarithmic cotangent bundle $T_{\mathbb{P}^2}^*(\log \mathcal{C})$ in a neighborhood of $\{J \neq 0\} \cap \{C = 0\}$ inside $\{J \neq 0\}$, as the transformation matrix remains regular when approaching this locus. The analogous argument holds on charts $U_1 = \{Z_1 \neq 0\}$ and $U_2 = \{Z_2 \neq 0\}$, completing the proof. □

Using Proposition 7.2 and considering the morphism $\sigma : (\mathbb{P}^2, \mathcal{C}) \rightarrow (\mathbb{P}^2, \mathcal{H})$, by functoriality, we obtain:

Proposition 7.4. *The bundle $E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C})$ is freely generated over $\{J \neq 0\}$ by the logarithmic jet differentials:*

$$(7.5) \quad \left(\left(\log \frac{A}{C} \right)' \right)^k \left(\left(\log \frac{B}{C} \right)' \right)^{m-3j-k} \left| \begin{array}{cc} \left(\log \frac{A}{C} \right)' & \left(\log \frac{B}{C} \right)' \\ \left(\log \frac{A}{C} \right)'' & \left(\log \frac{B}{C} \right)'' \end{array} \right|^j \quad (0 \leq j \leq \lfloor m/3 \rfloor, 0 \leq k \leq m-3j). \quad \square$$

Let $P \in \Gamma(\{J \neq 0\}, E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}(-t))$. To check and verify the global extendability of P , it suffices to demonstrate that P admits a holomorphic extension across the Jacobian divisor $\{J = 0\}$. Note that the intersection $\{J = 0\} \cap \{Z_0ABC = 0\}$ constitutes a Zariski-closed subset of codimension 2 in $\mathbb{P}^2$. By Hartog's extension theorem, it is thus sufficient to extend P to $\{J = 0\} \cap \{Z_0ABC \neq 0\}$. This extension problem necessitates an analysis of the transition formula for $E_{2,m}T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}(-t)$ from $\{J \neq 0\}$ to $\{Z_0ABC \neq 0\}$.

Our carefully chosen basis (7.5) induces a highly homogeneous representation of the logarithmic jet differentials and keeps the affine transition problem finite.

The remaining argument treats all Wronskian layers simultaneously. This is essential for the strengthened list, whose largest target has $m = 21$; a small- m pole bound does not determine the higher layers that occur there.

7.2. All-layer finite shape and exact C++ verification. In what follows, we will abbreviate the first-order jet transfer formulas from standard jets (x'_1, x'_2) to logarithmic jets $((\log \frac{a}{c})', (\log \frac{b}{c})')$ as:

$$\begin{aligned} \left(\log \frac{a}{c} \right)' &= \alpha x'_1 + \beta x'_2, & \alpha &:= \frac{a_1c - ac_1}{ac}, & \beta &:= \frac{a_2c - ac_2}{ac}, \\ \left(\log \frac{b}{c} \right)' &= \gamma x'_1 + \delta x'_2, & \gamma &:= \frac{b_1c - bc_1}{bc}, & \delta &:= \frac{b_2c - bc_2}{bc}. \end{aligned}$$

Because logarithmic jet differentials are by definition assumed to exist and to be holomorphic outside the divisor $\mathcal{C} = \{ABC = 0\}$, the presence of a, b, c (within the affine chart $\mathbb{C}^2 \subset \mathbb{P}^2$) in denominators does not produce any singularity.

The only possible singularity—which clarifies why Proposition 7.4 holds only on the complement $\mathbb{P}^2 \setminus \{J = 0\}$ —comes from inverting the transfer formulas from logarithmic jets back to standard jets:

$$(7.6) \quad \begin{aligned} x'_1 &= \frac{\delta}{\alpha\delta - \beta\gamma} \left(\log \frac{a}{c} \right)' - \frac{\beta}{\alpha\delta - \beta\gamma} \left(\log \frac{b}{c} \right)', \\ x'_2 &= -\frac{\gamma}{\alpha\delta - \beta\gamma} \left(\log \frac{a}{c} \right)' + \frac{\alpha}{\alpha\delta - \beta\gamma} \left(\log \frac{b}{c} \right)', \end{aligned}$$

and this requires the nonvanishing of the 2×2 determinant (see (7.3)):

$$(7.7) \quad \alpha\delta - \beta\gamma = \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \det M = \frac{D}{abc}, \quad \text{where} \quad D = \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \frac{J}{2Z_0^3}.$$

By a direct computation in the affine chart $\{Z_0 \neq 0\}$, this determinant happens to be a nonzero multiple of the Jacobian determinant J .

The transfer of the logarithmic Wronskian,

$$\overline{W}''' := \begin{vmatrix} \left(\log \frac{A}{C} \right)' & \left(\log \frac{B}{C} \right)' \\ \left(\log \frac{A}{C} \right)'' & \left(\log \frac{B}{C} \right)'' \end{vmatrix} = \begin{vmatrix} \left(\log \frac{a}{c} \right)' & \left(\log \frac{b}{c} \right)' \\ \left(\log \frac{a}{c} \right)'' & \left(\log \frac{b}{c} \right)'' \end{vmatrix},$$

to standard affine jet coordinates is more delicate, and is captured in the following Key Observation.

Key Observation 7.8. *When expressed in standard affine jet coordinates $(x'_1, x'_2, x''_1, x''_2)$, the logarithmic Wronskian takes the form:*

$$(7.9) \quad \overline{W}''' = (\alpha\delta - \beta\gamma) \begin{vmatrix} x'_1 & x'_2 \\ x''_1 & x''_2 \end{vmatrix} + \kappa x_1'^3 + \lambda x_1'^2 x'_2 + \mu x_1' x_2'^2 + \nu x_2'^3,$$

where the cubic terms in the first-order jets x'_1, x'_2 have coefficients $\kappa, \lambda, \mu, \nu$ given in terms of the second-order derivatives

$$a_{11}, a_{12}, a_{22}, \quad b_{11}, b_{12}, b_{22}, \quad c_{11}, c_{12}, c_{22},$$

and the coefficients $\alpha, \beta, \gamma, \delta$ as follows:

$$\begin{aligned}\kappa &:= -\gamma \frac{a_{11}}{a} + \alpha \frac{b_{11}}{b} - (\alpha - \gamma) \frac{c_{11}}{c} + \alpha^2 \gamma - \alpha \gamma^2, \\ \lambda &:= -\delta \frac{a_{11}}{a} - 2\gamma \frac{a_{12}}{a} + \beta \frac{b_{11}}{b} + 2\alpha \frac{b_{12}}{b} - (\beta - \delta) \frac{c_{11}}{c} - 2(\alpha - \gamma) \frac{c_{12}}{c} \\ &\quad + \alpha^2 \delta + 2\alpha \beta \gamma - 2\alpha \gamma \delta - \beta \gamma^2, \\ \mu &:= -2\delta \frac{a_{12}}{a} - \gamma \frac{a_{22}}{a} + 2\beta \frac{b_{12}}{b} + \alpha \frac{b_{22}}{b} - 2(\beta - \delta) \frac{c_{12}}{c} - (\alpha - \gamma) \frac{c_{22}}{c} \\ &\quad + 2\alpha \beta \delta - \alpha \delta^2 + \beta^2 \gamma - 2\beta \gamma \delta, \\ \nu &:= -\delta \frac{a_{22}}{a} + \beta \frac{b_{22}}{b} - (\beta - \delta) \frac{c_{22}}{c} \\ &\quad + \beta^2 \delta - \beta \delta^2.\end{aligned}$$

Proof. This follows by direct computation. $\square$

According to Key Vanishing Lemma II, it is enough to consider $m \leq 21$. On the affine chart $(x_1, x_2) \in \mathbb{C}^2 \subset \mathbb{P}^2$, away from the Jacobian zero-locus

$$\{D = 0\} = \{J = 0\} \cap \{Z_0 \neq 0\},$$

the basis in Proposition 7.4 gives the uniform expansion

$$(7.10) \quad \text{Jet}_{ABC} = \sum_{q=0}^{\lfloor m/3 \rfloor} \sum_{i+j=m-3q} P_{i,j}^{(q)} \left(\left(\log \frac{a}{c} \right)' \right)^i \left(\left(\log \frac{b}{c} \right)' \right)^j (\overline{W}''')^q.$$

Every coefficient $P_{i,j}^{(q)}$ is holomorphic away from $\{D = 0\}$.

The GAGA principle then ensures that every such coefficient is a rational expression whose denominator is an integral power of D . Our preliminary objective is to determine a uniform bound for that power in each Wronskian layer.

To this end, we consider the transfer formulas from logarithmic jets to standard jets. The first-order jet transfer formulas are already given in (7.6). Using (7.9), we have

$$W''' := \begin{vmatrix} x'_1 & x'_2 \\ x''_1 & x''_2 \end{vmatrix} = \frac{1}{\alpha\delta - \beta\gamma} \overline{W}''' - \frac{1}{\alpha\delta - \beta\gamma} \left(\kappa x_1'^3 + \lambda x_1'^2 x_2' + \mu x_1' x_2'^2 + \nu x_2'^3 \right),$$

where the cubic term $\kappa x_1'^3 + \lambda x_1'^2 x_2' + \mu x_1' x_2'^2 + \nu x_2'^3$ can be expanded using (7.6) as:

$$\begin{aligned}& \begin{pmatrix} x_1'^3 & x_1'^2 x_2' & x_1' x_2'^2 & x_2'^3 \end{pmatrix} \begin{pmatrix} \kappa \\ \lambda \\ \mu \\ \nu \end{pmatrix} \\ &= \left(\left(\left(\log \frac{a}{c} \right)' \right)^3 \quad \left(\left(\log \frac{a}{c} \right)' \right)^2 \left(\log \frac{b}{c} \right)' \quad \left(\log \frac{a}{c} \right)' \left(\left(\log \frac{b}{c} \right)' \right)^2 \quad \left(\left(\log \frac{b}{c} \right)' \right)^3 \right) \\ &\quad \cdot \frac{1}{(\alpha\delta - \beta\gamma)^3} \begin{pmatrix} \delta^3 & -\gamma\delta^2 & \gamma^2\delta & -\gamma^3 \\ -3\beta\delta^2 & \alpha\delta^2 + 2\beta\gamma\delta & -2\alpha\gamma\delta - \beta\gamma^2 & 3\alpha\gamma^2 \\ 3\beta^2\delta & -2\alpha\beta\delta - \beta^2\gamma & \alpha^2\delta + 2\alpha\beta\gamma & -3\alpha^2\gamma \\ -\beta^3 & \alpha\beta^2 & -\alpha^2\beta & \alpha^3 \end{pmatrix} \begin{pmatrix} \kappa \\ \lambda \\ \mu \\ \nu \end{pmatrix}.\end{aligned}$$

Let us examine this more closely. Define

$$\begin{pmatrix} \delta^3 & -\gamma\delta^2 & \gamma^2\delta & -\gamma^3 \\ -3\beta\delta^2 & \alpha\delta^2 + 2\beta\gamma\delta & -2\alpha\gamma\delta - \beta\gamma^2 & 3\alpha\gamma^2 \\ 3\beta^2\delta & -2\alpha\beta\delta - \beta^2\gamma & \alpha^2\delta + 2\alpha\beta\gamma & -3\alpha^2\gamma \\ -\beta^3 & \alpha\beta^2 & -\alpha^2\beta & \alpha^3 \end{pmatrix} \begin{pmatrix} \kappa \\ \lambda \\ \mu \\ \nu \end{pmatrix} =: \begin{pmatrix} \tau_{3,0} \\ \tau_{2,1} \\ \tau_{1,2} \\ \tau_{0,3} \end{pmatrix}.$$

Miracle Factorizations 7.11. *The four expressions:*

$$\begin{aligned}
\tau_{3,0} &:= \delta^3 \kappa - \gamma \delta^2 \lambda + \gamma^2 \delta \mu - \gamma^3 \nu \\
&= (\alpha \delta - \beta \gamma) (\dots), \\
\tau_{2,1} &:= -3 \beta \delta^2 \kappa + (\alpha \delta^2 + 2 \beta \gamma \delta) \lambda + (-2 \alpha \gamma \delta - \beta \gamma^2) \mu + 3 \alpha \gamma^2 \nu \\
&= (\alpha \delta - \beta \gamma) (\dots), \\
\tau_{1,2} &:= 3 \beta^2 \delta \kappa + (-2 \alpha \beta \delta - \beta^2 \gamma) \lambda + (\alpha^2 \delta + 2 \alpha \beta \gamma) \mu - 3 \alpha^2 \gamma \nu \\
&= (\alpha \delta - \beta \gamma) (\dots), \\
\tau_{0,3} &:= -\beta^3 \kappa + \alpha \beta^2 \lambda - \alpha^2 \beta \mu + \alpha^3 \nu \\
&= (\alpha \delta - \beta \gamma) (\dots),
\end{aligned}$$

are all multiples of $\alpha \delta - \beta \gamma$.

Proof. Substitute the displayed formulas for $\kappa, \lambda, \mu, \nu$ and divide in the polynomial ring generated by $\alpha, \beta, \gamma, \delta$ and the normalized second derivatives. Each division has zero remainder. The four quotients are displayed and independently checked by exact symbolic expansion in Appendix A. $\square$

Therefore:

$$\begin{pmatrix} \tau_{3,0} \\ \tau_{2,1} \\ \tau_{1,2} \\ \tau_{0,3} \end{pmatrix} =: (\alpha \delta - \beta \gamma) \begin{pmatrix} \theta_{3,0} \\ \theta_{2,1} \\ \theta_{1,2} \\ \theta_{0,3} \end{pmatrix},$$

and finally:

$$(7.12) \quad \begin{vmatrix} x'_1 & x'_2 \\ x''_1 & x''_2 \end{vmatrix} = \frac{1}{\alpha \delta - \beta \gamma} \begin{vmatrix} (\log \frac{a}{c})' & (\log \frac{b}{c})' \\ (\log \frac{a}{c})'' & (\log \frac{b}{c})'' \end{vmatrix} - \frac{1}{(\alpha \delta - \beta \gamma)^3} \sum_{i+j=3} \theta_{i,j} ((\log \frac{a}{c})')^i ((\log \frac{b}{c})')^j.$$

Recall that our main assumption is that, in the standard affine coordinates $(x_1, x_2) \in \mathbb{C}^2$, the logarithmic 2-jet differential Jet_{ABC} is holomorphic over $\mathbb{C}^2 \setminus \{abc = 0\}$. Therefore, it can be expressed as

$$(7.13) \quad \text{Jet}_{ABC} = \sum_{q=0}^{\lfloor m/3 \rfloor} \sum_{r+s=m-3q} \bar{P}_{r,s}^{(q)} x_1'^r x_2'^s (W''')^q,$$

where every $\bar{P}_{r,s}^{(q)}$ is holomorphic on $\mathbb{C}^2 \setminus \{abc = 0\}$. By the GAGA principle it is a rational function whose denominator may contain only a, b , and c .

We now compare (7.10) with (7.13). Substitution of (7.6) and (7.12) shows that the coefficients $\{P_{i,j}^{(q)}\}_{i+j=m-3q}$ can be written as rational expressions whose denominators consist of factors a, b, c (which are coprime to D) and D (see (7.7)). In Wronskian layer q , the maximal possible pole order with respect to D is $m - 2q$.

Indeed, an ordinary first derivative contributes at most one inverse power of D , while each occurrence of the lower cubic term in (7.12) trades three first-derivative weights for two additional inverse powers of D . For a monomial of total weight m in layer q , this gives precisely $D^{-(m-2q)}$. The argument is independent of q and therefore covers every $0 \leq q \leq \lfloor m/3 \rfloor$.

Consequently the invariant logarithmic 2-jet differential Jet_{ABC} in (7.10), of weighted order $m \leq 21$, has the following uniform all-layer form in homogeneous coordinates:

$$(7.14) \quad \text{Jet}_{ABC} = \sum_{q=0}^{\lfloor m/3 \rfloor} \sum_{i+j=m-3q} \frac{P_{i,j}^{(q)}(Z_0, Z_1, Z_2)}{J(Z_0, Z_1, Z_2)^{m-2q}} \left((\log \frac{A}{C})' \right)^i \left((\log \frac{B}{C})' \right)^j (\bar{W}''')^q,$$

where

$$(7.15) \quad n_q := m - 2q, \quad P_{i,j}^{(q)} \in \mathbb{C}[Z_0, Z_1, Z_2]_{d_q}, \quad d_q := 3n_q - t.$$

Indeed, homogeneity gives $d_q - 3(m - 2q) = -t$ in every layer. Thus (7.14) is not a truncation at a fixed Wronskian power: it is the complete finite shape, and it is precisely the shape enumerated by the C++ verifier. It represents a *meromorphic* (rational) section of

$$E_{2,m} T_{\mathbb{P}^2}^* (\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-t).$$

For Jet_{ABC} to be a genuine *holomorphic* logarithmic jet differential, the coefficients of all polynomials $P_{i,j}^{(q)}$ must satisfy a finite homogeneous linear system. The system records that, after the complete coupled transition to ordinary affine jets, every coefficient has the divisibility required to cancel the appropriate power of the Jacobian J . Cancellations are taken only after summing the contributions of all Wronskian layers to the same ordinary-jet monomial; testing the layers separately would give only a diagnostic, not the desired kernel computation.

The number of homogeneous coefficient variables is

$$(7.16) \quad N(m, t) = \sum_{q=0}^{\lfloor m/3 \rfloor} (m - 3q + 1) \binom{3(m - 2q) - t + 2}{2},$$

where a summand is omitted when $3(m - 2q) - t < 0$. In particular, $N(13, 9) = 12,550$, whereas the largest strengthened target $(m, t) = (21, 11)$ has $N(21, 11) = 81,840$ variables.

To streamline explicit computer-assisted computations, we adopt a strategic simplification by choosing specific coefficients $A_\bullet, B_\bullet, C_\bullet$ such that the three conics $\mathcal{C}$ are of Fermat type; for example:

$$(7.17) \quad A(Z) = 2Z_0^2 + Z_1^2 + Z_2^2, \quad B(Z) = Z_0^2 + 2Z_1^2 + Z_2^2, \quad C(Z) = Z_0^2 + Z_1^2 + 2Z_2^2.$$

Each of the three conics in (7.17) is smooth. Any two meet transversally, and the three have no common point: subtracting their equations gives $Z_0^2 = Z_1^2 = Z_2^2$, after which any one equation forces all three coordinates to vanish. Hence their union is a simple normal crossing divisor. This choice also gives $J = 32Z_0Z_1Z_2$, which makes the divisibility test particularly transparent.

The auxiliary Jacobian condition **(TP2)** nevertheless fails at finitely many points. We therefore test extension on both $\{Z_0ABC \neq 0\}$ and $\{Z_2ABC \neq 0\}$. On $\mathbb{P}^2 \setminus \mathcal{C}$, their union contains $\{J = 0\}$ except for the single point $[0 : 1 : 0]$. That omitted set has codimension two, so Hartogs' theorem completes the extension there.

More precisely, we work in the two affine charts:

$$\{Z_0 \neq 0\} \quad \text{and} \quad \{Z_2 \neq 0\},$$

with the coordinate substitutions:

$$(Z_0, Z_1, Z_2) \mapsto (1, x_1, x_2) \quad \text{and} \quad (Z_0, Z_1, Z_2) \mapsto (y_0, y_1, 1).$$

In these coordinates, J reads as $32x_1x_2$ or $32y_0y_1$. Thus the required cancellation is equivalent to divisibility by $(x_1x_2)^{m-2k}$ or $(y_0y_1)^{m-2k}$ in the coefficient of the ordinary Wronskian power W^k . These conditions form the finite homogeneous linear system computed below.

The exact matrix computed by the C++ verifier. On either affine chart, write again A, B, C for the affine equations, and put

$$H := ABC, \quad \widehat{L}_1 := B(CA' - AC'), \quad \widehat{L}_2 := A(CB' - BC').$$

Thus $H(\log(A/C))' = \widehat{L}_1$ and $H(\log(B/C))' = \widehat{L}_2$. If $W = x_1'x_2'' - x_1''x_2'$ is the ordinary affine Wronskian, define the cubic first-jet polynomial R_3 by

$$(7.18) \quad \widehat{L}_1(\widehat{L}_2)' - \widehat{L}_2(\widehat{L}_1)' = HDW + R_3.$$

Equivalently,

$$(7.19) \quad H^3 \overline{W}''' = H^2 DW + HR_3.$$

The program constructs R_3 from (7.18); it does not enter a precomputed formula for it.

Fix a source layer q in (7.14), and put

$$n_q = m - 2q, \quad s_q = m - 3q, \quad i + j = s_q, \quad \deg P_{i,j}^{(q)} = 3n_q - t.$$

After substituting (7.19), the contribution of $P_{i,j}^{(q)}$ to the coefficient of W^k , $0 \leq k \leq q$, is placed over the common denominator $H^{m-2k} J^{m-2k}$. Its numerator is exactly

$$(7.20) \quad \binom{q}{k} P_{i,j}^{(q)} \widehat{L}_1^i \widehat{L}_2^j D^k (HJ^2 R_3)^{q-k}.$$

For fixed k and a fixed velocity monomial, the program first sums (7.20) over *all* $q \geq k$ and only then tests divisibility by J^{m-2k} . This order matters: cancellation between different Wronskian layers is part of

the extension problem. A layer-by-layer rank test would therefore compute a different, generally stronger system.

Each scalar coefficient of each homogeneous polynomial $P_{i,j}^{(q)}$ is a column. A row is indexed by a chart, the power k , a velocity monomial, and a base monomial that is not divisible by the required power of the monomial Jacobian. The verifier generates these sparse columns on demand and performs exact column elimination over $\mathbb{F}_5$; it never materializes a dense matrix. No parity split, character decomposition, involution, root-of-unity filter, or symmetry reduction is used.

Before a rank computation, the self-test verifies the determinant identity for D , checks (7.19) directly on nonsingular jets in both charts, and checks the expected variable counts. The transition matrix has integer entries. Hence full column rank after reduction modulo 5 exhibits an integer maximal minor that is nonzero modulo 5; that minor is nonzero over $\mathbb{Q}$, and the original matrix has zero kernel over $\mathbb{C}$.

The source can be compiled and the largest target rerun from the root of the evidence archive as follows:

```
clang++ -std=c++17 -O3 -DNDEBUG -static \
  source/paper1_three_conics_full_verifier.cpp -o verifier
./verifier --self-test
./verifier --case 21 11 --max-seconds 3600 \
  --max-pivot-nonzeros 650000000 \
  --certificate certificates/recheck_m21_t11_p5.json
```

On Windows, the last two commands may be run with `verifier.exe`. Exit code 0 together with the verdict

PROVED_FULL_COLUMN_RANK_MOD_5

means that every expected column was processed and the exact nullity is zero. A time or memory guard returns

UNRESOLVED_RESOURCE_LIMIT

and is never accepted as a vanishing proof. The JSON files record ranks, pivot rows, and fingerprints for reproducibility; the auditable proof object is the exact source together with a successful deterministic rerun.

All eighteen matrices in Key Vanishing Lemma II have full column rank. For the largest case $(m, t) = (21, 11)$, the program processes 81,840 columns, finds rank 81,840, and returns nullity zero.¹

Remark 7.5. For the explicit Fermat-type configuration (7.17), our exact finite computations give the vanishings for $(m, t) = (1, 1), (2, 1), (3, 1), (4, 1), (5, 2)$. For $(m, t) = (5, 1)$, by contrast, they give

$$\dim H^0(\mathbb{P}^2, E_{2,5} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = 9.$$

Let $\eta_1, \dots, \eta_9$ be a basis of this space. The vanishings in weighted degrees 1, 2, 3, 4 exclude a common invariant jet generator of smaller positive weight. The standard minimal-weight argument therefore shows that the 9-dimensional space contains two algebraically independent members, say η and θ . If either pullback is nonzero, Theorem 1.2 gives the desired estimate with coefficient $m/t = 5$. If both pullbacks vanish, the common zero locus of η and θ has proper image in the first Semple stage; hence $f_{[1]}$ is algebraically degenerate and Theorem 1.1 applies. Thus the nine sections give a Second Main Theorem with constant 5 for the Fermat-type configuration (7.17). This special conclusion is not needed in the proof of Theorem 1.6.

Acknowledgements

We are grateful to Julien Duval for introducing us a decade ago to the problem of establishing a Second Main Theorem for three conics in $\mathbb{P}^2$. We thank Junjiro Noguchi for sharing his private notes and providing helpful comments. We thank Duc Viet Vu for offering valuable insights from a pluripotential perspective.

S.-Y. Xie would like to thank Frédéric Campana, Yunling Chen, Tao Cui, Hanlong Fang, Jie Liu, Mihai Păun, Erwan Rousseau, Yum-Tong Siu, Ruiran Sun, and Kang Zuo for inspiring discussions. S.-Y. Xie thanks Daniel Barlet and Renjie Lv for discussions concerning the one-jet vanishing problem. S.-Y. Xie and L. Hou also wish to thank Tianyuan Mathematics Research Center for its excellent environment.

We also thank Yunling Chen, Alexandre Eremenko, and Yi C. Huang for their suggestions on an earlier version. We thank Steven Lu and Katsutoshi Yamanoi for helpful comments.

¹The complete C++ evidence archive—source, mathematical explanation, certificates, hashes, and full logs—is available from <https://xiesongyan.github.io/>.

We are grateful to Eric Riedl for pointing out an error in Proposition 6.1 in an earlier version and for providing a concrete counterexample. This observation helped us correct the relevant argument.

We thank Pengchao Wang for pointing out two important errors in an earlier version.

The results of this article were first presented by S.-Y. Xie at *The Seminar on Several Complex Variables and Complex Geometry* (held from July 27 to August 2, 2025), which was organized by Xiangyu Zhou, Yum-Tong Siu, and John Erik Fornæss.

Funding

S.-Y. Xie acknowledges partial support from the National Key R&D Program of China under Grants No. 2023YFA1010500 and No. 2021YFA1003100, and from the National Natural Science Foundation of China under Grants No. 12288201 and No. 12471081, as well as support from the Xiaomi Young Talents Program. D.T. Huynh is supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under the grant number 101.04-2025.19.

APPENDIX A. EXPLICIT QUOTIENTS IN THE FOUR FACTORIZATION IDENTITIES

We complete the calculation in Miracle Factorizations 7.11. Set

$$\widehat{a}_{ij} := \frac{a_{ij}}{a}, \quad \widehat{b}_{ij} := \frac{b_{ij}}{b}, \quad \widehat{c}_{ij} := \frac{c_{ij}}{c}, \quad \Delta := \alpha\delta - \beta\gamma.$$

After substituting the formulas for $\kappa, \lambda, \mu, \nu$, direct expansion gives

$$\tau_{3,0} = \Delta\theta_{3,0}, \quad \tau_{2,1} = \Delta\theta_{2,1}, \quad \tau_{1,2} = \Delta\theta_{1,2}, \quad \tau_{0,3} = \Delta\theta_{0,3},$$

where

$$\begin{aligned} \theta_{3,0} &= \delta^2(\widehat{b}_{11} - \widehat{c}_{11}) - 2\delta\gamma(\widehat{b}_{12} - \widehat{c}_{12}) + \gamma^2(\widehat{b}_{22} - \widehat{c}_{22}), \\ \theta_{2,1} &= -\delta^2\widehat{a}_{11} + 2\delta\gamma\widehat{a}_{12} - \gamma^2\widehat{a}_{22} - 2\beta\delta\widehat{b}_{11} + 2(\alpha\delta + \beta\gamma)\widehat{b}_{12} - 2\alpha\gamma\widehat{b}_{22} \\ &\quad + (2\beta\delta + \delta^2)\widehat{c}_{11} - 2(\alpha\delta + \beta\gamma + \delta\gamma)\widehat{c}_{12} + (2\alpha\gamma + \gamma^2)\widehat{c}_{22} + \Delta^2, \\ \theta_{1,2} &= 2\beta\delta\widehat{a}_{11} - 2(\alpha\delta + \beta\gamma)\widehat{a}_{12} + 2\alpha\gamma\widehat{a}_{22} + \beta^2\widehat{b}_{11} - 2\alpha\beta\widehat{b}_{12} + \alpha^2\widehat{b}_{22} \\ &\quad - (\beta^2 + 2\beta\delta)\widehat{c}_{11} + 2(\alpha\beta + \alpha\delta + \beta\gamma)\widehat{c}_{12} - (\alpha^2 + 2\alpha\gamma)\widehat{c}_{22} - \Delta^2, \\ \theta_{0,3} &= -\beta^2(\widehat{a}_{11} - \widehat{c}_{11}) + 2\alpha\beta(\widehat{a}_{12} - \widehat{c}_{12}) - \alpha^2(\widehat{a}_{22} - \widehat{c}_{22}). \end{aligned}$$

For completeness, we also verified these identities by exact symbolic computation. The calculation was performed with SymPy 1.14.0 in the polynomial ring

$$\mathbb{Z}[\alpha, \beta, \gamma, \delta, \widehat{a}_{11}, \widehat{a}_{12}, \widehat{a}_{22}, \widehat{b}_{11}, \widehat{b}_{12}, \widehat{b}_{22}, \widehat{c}_{11}, \widehat{c}_{12}, \widehat{c}_{22}].$$

For each of the four expressions, exact division by Δ returned the quotient displayed above and remainder zero. A second expansion of $\tau_{i,j} - \Delta\theta_{i,j}$ returned the zero polynomial. Thus the factorization is an identity in the coefficient ring, not merely a numerical check at sampled values.

APPENDIX B. PROOF OF THE ONE-JET VANISHING THEOREM

Proof of Theorem 1.5. The proof has three steps. First, two natural logarithmic forms identify E as an elementary transform of the trivial rank-two bundle along a smooth ramification cubic. A Chern-class computation then determines the transform and allows us to prove that E is nef by the curve criterion. Finally, an intersection calculation on $\mathbb{P}(E)$ excludes every negative twist.

Choose homogeneous quadratic equations

$$\mathcal{C}_1 = \{A = 0\}, \quad \mathcal{C}_2 = \{B = 0\}, \quad \mathcal{C}_3 = \{C = 0\}.$$

For a generic triple, A, B, C have no common zero and therefore define a morphism

$$(B.1) \quad \sigma: \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \quad [Z_0 : Z_1 : Z_2] \longmapsto [A : B : C].$$

The morphism σ is proper. If one of its fibers were positive-dimensional, that fiber would contain a curve Γ_0 . The restriction of $\sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$ to Γ_0 would then be trivial, whereas

$$\sigma^*\mathcal{O}_{\mathbb{P}^2}(1) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$$

has positive degree on every curve. Thus σ is quasi-finite and hence, being proper, finite. If $\mathcal{H} = \{Y_0Y_1Y_2 = 0\}$ is the coordinate triangle in the target, then $\mathcal{C} = \sigma^{-1}(\mathcal{H})$. Hence the logarithmic forms

$$\eta_1 = \mathrm{dlog} \frac{A}{C}, \quad \eta_2 = \mathrm{dlog} \frac{B}{C}$$

induce a bundle morphism

$$\varphi: \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \longrightarrow E.$$

Let

$$J = \det \left(\frac{\partial(A, B, C)}{\partial(Z_0, Z_1, Z_2)} \right) \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)), \quad R = \{J = 0\},$$

and set

$$\Omega = Z_0 dZ_1 \wedge dZ_2 - Z_1 dZ_0 \wedge dZ_2 + Z_2 dZ_0 \wedge dZ_1.$$

A direct homogeneous computation gives

$$(B.2) \quad \eta_1 \wedge \eta_2 = \frac{J}{2} \frac{\Omega}{ABC}.$$

Thus R is the degeneracy divisor of φ . For a generic triple it is a smooth cubic; the relevant generic locus is shown to be nonempty in Remark B.1. Along R the determinant vanishes, so $\mathrm{rank} \varphi \leq 1$. If the rank were zero at some point, all four entries of a local matrix would belong to the maximal ideal there, and its determinant would vanish to order at least two. This contradicts the smoothness of R . Hence $\mathrm{rank} \varphi = 1$ everywhere on R . Consequently, with $i: R \hookrightarrow \mathbb{P}^2$, there is a line bundle L on R and an exact sequence

$$(B.3) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \xrightarrow{\varphi} E \longrightarrow i_*L \longrightarrow 0.$$

Indeed, elementary row and column operations reduce a local matrix of φ along R to $\mathrm{diag}(1, u)$, where $u = 0$ is a local equation of R .

The residue sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^2}^1 \longrightarrow E \longrightarrow \bigoplus_{j=1}^3 \mathcal{O}_{\mathcal{C}_j} \longrightarrow 0$$

gives

$$(B.4) \quad \mathrm{ch}(E) = 2 + 3H - \frac{9}{2}H^2, \quad c_1(E) = 3H, \quad c_2(E) = 9H^2.$$

Grothendieck–Riemann–Roch for the smooth cubic $i: R \hookrightarrow \mathbb{P}^2$ gives the explicit identity

$$\mathrm{ch}(i_*L) = i_* \left(\mathrm{ch}(L) \frac{\mathrm{td}(R)}{i^* \mathrm{td}(\mathbb{P}^2)} \right) = 3H + \left(\deg L - \frac{R^2}{2} \right) H^2 = 3H + \left(\deg L - \frac{9}{2} \right) H^2.$$

Comparison with (B.3) and (B.4) gives

$$(B.5) \quad \deg L = 0.$$

For completeness, the same Chern-character computation also gives

$$\mathrm{ch}(S^m E) = (m+1) + \frac{3m(m+1)}{2}H - \frac{9m(m+1)}{4}H^2.$$

Applying Hirzebruch–Riemann–Roch after twisting by $\mathcal{O}_{\mathbb{P}^2}(-t)$ proves formula (1.5).

We next prove that E is nef. By the curve criterion, it is enough to show that, for every morphism $v: B \rightarrow \mathbb{P}^2$ from a smooth projective curve, every quotient line bundle of v^*E has nonnegative degree. We first treat curves contained in R . Restrict (B.3) to R . Since R is Cartier,

$$\mathrm{Tor}_1^{\mathbb{P}^2}(i_*L, \mathcal{O}_R) \simeq L \otimes \mathcal{O}_R(-R),$$

and tensoring gives

$$(B.6) \quad 0 \longrightarrow L(-R) \longrightarrow \mathcal{O}_R^{\oplus 2} \longrightarrow E|_R \longrightarrow L \longrightarrow 0.$$

The constant-rank-one property shows that the first arrow is a subbundle inclusion. If M denotes the image of $\mathcal{O}_R^{\oplus 2} \rightarrow E|_R$, then

$$(B.7) \quad \begin{aligned} 0 &\longrightarrow L(-R) \longrightarrow \mathcal{O}_R^{\oplus 2} \longrightarrow M \longrightarrow 0, \\ 0 &\longrightarrow M \longrightarrow E|_R \longrightarrow L \longrightarrow 0. \end{aligned}$$

Both L and M are line bundles, and

$$(B.8) \quad \deg M = -\deg L(-R) = R^2 = 9.$$

It follows that $E|_R$ is nef. Indeed, for a quotient line bundle $E|_R \twoheadrightarrow Q$, either $M \rightarrow Q$ is nonzero, in which case $\deg Q \geq \deg M = 9$, or the quotient factors through L , in which case $Q \simeq L$ and $\deg Q = 0$.

Let now $v: B \rightarrow \mathbb{P}^2$ be a nonconstant morphism from a smooth projective curve. If $v(B) \not\subset R$, tensoring (B.3) with $\mathcal{O}_B$ may produce $\mathrm{Tor}_1^{\mathbb{P}^2}(i_*L, \mathcal{O}_B)$. This is a torsion sheaf, and its image in the torsion-free sheaf $\mathcal{O}_B^{\oplus 2}$ is zero. Hence $v^*\varphi$ is injective; it is generically an isomorphism and fits into

$$(B.9) \quad 0 \longrightarrow \mathcal{O}_B^{\oplus 2} \longrightarrow v^*E \longrightarrow T \longrightarrow 0,$$

where T is torsion. For every quotient line bundle $v^*E \twoheadrightarrow Q$, the composite $\mathcal{O}_B^{\oplus 2} \rightarrow Q$ is nonzero; otherwise the quotient would factor through T , whereas $\mathrm{Hom}(T, Q) = 0$. Thus Q has a nonzero global section and $\deg Q \geq 0$. If $v(B) \subset R$, the same conclusion follows from the nefness of $E|_R$. Therefore E is nef.

Finally, use the quotient convention for

$$p: Y = \mathbb{P}_{\mathbb{P}^2}(E) \longrightarrow \mathbb{P}^2, \quad \xi = c_1(\mathcal{O}_Y(1)), \quad h = p^*H.$$

Then $p_*\mathcal{O}_Y(m) = S^m E$ for $m \geq 0$, and the nefness of E is equivalent to that of ξ . With the quotient convention, the projective-bundle relation is

$$\xi^2 - \xi p^*c_1(E) + p^*c_2(E) = 0.$$

Together with (B.4), it gives the intersection numbers

$$(B.10) \quad \xi^3 = c_1(E)^2 - c_2(E) = 0, \quad \xi^2 h = c_1(E) \cdot H = 3.$$

By the projection formula, a nonzero section in (1.6) would define an effective divisor $\Gamma \sim m\xi - th$ on Y . Since ξ is nef, the intersection of the two nef classes ξ, ξ with the effective surface Γ is nonnegative; hence $\xi^2 \cdot \Gamma \geq 0$. On the other hand,

$$\xi^2 \cdot \Gamma = m\xi^3 - t\xi^2 h = -3t < 0.$$

This contradiction proves the asserted vanishing. $\square$

Remark B.1 (Nonemptiness of the genericity conditions). The conditions used in the proof are Zariski open and nonempty. For example, take

$$A = Z_0^2 + Z_1 Z_2, \quad B = Z_1^2 + Z_2 Z_0, \quad C = Z_2^2 + Z_0 Z_1.$$

The conics are smooth, have no common point, and meet pairwise transversally. Indeed, if one coordinate vanishes, the three equations force all three to vanish; if none vanishes, multiplying $A = B = C = 0$ gives $(Z_0 Z_1 Z_2)^2 = -(Z_0 Z_1 Z_2)^2$, again a contradiction. For instance, if $A = B = 0$ and $\nabla A = \lambda \nabla B$, then

$$2Z_0 = \lambda Z_2, \quad Z_2 = 2\lambda Z_1, \quad Z_1 = \lambda Z_0.$$

If $Z_0 = 0$, all coordinates vanish; otherwise $\lambda^3 = 1$, and then $A = 3Z_0^2 \neq 0$, a contradiction. The other pairs follow cyclically. Moreover,

$$J = -2(Z_0^3 + Z_1^3 + Z_2^3 - 5Z_0 Z_1 Z_2),$$

which is a smooth Hesse cubic. Indeed, a common zero of its three partial derivatives would satisfy

$$3Z_0^2 = 5Z_1 Z_2, \quad 3Z_1^2 = 5Z_2 Z_0, \quad 3Z_2^2 = 5Z_0 Z_1.$$

Multiplication gives $27(Z_0 Z_1 Z_2)^2 = 125(Z_0 Z_1 Z_2)^2$ when no coordinate vanishes, while if one coordinate vanishes, the three identities force all of them to vanish. Both alternatives are impossible in $\mathbb{P}^2$.

APPENDIX C. A COHOMOLOGICAL OBSERVATION

This appendix was authored by Pengchao Wang

(School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China).

Email: 2200010773@stu.pku.edu.cn.

Let $\mathcal{C} \subset \mathbb{P}^2$ be three generic smooth conics with simple normal crossings. Let

$$\begin{aligned} 0 \subset S^m T_{\mathbb{P}^2}^*(\log \mathcal{C}) &= F^0(E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C})) \\ &\subset F^1(E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C})) \subset \dots \\ &\subset F^{\lfloor m/3 \rfloor}(E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C})) \\ &= E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \end{aligned}$$

be the ascending filtration of subsheaves that defines the grading (2.3). We set $F^k(E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C})) = 0$ for $k \leq -1$. From (2.3) we obtain:

Proposition C.1. *For every $0 \leq k \leq m/3$, the following sequence is exact:*

$$\begin{aligned} 0 \longrightarrow F^{k-1}(E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C})) &\longrightarrow F^k(E_{2,m} T_{\mathbb{P}^2}^*(\log \mathcal{C})) \\ &\longrightarrow S^{m-3k} T_{\mathbb{P}^2}^*(\log \mathcal{C}) \otimes \bar{K}_{\mathbb{P}^2}^{\otimes k} \longrightarrow 0. \end{aligned}$$

For convenience, from now on we write $\bar{T}_{\mathbb{P}^2}^* := T_{\mathbb{P}^2}^*(\log \mathcal{C})$ and $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^2}(1)$.

Proposition C.2. *For integer $q \geq 0$ and $r \in \{1, 2\}$, the following vanishing holds:*

$$H^0(\mathbb{P}^2, E_{2,3q+r} \bar{T}_{\mathbb{P}^2}^* \otimes \mathcal{O}_{\mathbb{P}^2}(-(3q+r))) = 0,$$

provided that $H^0(\mathbb{P}^2, S^{3\ell+r} \bar{T}_{\mathbb{P}^2}^ \otimes \mathcal{O}_{\mathbb{P}^2}(-(3\ell+r))) = 0$ for all $\ell = 0, 1, \dots, q$.*

Proof. By Proposition C.1, for every integer $k = 0, 1, \dots, q$, we have a short exact sequence:

$$\begin{aligned} 0 \longrightarrow F^{k-1}(E_{2,3q+r} \bar{T}_{\mathbb{P}^2}^*) \otimes \mathcal{O}(-(3q+r)) \\ \longrightarrow F^k(E_{2,3q+r} \bar{T}_{\mathbb{P}^2}^*) \otimes \mathcal{O}(-(3q+r)) \\ \longrightarrow S^{3q+r-3k} \bar{T}_{\mathbb{P}^2}^* \otimes \bar{K}_{\mathbb{P}^2}^{\otimes k} \otimes \mathcal{O}(-(3q+r)) \longrightarrow 0. \end{aligned}$$

Note that the last term $S^{3q+r-3k} \bar{T}_{\mathbb{P}^2}^* \otimes \bar{K}_{\mathbb{P}^2}^{\otimes k} \otimes \mathcal{O}(-(3q+r)) \cong S^{3(q-k)+r} \bar{T}_{\mathbb{P}^2}^* \otimes \mathcal{O}(-(3(q-k)+r))$ has no global section by our assumption. Considering the induced long exact sequence in cohomology, we obtain

$$\begin{aligned} H^0(\mathbb{P}^2, F^k(E_{2,3q+r} \bar{T}_{\mathbb{P}^2}^*) \otimes \mathcal{O}(-(3q+r))) \\ \cong H^0(\mathbb{P}^2, F^{k-1}(E_{2,3q+r} \bar{T}_{\mathbb{P}^2}^*) \otimes \mathcal{O}(-(3q+r))). \end{aligned}$$

Iterating backwards for $k = q, q-1, \dots, 1$ gives the desired vanishing, since by assumption

$$H^0(\mathbb{P}^2, S^{3q+r} \bar{T}_{\mathbb{P}^2}^* \otimes \mathcal{O}(-(3q+r))) = 0.$$

□

Remark C.3. For $r = 0$ the argument fails: when $q = k$, the term

$$S^{3q+r-3k} \bar{T}_{\mathbb{P}^2}^* \otimes \bar{K}_{\mathbb{P}^2}^{\otimes k} \otimes \mathcal{O}(-(3q+r)) \cong \mathcal{O}_{\mathbb{P}^2}$$

has one-dimensional global sections.

AUTHOR CONTRIBUTIONS AND AI-ASSISTED PREPARATION

Dinh Tuan Huynh proposed the original problem and the initial pole-order-3 route, and carried out the Riemann–Roch computations. Lei Hou and Song-Yan Xie discovered the exponent bound that reduces the extension problem to finite linear algebra; they developed the theoretical proof of the Key Vanishing Lemmas, implemented the Maple verification of Key Vanishing Lemma I, and selected the Fermat-type conics used for Key Vanishing Lemma II. Starting from that theory, Joël Merker improved the efficiency of the finite algorithm and implemented part of the verification of Key Vanishing Lemma II in Maple, using high-performance computing resources in Orsay. The remaining obstruction was not the available hardware: at this scale, Maple’s modular linear algebra could not fully exploit the machine’s computational capacity, and the strengthened cases required here remained incomplete. With coding assistance from GPT-5.6-sol, Lei Hou and Song-Yan Xie then developed a more efficient exact full-coupled C++ implementation. They checked its fidelity to the mathematical reduction and its certificates, and completed all strengthened cases of Key Vanishing Lemma II on a standard laptop with 32 GB of memory and 1 TB of storage.

The one-jet vanishing problem settled in Theorem 1.5 was formulated by the authors. Generative AI helped derive an initial proof, which the authors checked line by line, corrected where necessary, and integrated into the manuscript. Generative AI was also used principally for C++ code development and language polishing, and assisted an author-directed exploration in July 2026 of the unused pole-order-4 variant mentioned in the Introduction. All other mathematical ideas, specifications, and arguments are due to the authors. The authors verified the proof of Theorem 1.5, the code, certificates, final text, and computations, and accept full responsibility for them.

REFERENCES

- [1] Vyacheslav Babets. Theorems of Picard type for holomorphic mappings. *Sibirsk. Mat. Zh.*, 25(2):35–41, 1984. ↑ 1
- [2] François Berteloot and Julien Duval. Sur l’hyperbolicité de certains complémentaires. *Enseign. Math.* (2), 47(3–4):253–267, 2001. ↑ 1, 23
- [3] André Bloch. Sur les systèmes de fonctions uniformes satisfaisant à l’équation d’une variété algébrique dont l’irrégularité dépasse la dimension. *J. Math. Pures Appl.* (9), 5:19–66, 1926. ↑ 3
- [4] Fedor Alekseevich Bogomolov. Holomorphic tensors and vector bundles on projective manifolds. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(6):1227–1287, 1439, 1978. ↑ 10
- [5] Damian Brotbek. Symmetric differential forms on complete intersection varieties and applications. *Math. Ann.*, 366(1–2):417–446, 2016. ↑ 5
- [6] Damian Brotbek and Lionel Darondeau. Complete intersection varieties with ample cotangent bundles. *Invent. Math.*, 212(3):913–940, 2018 (arXiv:1511.04709). ↑ 5
- [7] Henri Cartan. Sur les zéros des combinaisons linéaires de p fonctions holomorphes données. *Mathematica*, 7:80–103, 1933. ↑ 2
- [8] Yunling Chen. *Asymptotic equality of Cartan’s second main theorem and uniqueness results from hyperbolic perspectives*. PhD thesis, Tsinghua University, 2025. ↑ 10
- [9] Yunling Chen, John Erik Fornæss, and Song-Yan Xie. Generating all Ahlfors currents by a single entire curve. *arXiv: 2511.12473*, 2025. ↑ 2
- [10] Zhangchi Chen, Dinh Tuan Huynh, Ruiran Sun, and Song-Yan Xie. Entire holomorphic curves into $\mathbb{P}^n(\mathbb{C})$ intersecting $n + 1$ general hypersurfaces. *Science China Mathematics*, 2025. ↑ 2, 23
- [11] Herbert Clemens. Curves on generic hypersurfaces. *Ann. Sci. École Norm. Sup.* (4), 19(4):629–636, 1986. ↑ 14
- [12] Gregory M. Constantine and Thomas H. Savits. A multivariate Faà di Bruno formula with applications. *Trans. Amer. Math. Soc.*, 348(2):503–520, 1996. ↑ 8
- [13] Pietro Corvaja and Umberto Zannier. Some cases of Vojta’s conjecture on integral points over function fields. *J. Algebraic Geom.*, 17(2):295–333, 2008. ↑ 23
- [14] Tao Cui, Lei Hou, Jianjun Liu, and Song-Yan Xie. A symmetry method for key vanishing lemmas and optimal 2-jet thresholds for hyperbolicity, 2026. ↑ 6, 7
- [15] Lionel Darondeau. Slanted vector fields for jet spaces. *Math. Z.*, 282(1–2):547–575, 2016. ↑ 14
- [16] Olivier Debarre. Varieties with ample cotangent bundle. *Compos. Math.*, 141(6):1445–1459, 2005. ↑ 5
- [17] Jean-Pierre Demailly. Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 285–360. Amer. Math. Soc., Providence, RI, 1997. ↑ 3, 8, 9, 10, 12, 17
- [18] Jean-Pierre Demailly and Jawher El Goul. Hyperbolicity of generic surfaces of high degree in projective 3-space. *Amer. J. Math.*, 122(3):515–546, 2000. ↑ 1, 5, 7, 10, 11, 12, 13, 17
- [19] Simone Diverio, Jo’el Merker, and Erwan Rousseau. Effective algebraic degeneracy. *Invent. Math.*, 180(1):161–223, 2010. ↑ 16
- [20] Gerd-Eberhard Dethloff and Steven Shin-Yi Lu. Logarithmic jet bundles and applications. *Osaka J. Math.*, 38(1):185–237, 2001. ↑ 8, 9, 10
- [21] Gerd-Eberhard Dethloff, Georg Schumacher, and Pit-Mann Wong. Hyperbolicity of the complements of plane algebraic curves. *American Journal of Mathematics*, 117:573–599, 1995. ↑ 1
- [22] Gerd-Eberhard Dethloff, Georg Schumacher, and Pit-Mann Wong. On the hyperbolicity of the complements of curves in algebraic surfaces: the three-component case. *Duke Math. J.*, 78(1):193–212, 1995. ↑ 1, 3
- [23] Julien Duval. Around Brody Lemma. In *Hyperbolicity properties of algebraic varieties*, volume 56 of *Panor. Synthèses*, pages 1–12. Soc. Math. France, Paris, [2021] ©2021. ↑ 4
- [24] Lawrence Ein. Subvarieties of generic complete intersections. *Invent. Math.*, 94(1):163–169, 1988. ↑ 14
- [25] Jawher El Goul. Logarithmic jets and hyperbolicity. *Osaka J. Math.*, 40(2):469–491, 2003. ↑ 1, 4, 8, 10, 11, 12, 13, 18
- [26] Ichiro Enoki. Stability and negativity for tangent sheaves of minimal Kähler spaces. In *Geometry and analysis on manifolds (Katata/Kyoto, 1987)*, volume 1339 of *Lecture Notes in Math.*, pages 118–126. Springer, Berlin, 1988. ↑ 10
- [27] Alexandre Eremenko and Mikhail Sodin. Distribution of values of meromorphic functions and meromorphic curves from the standpoint of potential theory. *Algebra i Analiz*, 3(1):131–164, 1991. ↑ 2
- [28] William Fulton and Joe Harris. *Representation Theory: A First Course*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, 1991. ↑ 12
- [29] Hans Grauert. Jetmetriken und hyperbolische Geometrie. *Math. Z.*, 200(2):149–168, 1989. ↑ 1

- [30] Mark Green and Phillip Griffiths. Two applications of algebraic geometry to entire holomorphic mappings. In *The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979)*, pages 41–74. Springer, New York-Berlin, 1980. [↑ 3](#)
- [31] Ji Guo, Chia-Liang Sun, and Julie Tzu-Yueh Wang. On the d th roots of exponential polynomials and related problems arising from the Green-Griffiths-Lang conjecture. *J. Geom. Anal.*, 31(5):5201–5218, 2021. [↑ 23](#)
- [32] Ji Guo, Chia-Liang Sun, and Julie Tzu-Yueh Wang. A truncated second main theorem for algebraic tori with moving targets and applications. *J. Lond. Math. Soc. (2)*, 106(4):3670–3686, 2022. [↑ 23](#)
- [33] Robin Hartshorne. *Algebraic geometry*, volume No. 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1977. [↑ 13](#)
- [34] F. Hirzebruch. Arrangements of lines and algebraic surfaces. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 113–140. Birkhäuser, Boston, MA, 1983. [↑ 4](#)
- [35] Lei Hou, Dinh Tuan Huynh, Joël Merker, and Song-Yan Xie. Improved degree bounds for hyperbolicity of surfaces and curve complements, 2026. arXiv: [2603.14881 \[math.CV\]](#). [↑ 7](#)
- [36] Lei Hou, Pengchao Wang, and Song-Yan Xie. Invariant two-jets and effective hyperbolicity for complements of two plane curves, 2026. Preprint. [↑ 6](#)
- [37] Dinh Tuan Huynh, Duc-Viet Vu, and Song-Yan Xie. Entire holomorphic curves into projective spaces intersecting a generic hypersurface of high degree. *Ann. Inst. Fourier (Grenoble)*, 69(2):653–671, 2019. [↑ 3, 5, 21](#)
- [38] Dinh Tuan Huynh and Song-Yan Xie. On Ahlfors currents. *J. Math. Pures Appl. (9)*, 156:307–327, 2021. [↑ 2](#)
- [39] Shoshichi Kobayashi. *Hyperbolic manifolds and holomorphic mappings*, volume 2 of *Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1970. [↑ 1](#)
- [40] Steven Shin-Yi Lu and Shing-Tung Yau. Holomorphic curves in surfaces of general type. *Proc. Nat. Acad. Sci. U.S.A.*, 87(1):80–82, 1990. [↑ 5, 17](#)
- [41] Michael McQuillan. Diophantine approximations and foliations. *Inst. Hautes Études Sci. Publ. Math.*, (87):121–174, 1998. [↑ 2](#)
- [42] Joël Merker. Jets de Demailly-Semple d’ordres 4 et 5 en dimension 2. *Int. J. Contemp. Math. Sci.*, 3(17-20):861–933, 2008. [↑ 5](#)
- [43] Joël Merker. Low pole order frames on vertical jets of the universal hypersurface. *Ann. Inst. Fourier (Grenoble)*, 59(3):1077–1104, 2009. [↑ 14](#)
- [44] Joël Merker. Algebraic differential equations for entire holomorphic curves in projective hypersurfaces of general type: optimal lower degree bound. In *Geometry and analysis on manifolds*, volume 308 of *Progr. Math.*, pages 41–142. Birkhäuser/Springer, Cham, 2015. [↑ 8, 12](#)
- [45] Yoichi Miyaoka. Algebraic surfaces with positive indices. In *Classification of algebraic and analytic manifolds (Katata, 1982)*, volume 39 of *Progr. Math.*, pages 281–301. Birkhäuser Boston, Boston, MA, 1983. [↑ 5, 17](#)
- [46] Michael Nakamaye. Stable base loci of linear series. *Math. Ann.*, 318(4):837–847, 2000. [↑ 5](#)
- [47] Junjiro Noguchi. Holomorphic curves in algebraic varieties. *Hiroshima Math. J.*, 7(3):833–853, 1977. [↑ 2](#)
- [48] Junjiro Noguchi. Supplement to “holomorphic curves in algebraic varieties”. *Hiroshima Math. J.*, 10(1):229–231, 1980. [↑ 2](#)
- [49] Junjiro Noguchi. Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties. *Nagoya Math. J.*, 83:213–233, 1981. [↑ 2](#)
- [50] Junjiro Noguchi. Logarithmic jet spaces and extensions of de Franchis’ theorem. In *Contributions to several complex variables*, Aspects Math., E9, pages 227–249. Friedr. Vieweg, Braunschweig, 1986. [↑ 8](#)
- [51] Junjiro Noguchi and Jörg Winkelmann. *Nevanlinna theory in several complex variables and Diophantine approximation*, volume 350 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Tokyo, 2014. [↑ 2](#)
- [52] Junjiro Noguchi, Jörg Winkelmann, and Katsutoshi Yamanoi. The second main theorem for holomorphic curves into semi-abelian varieties. *Acta Math.*, 188(1):129–161, 2002. [↑ 2](#)
- [53] Junjiro Noguchi, Jörg Winkelmann, and Katsutoshi Yamanoi. Degeneracy of holomorphic curves into algebraic varieties. *J. Math. Pures Appl.*, 88:293–306, 2007. [↑ 1](#)
- [54] Mihai Păun. Vector fields on the total space of hypersurfaces in the projective space and hyperbolicity. *Math. Ann.*, 340(4):875–892, 2008. [↑ 1, 5, 14, 15](#)
- [55] Mihai Păun and Nessim Sibony. Value distribution theory for parabolic Riemann surfaces. In *Hyperbolicity properties of algebraic varieties*, volume 56 of *Panor. Synthèses*, pages 13–72. Soc. Math. France, Paris, [2021] ©2021. [↑ 2](#)
- [56] Erwan Rousseau. Weak analytic hyperbolicity of complements of generic surfaces of high degree in projective 3-space. *Osaka J. Math.*, 44(4):955–971, 2007. [↑ 14](#)
- [57] Erwan Rousseau. Weak analytic hyperbolicity of generic hypersurfaces of high degree in $\mathbb{P}^4$. *Ann. Fac. Sci. Toulouse Math. (6)*, 16(2):369–383, 2007. [↑ 14](#)
- [58] Erwan Rousseau. Logarithmic vector fields and hyperbolicity. *Nagoya Math. J.*, 195:21–40, 2009. [↑ 1, 5, 8, 14, 15, 16](#)
- [59] Min Ru. A defect relation for holomorphic curves intersecting hypersurfaces. *Amer. J. Math.*, 126(1):215–226, 2004. [↑ 2](#)
- [60] Min Ru. Holomorphic curves into algebraic varieties. *Ann. of Math.*, 169:255–267, 2009. [↑ 2](#)
- [61] Min Ru. *Nevanlinna theory and its relation to Diophantine approximation*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, [2021] ©2021. [↑ 2, 3](#)
- [62] Min Ru and Julie Tzu-Yueh Wang. Defect relation of $n + 1$ components through the GCD method. *Pacific J. Math.*, 338(2):349–371, 2025. [↑ 2](#)
- [63] Min Ru and Julie Tzu-Yueh Wang. Vojta’s abc conjecture for entire curves in toric varieties highly ramified over the boundary. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2026. [↑ 2](#)
- [64] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier (Grenoble)*, 6:1–42, 1955/56. [↑](#)

- [65] Yum-Tong Siu. Hyperbolicity in complex geometry. In *The legacy of Niels Henrik Abel*, pages 543–566. Springer, Berlin, 2004. ↑ 1, 5, 13, 14, 15, 16
- [66] Yum-Tong Siu and Sai-Kee Yeung. A generalized Bloch’s theorem and the hyperbolicity of the complement of an ample divisor in an abelian variety. *Math. Ann.*, 306(4):743–758, 1996. ↑ 3
- [67] Yum-Tong Siu and Sai-Kee Yeung. Hyperbolicity of the complement of a generic smooth curve of high degree in the complex projective plane. *Invent. Math.*, 124(1-3):573–618, 1996. ↑ 1, 5
- [68] Hajime Tsuji. Stability of tangent bundles of minimal algebraic varieties. *Topology*, 27(4):429–442, 1988. ↑ 10
- [69] Eckart Viehweg and Kang Zuo. Base spaces of non-isotrivial families of smooth minimal models. In *Complex geometry (Göttingen, 2000)*, pages 279–328. Springer, Berlin, 2002. ↑ 4
- [70] Claire Voisin. On a conjecture of Clemens on rational curves on hypersurfaces. *J. Differential Geom.*, 44(1):200–213, 1996. ↑ 14
- [71] Hao Wu and Song-Yan Xie. Entire curves generating all shapes of Nevanlinna currents. *J. Lond. Math. Soc. (2)*, 111(5):Paper No. e70177, 25, 2025. ↑ 2
- [72] Song-Yan Xie. On the ampleness of the cotangent bundles of complete intersections. *Invent. Math.*, 212(3):941–996, 2018 (arXiv:1510.06323). ↑ 5
- [73] Song-Yan Xie. Entire curves producing distinct Nevanlinna currents. *Int. Math. Res. Not. IMRN*, (10):8350–8362, 2024. ↑ 2
- [74] Mikhail Zaidenberg. Stability of hyperbolic embeddedness and the construction of examples. *Mat. Sb. (N.S.)*, 135(177)(3):361–372, 415, 1988. ↑ 1

ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCES, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA
Email address: houlei@amss.ac.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF EDUCATION, HUE UNIVERSITY, HUE CITY 530000, VIETNAM
Email address: dinh Tuan Huynh@hueuni.edu.vn

LABORATOIRE DE MATHÉMATIQUES D’ORSAY, CNRS, UNIVERSITÉ PARIS-SACLAY, 91405 ORSAY CEDEX, FRANCE
Email address: joel.merker@universite-paris-saclay.fr

STATE KEY LABORATORY OF MATHEMATICAL SCIENCES, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA; SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF CHINESE ACADEMY OF SCIENCES, BEIJING 100049, CHINA
Email address: xiesongyan@amss.ac.cn