HOLE PROBABILITIES OF RANDOM ZEROS ON COMPACT RIEMANN SURFACES

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ABSTRACT. We establish an estimate for the convergence speed of hole probabilities for zeros of random holomorphic sections on compact Riemann surfaces.

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CONTENTS

1. INTRODUCTION

The study of zeros in random polynomials began with Waring [56], who first used probabilistic methods to determine the number of imaginary zeros in algebraic polynomials. Significant progress was made in the 1930s by Bloch and Pólya [9], who provided rigorous estimates for the expected number of real roots in certain classes of random algebraic polynomial equations. This foundational work spurred further research by Littlewood, Offord, Hammersley, Kac, Erdős, and Turán, among others. For a comprehensive historical overview, see [7, 46].

In this article, we investigate the distribution of zeros for random sections of holomorphic line bundles over compact Riemann surfaces. Our work extends the foundational results of Shiffman-Zelditch [53], incorporating recent advances from [17] to establish precise estimates for the convergence rate of hole probabilities. The inspiration for our approach comes from the work of Nishry-Wennman [39] on hole probabilities of Gaussian entire functions, though our methodology differs in several key aspects and represents a distinct contribution to the problem.

Let X be a compact Riemann surface with a Kähler form ω_0, and let $\mathcal{L}$ be a positive holomorphic line bundle on X with $\deg(\mathcal{L}) \geq 1$. We fix a Hermitian metric h on $\mathcal{L}$ such that the Chern curvature form $c_1(\mathcal{L}, \mathfrak{h})$ is strictly positive. The normalized $(1,1)$-form

$$\omega := c_1(\mathcal{L}, \mathfrak{h}) / \deg(\mathcal{L})$$

serves as a probability measure on X, since $\int_X \omega = 1$.

For every positive integer n, the n-th power $\mathcal{L}^n := \mathcal{L}^{\otimes n}$ of the line bundle $\mathcal{L}$ inherits a natural metric $\mathfrak{h}_n$ induced by h. Specifically, for local sections s of $\mathcal{L}$ around a point $x \in X$, we have $\|s^{\otimes n}\|_{\mathfrak{h}_n}(x) := \|s\|^n_{\mathfrak{h}}(x)$. Let $(\cdot,\cdot)_n$ denote the pointwise Hermitian inner product corresponding to the Hermitian metric $\mathfrak{h}_n$.

On the space $H^0(X, \mathcal{L}^n)$ of global holomorphic sections of $\mathcal{L}^n$, we define a global Hermitian inner product as follows:

$$\langle s_1, s_2 \rangle_n := \int_X (s_1, s_2)_n \, \omega_0 \quad \text{for} \quad s_1, s_2 \in H^0(X, \mathcal{L}^n).$$

By the Riemann-Roch theorem, the global sections of $\mathcal{L}^n$ are abundant for sufficiently large n. Specifically, we have

$$\dim_{\mathbb{C}} H^0(X, \mathcal{L}^n) = n \cdot \deg(\mathcal{L}) - g + 1 \gg 1.$$

On the projectivized space $\mathbb{P}H^0(X, \mathcal{L}^n)$, we consider the Fubini-Study volume form V_n^{FS} induced by $\langle \cdot, \cdot \rangle_n$.

The zero set Z_s of a section $s \in H^0(X, \mathcal{L}^n) \setminus \{0\}$, or of an element $[s] \in \mathbb{P}H^0(X, \mathcal{L}^n)$, consists of $n \cdot \deg(\mathcal{L})$ points, counting multiplicities. Let $[Z_s]$ denote the sum of Dirac masses at the points in Z_s. The Poincaré-Lelong formula states that

$$[Z_s] = \mathrm{dd^c} \log\|s\|_{\mathfrak{h}_n} + n \cdot \deg(\mathcal{L}) \, \omega.$$

The empirical probability measure of Z_s is defined as the unit mass normalization

$$[\![Z_s]\!] := (n \cdot \deg(\mathcal{L}))^{-1} [Z_s].$$

The distribution of $[\![Z_s]\!]$ for $s \in H^0(X, \mathcal{L}^n) \setminus \{0\}$ with respect to the complex Gaussian measure on $H^0(X, \mathcal{L}^n)$ associated with $\langle \cdot, \cdot \rangle_n$ coincides with the distribution of $[\![Z_s]\!]$ for $[s] \in \mathbb{P}H^0(X, \mathcal{L}^n)$ with respect to the Fubini-Study volume form V_n^{FS} (cf. [50, Section 2]).

The study of the zero distribution of random holomorphic sections has a rich literature; see, for example, [3, 4, 5, 6, 8, 15, 16, 18, 19, 20, 21, 34, 41, 45, 47, 48, 50, 51, 52].

Notably, a celebrated theorem by Shiffman and Zelditch [48] implies that, under the Fubini-Study volume form V_n^{FS}, the zeros of sections in $\mathbb{P}H^0(X, \mathcal{L}^n)$ are equidistributed with respect to ω. Specifically, for any smooth test function ϕ on X, we have

$$\lim_{n \to \infty} \int_{\mathbb{P}H^0(X, \mathcal{L}^n)} \langle [\![Z_s]\!], \phi \rangle \, V_n^{\mathrm{FS}}(s) = \int_X \phi \, \omega.$$

Our goal is to study the hole probabilities of this distribution, following the framework established in [17, 53, 58, 59]. For an open subset D in X with $\overline{D} \neq X$, we define, for each large n, the *hole event*

$$H_{n,D} := \big\{ [s] \in \mathbb{P}H^0(X, \mathcal{L}^n) \mid Z_s \cap D = \varnothing \big\},$$

which concerns holomorphic sections of $\mathcal{L}^n$ that vanish nowhere on D.

To state our main result, we introduce a key functional $\mathcal{I}_{\omega,D}$ that was presented in [17], with its initial model proposed by Zeitouni and Zelditch [58] (see also [26, 39, 59]). Let $\mathcal{M}(X \setminus D)$ denote the set of probability measures supported on $X \setminus D$. We define

$$\mathcal{I}_{\omega,D}(\mu) := -\int_X U_\mu\, \omega - \int_X U_\mu\, \mathrm{d}\mu = -\int_X U^\star_\mu\, \mathrm{d}\mu + 2\max U^\star_\mu, \quad \forall\, \mu \in \mathcal{M}(X \setminus D). \tag{1.1}$$

Here, U_μ (resp. $U^\star_\mu$) is the ω-potential of type M (resp. type I) of μ, as defined in Section 2.2. When D is non-empty, $\mathcal{I}_{\omega,D}$ is strictly positive.

For convenience, we abbreviate $\mathcal{I}_D := \mathcal{I}_{\omega,D}$, since ω is fixed throughout this article. A key property of $\mathcal{I}_D$ is that it admits a unique minimizer $\nu_D \in \mathcal{M}(X \setminus D)$, which coincides with the limit of the conditional expectations $\mathbf{E}\big(\llbracket Z_s \rrbracket \mid H_{n,D}\big)$ of the empirical probability measures $\llbracket Z_s \rrbracket$ for the hole events $[s] \in H_{n,D}$ as n tends to infinity. See [17].

A striking phenomenon is that the support of ν_D avoids an open "forbidden set" distinct from D (cf. [17]). This behavior was first observed in the context of Gaussian entire functions [26, 39], where it revealed deep connections between zero distributions and potential theory.

Recall that an open set E in X is said to have a smooth boundary ∂E if, for every $x \in \partial E$, there exists a neighborhood B of x and a smooth diffeomorphism $\psi : B \to \mathbb{D}$, where $\mathbb{D}$ is the unit disc in $\mathbb{C}$, such that

$$\psi(E \cap B) = \mathbb{D} \cap \{\mathrm{Im}(z) > 0\} \quad \text{and} \quad \psi(\partial E \cap B) = \mathbb{D} \cap \mathbb{R}.$$

For technical convenience, we assume from now on that the hole D has a **smooth boundary** and that ∂D **consists of only finitely many components**. Our main result concerns the exponential decay of the hole probabilities $\mathbf{P}_n(H_{n,D})$.

Theorem 1.1. *There exist a constant $C_D > 0$ independent of n, and a constant $C > 0$ independent of D, n, such that*

$$-C_D \frac{\log n}{n} \le \frac{1}{n^2 \deg(\mathcal{L})^2} \log \mathbf{P}_n(H_{n,D}) + \min \mathcal{I}_D \le C \frac{\log n}{n} \tag{1.2}$$

for all $n \ge n_0$, where n_0 depends on D.

It is surprising that the upper bound error is independent of D. Here, we assume $n \ge n_0$ to ensure that $\mathbf{P}_n(H_{n,D}) \neq 0$, since when $X \neq \mathbb{P}^1$, the hole event $H_{n,D}$ can be empty for certain small $n \in \mathbb{Z}_+$. This is also the reason why the lower bound error should depends on D.

Note that the estimate (1.2) refines the following hole probability asymptotic formula by providing the **convergence speed**:

$$\lim_{n\to\infty} \frac{1}{n^2 \deg(\mathcal{L})^2} \log \mathbf{P}_n(H_{n,D}) = -\min \mathcal{I}_D, \tag{1.3}$$

which was established in [17], as a by-product.

As a consequence of Theorem 1.1, in dimension one, we recover an estimate established by Shiffman-Zelditch-Zrebiec [53] on the exponential decay $\exp(-c_D n^{t+1})$ of the hole probabilities with respect to a fixed open domain D in a projective manifold of any dimension $t \ge 1$. For the case where $X = \mathbb{P}^t$ with $t \ge 1$ and equipped with the Fubini-Study metric, some precise asymptotic formulas for the hole probabilities of polydiscs were established by Zrebiec [61] and Zhu [60].

Nevertheless, even in the simplest case where $(X, \mathcal{L}, \omega) = (\mathbb{P}^1, \mathscr{O}(1), \omega_{\mathrm{FS}})$, the convergence speed $O(\log n/n)$ in Theorem 1.1 is new and is expected to be optimal. See Question 11.1 for further discussion. In particular, when $\deg(\mathcal{L}) = 1$, there might exist an asymptotic expansion formula (compare with (11.1))

$$\mathbf{P}_n(H_{n,D}) = \exp\left[C_1' n^2 + C_2' n \log n + C_3' n + o(n)\right], \qquad \text{as} \quad n \to +\infty, \tag{1.4}$$

where $C_1' = -\min \mathcal{I}_D$, and C_2' and C_3' are constants depending on D.

Theorem 1.1 provides strong evidence for the existence of such a C_2'. Determining C_2', C_3', and the lower-order terms in the conjectural formula (1.4) would be a challenging research project.

Our second result concerns the asymptotic behavior of hole probabilities for discs with varying radii. Fix a radius $r_0 > 0$ such that any open disc $\mathbb{B}(x, r_0)$, centered at $x \in X$ with radius r_0 with respect to the given metric ω_0, is not the entire space X.

Theorem 1.2. *There exist constants $C > c > 0$ independent of x and r, such that*

$$-Cr^4 \leq \lim_{n\to\infty} \frac{1}{n^2 \deg(\mathcal{L})^2} \log \mathbf{P}_n(H_{n,\mathbb{B}(x,r)}) \leq -cr^4, \quad \forall x \in X, \quad \forall 0 < r < r_0. \tag{1.5}$$

Previously, such hole probabilities with a parameter r on the hole size were not known for zeros of random holomorphic sections, except in some very special cases where *tour de force* calculations are possible (see [60, 61] for such exceptions in dimension one and higher). The estimate (1.5) is inspired by several classical works in recent decades.

In the setting of Gaussian entire functions $F(z) := \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n!}} z^n$ with independent and identically distributed (i.i.d.) standard complex Gaussian coefficients a_n, Sodin and Tsirelson [55] established that the hole probability with respect to a disc of radius r decays as $\exp(-cr^4)$ as $r \to \infty$. An optimal constant c was obtained by Nishry [37]. Recently, Buckley, Nishry, Peled, and Sodin [14] studied the hole probabilities for zeros of hyperbolic Gaussian Taylor series with finite radii of convergence. See also [10, 11, 12, 13, 25, 26, 27, 31, 32, 38, 39, 42, 49].

We now discuss the insights and ideas underlying the proofs of Theorems 1.1 and 1.2. The functional $\mathcal{I}_D$ plays a central role in our approach. It was first introduced by Zeitouni and Zelditch [58] (for $X = \mathbb{P}^1$ and $D = \varnothing$) and Zelditch [59] (for $D = \varnothing$) to study large deviations of random zeros of line bundle sections on compact Riemann surfaces. Ghosh and Nishry [26] later developed a similar functional $\mathcal{I}_D$ to investigate the hole event problem for Gaussian entire functions on $\mathbb{C}$. While their work addressed a different setting and employed distinct methods, it provided significant inspiration for the current formulation. More recently, Dinh, Ghosh, and the first named author [17] refined $\mathcal{I}_D$ into its present form, which differs substantially from the original proposal. This refinement was crucial for resolving the equidistribution problem of hole events and establishing the exponential decay (1.3) of the hole probabilities.

The proof of Theorem 1.1 reduces to the following two key steps:

- ($\heartsuit$) Construct holomorphic sections $s \in H^0(X, \mathcal{L}^n)$ whose zeros avoid the hole D.
- ($\diamondsuit$) Demonstrate that such sections $[s]$ form a subset of $\mathbb{P}H^0(X, \mathcal{L}^n)$ with "large" volume.

In general, constructing "desired" holomorphic sections is widely recognized as a notoriously challenging task, lying at the heart of several prominent problems in algebraic and complex geometry (cf. [54]). It is worth noting that, in most cases, one relies on three classical approaches:

(1) Methods inspired by electrostatic potentials.
(2) Differential geometric methods involving positive curvature.
(3) L^2-methods for solving $\overline{\partial}$-equations.

While (2) and (3) are applicable in any dimension, (1) is effective only in dimension one. Our approach aligns with (1). The starting point is Zelditch's density formula (2.14) (cf. [59, Theorem 2]), whose proof relies on a certain bosonization formula [2] and ingenious calculations.

The functional $\mathcal{I}_D$ was introduced by Dinh, Ghosh, and the first named author in [17]. This functional is derived from the key exponent in Zelditch's formula and acts as a rate function for a large deviation principle governing the empirical measures $[\![Z_s]\!]$ of zeros of random holomorphic sections $[s] \in H_{n,D}$.

The underlying intuition is that the zeros of a random section in the hole event $H_{n,D}$ repel each other, analogous to the behavior of electrons. Consequently, for most $[s] \in H_{n,D}$, the zero sets Z_s yield empirical measures $[\![Z_s]\!]$ that are very close to the minimizer ν_D of the functional $\mathcal{I}_D$ (cf. [17, Theorem 2]).

In practice, the upper bound in Theorem 1.1 is relatively straightforward to establish using Zelditch's density formula (2.14). The true challenge lies in obtaining the lower bound.

Our strategy for addressing ($\heartsuit$) and ($\diamondsuit$) consists of two main components:

($\spadesuit$) First, we seek some $[s] \in H_{n,D}$ such that a certain necessary modification $\widehat{\mathcal{I}}_{n,D}$ of $\mathcal{I}_D$ satisfies $\widehat{\mathcal{I}}_{n,D}([\![Z_s]\!]) \approx \mathcal{I}_D(\nu_D)$ for the empirical measure $[\![Z_s]\!]$. This modification $\widehat{\mathcal{I}}_{n,D}$ is required because $\mathcal{I}_D(\mu) = +\infty$ for any probability measure μ that assigns positive mass to a single point.

($\clubsuit$) Next, we demonstrate that by applying arbitrary mild perturbations $Z_{s'}$ of $Z_s \subset X \setminus D$ (while still satisfying the Abel-Jacobi equation (2.5)), we can generate many sections $[s'] \in H_{n,D}$ that collectively form a subset of large volume.

The requirement for extreme closeness in Step ($\spadesuit$) ensures that, after applying these mild perturbations, the empirical measures $[\![Z_{s'}]\!]$ continue to satisfy $\widehat{\mathcal{I}}_{n,D}([\![Z_{s'}]\!]) \approx \mathcal{I}_D(\nu_D)$. As a result, such sections $[s'] \in H_{n,D}$ can form a subset of large volume, as predicted by Zelditch's density formula (2.14).

When $g = 0$, such a blueprint can be realized without difficulty; see Remark 9.2 below. However, when $g \geq 1$, a fundamental difficulty arises, as pointed out by Zelditch [59]. For simplicity, assume $n > g$ and $\deg(\mathcal{L}) = 1$. Note that the number of zeros of every $[s] \in \mathbb{P}H^0(X, \mathcal{L}^n)$ is precisely n, which exceeds the dimension $n - g$ of $\mathbb{P}H^0(X, \mathcal{L}^n)$. Write the zeros of s in any order as $p_1, \ldots, p_{n-g}, q_1, \ldots, q_g$. For a generic s, by Abel's theorem, the last g zeros $q_1, \ldots, q_g$ are uniquely determined by the first $n-g$ zeros $p_1, \ldots, p_{n-g}$. However, even if all the p_j's lie outside the hole D, it is not guaranteed that the q_j's also lie in $X \setminus D$. In other words, there is no direct way to parameterize $H_{n,D}$.

To achieve ($\spadesuit$) for any $g \geq 1$, our approach proceeds as follows:

($\spadesuit$1) Temporarily disregarding the Abel-Jacobi equation (2.5), we aim to identify an empirical measure $\delta_{\mathbf{p}} = \frac{1}{n \deg \mathcal{L}}(\delta_{p_1} + \cdots + \delta_{p_{n \deg \mathcal{L}}})$, where $\mathbf{p} = p_1 + \cdots + p_{n \deg \mathcal{L}} \in$

$(X \setminus D)^{(n \deg \mathcal{L})}$, ensuring that the functional value $\widehat{\mathcal{I}}_{n,D}(\delta_{\mathbf{p}})$ closely approximates $\mathcal{I}_D(\nu_D)$.

(♠2) We then delicately perturb $\mathbf{p}$ to a nearby configuration $\mathbf{p}' \in (X \setminus D)^{(n \deg \mathcal{L})}$ in order to satisfy the Abel-Jacobi equation (2.5). Throughout this adjustment, we maintain that $\widehat{\mathcal{I}}_{n,D}(\delta_{\mathbf{p}'})$ remains nearly identical to $\widehat{\mathcal{I}}_{n,D}(\delta_{\mathbf{p}})$. This allows us to establish a valid section $[s] \in H_{n,D}$ with $[\![Z_s]\!] = \delta_{\mathbf{p}'}$, ensuring that $\widehat{\mathcal{I}}_{n,D}([\![Z_s]\!]) = \widehat{\mathcal{I}}_{n,D}(\delta_{\mathbf{p}'})$ closely matches $\mathcal{I}_D(\nu_D)$.

Let us highlight three major contributions of this article:

(i) We establish the Lipschitz regularity of the quasi-potential U_{ν_D} of ν_D (Section 3). This result is crucial for the subsequent analysis and relies on the structure theorem [17, Theorem 1.1] of the equilibrium measure ν_D.

(ii) By adapting the theory of Fekete points on compact Riemann surfaces (Section 5), we demonstrate that the points $p_1, \ldots, p_{n \deg \mathcal{L}} \in X \setminus D$ obtained in Step (♠1) are sufficiently separated (Section 6). This adaptation is inspired by the work of Nishry and Wennman [39].

(iii) We introduce a novel perturbation method (Section 7) to accomplish Step (♠2) (Section 8). Within the proof, Lemma 7.1 addresses the core challenge by rendering the constraint imposed by the Abel-Jacobi equation (2.5) asymptotically negligible as $n \to \infty$.

A significant distinction between our approach and that of [39] lies in our reliance on insights concerning the equilibrium measure ν_D, as established in [17, Theorem 1.1]. By fully leveraging potential theory, we eliminate the need to assume that the hole D is simply connected. In contrast, Nishry and Wennman [39] adopted a variational method with a PDE-oriented perspective, which required them to assume the simple connectedness of D.

In Section 2, we present some useful background. The proof of Theorem 1.1 is given in Section 9, and the proof of Theorem 1.2 is provided in Section 10. Finally, we discuss some open problems in Section 11.

Notations:

$O(|f|)$ Represents a term that is bounded by $C \cdot |f|$ for some positive constant C, where C may depend on X, ω_0, ω, D, but not on any other parameters.

$\mathbb{D}(x,r) \subset \mathbb{C}$ Denotes the open disc centered at $x \in \mathbb{C}$ with radius $r > 0$. We abbreviate $\mathbb{D}(0,r)$ as $\mathbb{D}_r$, and $\mathbb{D}_1$ as $\mathbb{D}$.

$\mathbb{B}(x,r) \subset X$ Denotes the open disc centered at $x \in X$ with radius $r > 0$ with respect to the given metric ω_0.

2. Preliminaries on Abel-Jacobi Theory and Quasi-Potentials

2.1. **Abel-Jacobi Theory.** Let X be a compact Riemann surface. A *divisor* on X is a finite formal linear combination $\mathcal{D} := \sum a_j x_j$, where $a_j \in \mathbb{Z}$ and $x_j \in X$. If $a_j \geq 0$ for all j, then we call $\mathcal{D}$ an *effective divisor*. The degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is defined as the total sum $\sum a_j$. A divisor $\mathcal{D}$ on X can be associated with a line bundle $\mathscr{O}(\mathcal{D})$ (cf. [29]).

For a nonzero meromorphic function f on X (or a nontrivial holomorphic section s of a line bundle $\mathcal{L}$), we denote by $(f) := (f)_0 - (f)_\infty$ (resp. (s)) the divisor defined by the zeros $(f)_0$ and the poles $(f)_\infty$ of f (resp. the zeros of s). Such a divisor $\mathcal{D} = (f)$ is called a *principal divisor*, and it always has degree 0.

Consider a canonical basis $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ of $H_1(X, \mathbb{Z})$ with the intersection numbers $\alpha_i \cdot \beta_j = \delta_{ij}$ and $\alpha_{j_1} \cdot \alpha_{j_2} = \beta_{j_1} \cdot \beta_{j_2} = 0$ for $j_1 \neq j_2$. Let $\{\phi_1, \ldots, \phi_g\}$ be a basis for the complex vector space of holomorphic 1-forms on X. Denote by $\Phi = (\phi_1, \ldots, \phi_g)^\intercal$ a column vector of length g. The $g \times 2g$ matrix

$$\Omega := \Big(\int_{\alpha_1} \Phi, \ \ldots, \int_{\alpha_g} \Phi, \int_{\beta_1} \Phi, \ \ldots, \int_{\beta_g} \Phi \Big)$$

is called the *period matrix*. The $2g$ columns of Ω are linearly independent over $\mathbb{R}$, and hence generate a lattice Λ in $\mathbb{C}^g$ (cf. [29, 30]). Define the *Jacobian variety* of X by

$$\operatorname{Jac}(X) := \mathbb{C}^g / \Lambda, \tag{2.1}$$

which is a g-dimensional complex torus.

Fix a base point $p_\star \in X$. The *Abel-Jacobi map* $\mathcal{A} : X \to \operatorname{Jac}(X)$ associated with $p_\star$ is defined by

$$\mathcal{A}(x) := \Big(\int_{p_\star}^x \phi_1, \ \ldots, \int_{p_\star}^x \phi_g \Big)^\intercal \pmod \Lambda, \qquad \forall x \in X. \tag{2.2}$$

The resulting point in $\operatorname{Jac}(X)$ is independent of the choice of path from $p_\star$ to x after taking the modulo Λ.

Let $X^{(m)}$ be the m-th symmetric product of X. Points in $X^{(m)}$ correspond one-to-one to effective divisors $p_1 + \cdots + p_m$ of degree m, where the points $p_1, \ldots, p_m \in X$ are not necessarily distinct. It is well known that $X^{(m)}$ inherits a complex structure from X^m. Define the canonical projection $\pi_m : X^m \to X^{(m)}$ by

$$\pi_m(p_1, \ldots, p_m) = p_1 + \cdots + p_m. \tag{2.3}$$

Let $\mathscr{R}_m := \{(p_1, \ldots, p_m) : p_j = p_\ell \text{ for some } j \neq \ell\}$ be the ramification subvariety. Near a point $p_1 + \cdots + p_m \notin \pi_m(\mathscr{R}_m)$, the holomorphic coordinates of $X^{(m)}$ are given by the local coordinates of $p_1, \ldots, p_m \in X$. We will denote $A_1 \otimes \cdots \otimes A_n := \pi_n(A_1 \times \cdots \times A_n)$ for $A_j \subset X$.

For every $t \geq 1$, define a holomorphic map $\mathcal{A}_t : X^{(t)} \to \operatorname{Jac}(X)$ by

$$\mathcal{A}_t(p_1 + \cdots + p_t) := \sum_{j=1}^t \mathcal{A}(p_j) = \sum_{j=1}^t \Big(\int_{p_\star}^{p_j} \phi_1, \ldots, \int_{p_\star}^{p_j} \phi_g \Big)^\intercal \pmod \Lambda.$$

Then, $\mathcal{A}_1 = \mathcal{A}$ is injective, and $\mathcal{A}_g$ is surjective by the Jacobi inversion theorem. Moreover, $\mathcal{A}_g$ is *birational*. Specifically, there exist a hypersurface $\mathscr{W} \subsetneq X^{(g)}$ and a codimension-2 analytic set $W_g^1 := \mathcal{A}_g(\mathscr{W}) \subsetneq \operatorname{Jac}(X)$ such that the restriction

$$\mathcal{A}_g : X^{(g)} \setminus \mathscr{W} \longrightarrow \operatorname{Jac}(X) \setminus W_g^1$$

is biholomorphic. The subvariety $\mathscr{W}$ consists of critical points of $\mathcal{A}_g$, and W_g^1 is called the *Wirtinger subvariety*. For any $q_1 + \cdots + q_g \in \mathscr{W}$, one has $\dim H^0(X, \mathscr{O}(q_1 + \cdots + q_g)) \geq 2$ (cf. [29, 30]). When $g = 1$, $\mathscr{W}$ is empty.

Abel's theorem states that $\mathcal{A}_t(p_1 + \cdots + p_t) = \mathcal{A}_t(p'_1 + \cdots + p'_t)$ if and only if $p_1 + \cdots + p_t - p'_1 - \cdots - p'_t$ is a principal divisor. In particular, there exists a holomorphic section $s \in H^0(X, \mathscr{O}(p_1 + \cdots + p_t))$ with $(s) = p'_1 + \cdots + p'_t$.

By the Riemann-Roch theorem, any line bundle on X of degree at least g admits a nontrivial global section s, and hence is linearly equivalent to $\mathscr{O}(\mathcal{D})$ for some effective divisor $\mathcal{D} = (s)$. For large n such that $m := n \cdot \deg(\mathcal{L}) - g > 0$, for any $p_1 + \cdots + p_m \in X^{(m)}$, we can apply this observation to the line bundle $\mathcal{L}^n \otimes \mathscr{O}(-(p_1 + \cdots + p_m))$ of degree g to see that there exists some $q_1 + \cdots + q_g \in X^{(g)}$ such that

$$\mathscr{O}(q_1 + \cdots + q_g) \simeq \mathcal{L}^n \otimes \mathscr{O}(-(p_1 + \cdots + p_m)). \tag{2.4}$$

Equivalently, if $\mathcal{L}^n \simeq \mathscr{O}(w_1 + \cdots + w_{n\cdot\deg(\mathcal{L})})$, then

$$\mathcal{A}_m(p_1+\cdots+p_m)+\mathcal{A}_g(q_1+\cdots+q_g) = \mathcal{A}_{n\cdot\deg(\mathcal{L})}(w_1+\cdots+w_{n\cdot\deg(\mathcal{L})}) =: \mathcal{A}_{n\cdot\deg(\mathcal{L})}(\mathcal{L}^n). \tag{2.5}$$

Here, $\mathcal{A}_{n\cdot\deg(\mathcal{L})}(\mathcal{L}^n)$ is well-defined because of Abel's theorem. Moreover, the choice of $q_1 + \cdots + q_g$ is unique if $\dim H^0(X, \mathscr{O}(q_1 + \cdots + q_g)) = 1$, or equivalently, if $\mathcal{A}_{n\cdot\deg(\mathcal{L})}(\mathcal{L}^n) - \mathcal{A}_m(p_1 + \cdots + p_m) \notin W_g^1$.

In this article, we will assume that $\deg(\mathcal{L}) = 1$. This assumption is justified by the following well-known fact, which is a corollary of the Jacobi inversion theorem.

Lemma 2.1 (cf. [17]–Lemma 6.4)**.** *Let $\mathcal{L}'$ be a positive line bundle on X of degree $d > 1$. Then there exists a positive line bundle $\mathcal{L}$ of degree 1 such that $\mathcal{L}^d \simeq \mathcal{L}'$.*

This lemma allows us to reduce the general case to the case where the degree of the line bundle is 1. Any results obtained for $\mathcal{L}$ of degree 1 can be extended to the line bundle $\mathcal{L}'$ by considering appropriate powers of $\mathcal{L}$.

2.2. **Quasi-potentials.** A function ϕ on X with values in $\mathbb{R} \cup \{-\infty\}$ is called *quasi-subharmonic* if, locally, it can be written as the difference of a subharmonic function and a smooth function. If ϕ is quasi-subharmonic, then there exists a constant $c \geq 0$ such that $\mathrm{dd^c}\phi \geq -c\omega$ in the sense of currents. When $c = 1$, ϕ is called an *ω-subharmonic function*, and $\mathrm{dd^c}\phi + \omega$ is a probability measure on X by Stokes' formula.

For any probability measure μ on X, we can write $\mu = \omega + \mathrm{dd^c} U_\mu$, where U_μ is the unique quasi-subharmonic function such that $\max U_\mu = 0$. We call U_μ the *ω-potential of type M* of μ (M stands for "maximum"). There is an alternative way to normalize the potential $U_\mu^\star$ by requiring that $\int_X U_\mu^\star\, \omega = 0$. We call $U_\mu^\star$ the *ω-potential of type I* of μ (I stands for "integral"). To construct ω-potentials, we use the following Green functions.

Lemma 2.2 (cf. [33]–Theorem 2.2, [17]–Lemma 2.1)**.** *There exists a Green function $G(x, y)$ of (X, ω) defined on $X \times X$, which is smooth outside the diagonal, such that for every $x \in X$,*

$$\int_X G(x, \cdot)\, \omega = 0 \qquad \textit{and} \qquad \mathrm{dd^c} G(x, \cdot) = \delta_x - \omega.$$

Moreover, the function $\varrho(x, y) := G(x, y) - \log \operatorname{dist}(x, y)$ is Lipschitz.

For any probability measure μ on X, the aforementioned ω-potentials are given by

$$U_\mu^\star(x) := \int_X G(x, \cdot)\, \mathrm{d}\mu \quad \text{and} \quad U_\mu := U_\mu^\star - \max U_\mu^\star. \tag{2.6}$$

In particular, we see that $U_\mu^\star$ is uniformly bounded from above.

In fact, one can define the ω-potential $U_\nu^\star$ for any positive or signed measure ν on X using the above formula, which will be used in the study of Fekete points later. For any two positive or signed measures μ_1, μ_2 on X, it is clear that $U^\star_{\mu_1+\mu_2} = U^\star_{\mu_1} + U^\star_{\mu_2}$. Moreover, by Stokes' formula, one can show (cf. [17, Lemma 2.3]) that

$$\int_X U^\star_{\mu_1}\, \mathrm{d}\mu_2 = \int_X U^\star_{\mu_2}\, \mathrm{d}\mu_1, \tag{2.7}$$

and

$$\int_X U^\star_{\mu_1}\, \mathrm{d}\mu_1 \le 0. \tag{2.8}$$

There is another geometric description for the ω-potentials. For every $p \in X$, the holomorphic line bundle $\mathscr{O}(p)$ associated with the one-point divisor p has a canonical section $\mathbf{1}_{\mathscr{O}(p)}$. By the Poincaré-Lelong formula and the $\partial\bar\partial$-lemma, we can find a smooth Hermitian metric $\mathfrak{h}_p$ for every p such that the curvature form is ω and the potential reads as

$$U_{\delta_p}(z) = \log \left\|\mathbf{1}_{\mathscr{O}(p)}(z)\right\|_{\mathfrak{h}_p} + C_{\mathfrak{h}_p}, \tag{2.9}$$

where $C_{\mathfrak{h}_p}$ is a normalized constant subject to the condition $\max U_{\delta_p} = 0$. This $\mathfrak{h}_p$ is unique up to a multiplicative factor. To ensure uniqueness, we adopt the normalization $\int_X \log\|\mathbf{1}_{\mathcal{O}(x)}(z)\|_{\mathfrak{h}_x}\, \omega(z) = 0$ for convenience.

For every point $\mathbf{p} := p_1 + \cdots + p_m \in X^{(m)}$, $m \ge 1$, we denote by

$$\delta_{\mathbf{p}} := \frac{1}{m}(\delta_{p_1} + \cdots + \delta_{p_m}) \tag{2.10}$$

the empirical probability measure of $\mathbf{p}$. Similarly, we have

$$U_{\delta_{\mathbf{p}}}(z) = \frac{1}{m} \log \left\|\mathbf{1}_{\mathscr{O}(\mathbf{p})}(z)\right\|_{\mathfrak{h}_{\mathbf{p}}} + C_{\mathfrak{h}_{\mathbf{p}}}, \tag{2.11}$$

where $\mathbf{1}_{\mathscr{O}(\mathbf{p})} := \bigotimes_{j=1}^m \mathbf{1}_{\mathscr{O}(p_j)}$, $\mathfrak{h}_{\mathbf{p}} := \bigotimes_{j=1}^m \mathfrak{h}_{p_j}$, and $C_{\mathfrak{h}_{\mathbf{p}}}$ is a normalized constant subject to the condition $\max U_{\delta_{\mathbf{p}}} = 0$.

2.3. **Density of Zeros.** From now on, we fix a positive line bundle $\mathcal{L}$ on a compact Riemann surface X with degree 1. Let $n > g$ be an integer, and set $m := n - g$. Define the *exceptional subvariety*

$$\mathscr{H}_m := \left\{p_1 + \cdots + p_m \in X^{(m)} : \mathcal{A}_n(\mathcal{L}^n) - \mathcal{A}_m(p_1, \ldots, p_m) \in W^1_g\right\}$$

of $X^{(m)}$. Outside $\mathscr{H}_m$, we have a holomorphic map

$$\mathcal{B}_m : X^{(m)} \setminus \mathscr{H}_m \longrightarrow X^{(g)}$$

given by $\mathcal{B}_m(p_1 + \cdots + p_m) := q_1 + \cdots + q_g$ in (2.4) or (2.5). The image of $\mathcal{B}_m$ is $X^{(g)} \setminus \mathscr{W}$. Consequently, we obtain a holomorphic map

$$\mathscr{A}_m : X^{(m)} \setminus \mathscr{H}_m \longrightarrow \mathbb{P}H^0(X, \mathcal{L}^n),$$

defined by

$$\mathscr{A}_m(p_1 + \cdots + p_m) := [s],$$

where $s \in H^0(X, \mathcal{L}^n)$ satisfies $(s) = p_1 + \cdots + p_m + \mathcal{B}_m(p_1 + \cdots + p_m)$. The map $\mathscr{A}_m$ is generically $\binom{n}{m}$-to-one (c.f. [59, Proposition 3]). For a measurable set $A \subset \mathcal{M}(X)$, if we set

$$A_{\mathbb{P}} := \left\{[s] \in \mathbb{P}H^0(X, \mathcal{L}^n) : [\![Z_s]\!] \in A\right\},$$

then

$$\mathbf{P}_n(A_{\mathbb{P}}) = \int_{A_{\mathbb{P}}} V_n^{\mathrm{FS}} = \binom{n}{m}^{-1} \int_{\mathscr{A}_m^{-1}(A_{\mathbb{P}})} \mathscr{A}_m^*(V_n^{\mathrm{FS}}). \tag{2.12}$$

Write $\mathbf{q} := \mathcal{B}_m(\mathbf{p}) = q_1 + \cdots + q_g \in X^{(g)}$. We define

$$\mathcal{E}_m(\mathbf{p}) := \frac{1}{m^2} \sum_{j_1 \neq j_2} G(p_{j_1}, p_{j_2}) \quad \text{and} \quad \mathcal{F}_m(\mathbf{p}) := \log \left\| e^{U^\star_{\delta_{\mathbf{p}}} + m^{-1} g U^\star_{\delta_{\mathbf{q}}}} \right\|_{L^{2m}(\omega_0)}. \tag{2.13}$$

Here, G is the Green function of (X, ω), and $U^\star_{\delta_{\mathbf{p}}}, U^\star_{\delta_{\mathbf{q}}}$ are the ω-potentials of type I.

Zelditch [59, Theorem 2] established the following explicit formula for $\mathscr{A}_m^*(V_n^{\mathrm{FS}})$ (see also [17, Section 4]).

Proposition 2.3. *For all $n \geq 2g$, on $X^{(m)} \setminus \mathscr{H}_m$, one has*

$$\mathscr{A}_m^*(V_n^{\mathrm{FS}}) = C_n \exp\left[m^2 \Big(\mathcal{E}_m(\mathbf{p}) - \frac{2(m+1)}{m} \mathcal{F}_m(\mathbf{p}) \Big) \right] \kappa_n(\mathbf{p}). \tag{2.14}$$

Here, $\{C_n\}$ are some normalized positive constants, and κ_n are positive smooth (m,m)-forms on $X^{(m)} \setminus \mathscr{H}_m$ given by

$$\kappa_n(\mathbf{p}) := \Upsilon(\mathbf{p}) \xi(\mathbf{q}) \prod_{j=1}^m \prod_{l=1}^g \mathrm{dist}(p_j, q_l)^2 \, i\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \cdots \wedge i\mathrm{d}z_m \wedge \mathrm{d}\overline{z}_m,$$

where $e^{-O(m)} \leq \Upsilon(\mathbf{p}) \leq e^{O(m)}$, and $\xi(\mathbf{q})$ is unbounded on $X^{(g)} \setminus \mathscr{W}$ such that $\xi(\mathbf{q}) \geq c$ for some $c > 0$ and

$$\int_{X^{(m)} \setminus \mathscr{H}_m} \xi(\mathbf{q}) \, i\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \cdots \wedge i\mathrm{d}z_m \wedge \mathrm{d}\overline{z}_m \leq e^{O(m)}.$$

Let us elucidate the aforementioned formula. For points $\mathbf{p}$ not lying within $\pi_m(\mathscr{R}_m)$ (as defined in equation (2.3)), we employ local coordinates z_j around each p_j, for $j = 1, \ldots, m$. These coordinates z_j are chosen to reflect distances compatible with the metric induced by ω_0. Consequently, this construction yields a coordinate system $(z_1, \ldots, z_m)$ for points $\mathbf{p} \in X^{(m)} \setminus \pi_m(\mathscr{R}_m)$.

For points $\mathbf{p}$ that lie within $\pi_m(\mathscr{R}_m)$ yet outside of $\mathscr{H}_m$, it is observed that $\mathcal{E}_m(\mathbf{p})$ tends towards negative infinity. Consequently, we set $\mathscr{A}_m(V_n^{\mathrm{FS}})$ to be identically zero on the set $\pi_m(\mathscr{R}_m) \setminus \mathscr{H}_m$. This definition reflects the fact that the probability of encountering sections with multiple zeros is negligible, effectively zero.

Lemma 2.4 ([17]–Lemma 4.2). *The following holds:*

$$\sup_{\mathbf{p} \in X^{(m)}} \left| \mathcal{F}_m(\mathbf{p}) - \max U^\star_{\delta_{\mathbf{p}}} \right| = O\Big(\frac{\log m}{m} \Big) \quad \textit{as} \quad m \to \infty.$$

The above result will be used to show that the asymptotic behavior of the hole probabilities $\mathbf{P}_n(H_{n,D})$ are not sensitive to ω_0 as $n \to +\infty$.

To apply Zelditch's density formula (2.14) to study hole probabilities, we need to compute its integration over the set $\mathscr{A}_m^{-1}(H_{n,D})$. Thus, we put $\mathcal{R}_{m,D} := \mathscr{A}_m^{-1}(H_{n,D})$, which can be rephrased as

$$\mathcal{R}_{m,D} = \big\{ \mathbf{p} \in (X \setminus D)^{(m)} \setminus \mathscr{H}_m : \mathbf{q} = \mathcal{B}_m(\mathbf{p}) \in (X \setminus D)^{(g)} \big\}.$$

We also define the following subset of $\mathcal{R}_{m,D}$ consisting of "separated" elements:

$$\mathcal{S}_{m,D} := \big\{ \mathbf{p} \in \mathcal{R}_{m,D} : \operatorname{dist}(p_{j_1}, p_{j_2}) \geq m^{-4} \text{ for } 1 \leq j_1 \neq j_2 \leq m, \\ \operatorname{dist}(p_j, q_l) \geq m^{-1} \text{ for } 1 \leq j \leq m, 1 \leq l \leq g \big\}, \tag{2.15}$$

where $\mathbf{p} := p_1 + \cdots + p_m$ and $\mathbf{q} := q_1 + \cdots + q_g$. This $\mathcal{S}_{m,D}$ will play an important role in the proof of Theorem 1.1.

The key point in the proof of Theorem 1.1 are to obtain upper and lower bounds for the functional $\mathcal{E}_m(\mathbf{p}) - 2\max U^{\star}_{\delta_{\mathbf{p}}}$.

3. Regularity of the Equilibrium Measure

By [17, Theorem 1], we know that the minimizer of $\mathcal{I}_D$ is $\nu_D := \omega|_{S_D} + \nu_{\partial D}$, where $S_D := \{U_{\nu_D} = 0\} \setminus \overline{D}$ and $\nu_{\partial D}$ is a non-vanishing positive measure on ∂D. Moreover, if ν_D has positive mass on some component of ∂D, then $\operatorname{supp}(\nu_D)$ contains this component and U_{ν_D} is strictly negative on this component of ∂D.

In this section, our goal is to prove the following regularity property of $\nu_{\partial D}$, which will be used for studying the Fekete points in the next section.

Proposition 3.1. *For any arc A on ∂D with $\nu(A) > 0$, one has*

$$\nu_D(A) \leq C_D \cdot \operatorname{length}(A)$$

for some $C_D > 0$ independent of A.

Recall the following result from [17, Proposition 3.1].

Proposition 3.2. *Let h be a continuous function on $\overline{D}$ and harmonic on D. Assume that $h + U_{\nu_D} \leq 0$ on $\overline{D}$, then*

$$\int_{\partial D} h \, \mathrm{d}\nu_D \leq 0.$$

Suppose that the inequality of Proposition 3.1 fails. By the following lemma, we will construct a harmonic function h such that $h + U_{\nu_D} \leq 0$ on $\overline{D}$ while $\int_{\partial D} h \, \mathrm{d}\nu_D > 0$, which leads to a contradiction. One key point is that we need to construct a simply connected domain to apply Possion formula.

Lemma 3.3. *Let μ be a probability measure on ∂D and z be a point in ∂D. Suppose that there exists a sequence $\{r_j\}_{j\geq 1}$ of positive radii shrinking to 0 such that $\mu(\mathbb{B}(z, r_j))/r_j$ tends to infinity. Then there exist a positive sequence $\{\beta_j\}_{j\geq 1}$ tending to infinity and a sequence of functions $\{F_j\}_{j\geq 1}$ which are harmonic on D and continuous on ∂D, such that $\beta_j r_j$ tends to 0 and such that for large enough j, $F_j < 0$ on $\overline{D} \setminus \mathbb{B}(z, \beta_j r_j)$ while $\int_{\partial D} F_j \, \mathrm{d}\mu > 0$.*

Proof. Take a small $\gamma > 0$ such that $\mathbb{B}(z, \gamma)$ intersects only one connected component of D. Reducing γ if necessary, we can draw a simply connected subset $D' \subset D$ with a smooth boundary such that $D \cap \mathbb{B}(z, \gamma) \subset D'$.

Denote $\alpha_j := \mu(\mathbb{B}(z, r_j))$. Take $\beta_j := 1/\sqrt{\alpha_j r_j}$, so that $\beta_j, \beta_j \alpha_j$ tend to infinity, while $\beta_j r_j, \beta_j r_j^2/\alpha_j$ tends to zero. Let g_j be a smooth cut-off function on ∂D such that

$$g_j = 1 \text{ on } \partial D \cap \mathbb{B}(z, r_j) \quad \text{and} \quad g_j = 0 \text{ on } \partial D \setminus \mathbb{B}(z, 2r_j). \tag{3.1}$$

Set $f_j := g_j - \alpha_j/2$. Let F_j, G_j be the harmonic extensions of f_j, g_j respectively to D, whose existence is guaranteed by [57, Theorem I.12] since ∂D contains finitely many components. It is clear that

$$\int_{\partial D} F_j \,\mathrm{d}\mu = \int_{\partial D} g_j \,\mathrm{d}\mu - \alpha_j/2 \geq \alpha_j - \alpha_j/2 > 0.$$

Now we prove that $F_j < 0$ on $\overline{D} \setminus \mathbb{B}(z, \beta_j r_j)$ for large j by means of the maximum principle and the auxiliary simply connected subdomain $D' \subset D$. By throwing out some finite j if necessary, we can assume that all $\beta_j r_j < \gamma$, hence $\partial\mathbb{B}(z, \beta_j r_j) \cap D$ coincides with $\partial\mathbb{B}(z, \beta_j r_j) \cap D'$ by our choice of D'.

Now we consider the restricted function of G_j on $\partial D'$, denoted by h_j. The key observation is that the support of h_j is contained in the union of $\partial D' \cap \mathbb{B}(z, 2r_j)$ and $\partial D' \setminus \partial D$, while these two segments keep a relatively large distance compared with $\beta_j r_j$. This feature provides the opportunity for manipulating the Poisson formula on D'.

By the Riemann mapping theorem and Kellogg's theorem (c.f. [24, p. 62, Theorem 4.3]), we can find a biholomorphism $\phi : D' \to \mathbb{D}$ such that both ϕ and ϕ^{-1} can be continuously extended to the boundaries and are K-Lipschitz with some positive constant K. Since G_j solves the Dirichlet problem on D' with the boundary data h_j, by the Poisson formula, we have

$$G_j(w) = \frac{1}{2\pi} \int_0^{2\pi} h_j \circ \phi^{-1}(e^{i\theta}) \frac{1 - |\phi(w)|^2}{|e^{i\theta} - \phi(w)|^2} \,\mathrm{d}\theta$$

for every w in $\partial\mathbb{B}(z, \beta_j r_j) \cap D'$. We only need to integrate against two kinds of θ, either $\phi^{-1}(e^{i\theta}) \in \partial D' \cap \mathbb{B}(z, 2r_j)$ or $\phi^{-1}(e^{i\theta}) \in \partial D' \setminus \partial D$, since $h_j \circ \phi^{-1}(e^{i\theta})$ vanishes otherwise.

Using the fact that ϕ is K-Lipschitz, we have

$$1 - |\phi(w)|^2 \leq 2(|\phi(z)| - |\phi(w)|) \leq 2|\phi(z) - \phi(w)| \leq 2\mathsf{K}\beta_j r_j. \tag{3.2}$$

Moreover, for every $e^{i\theta}$ in $\phi(\partial D' \cap \mathbb{B}(z, 2r_j))$, we have (recalling $\beta_j \to \infty$)

$$|e^{i\theta} - \phi(w)| \geq \mathsf{K}^{-1} \operatorname{dist}(\phi^{-1}(e^{i\theta}), w) \geq \mathsf{K}^{-1}\big(\operatorname{dist}(w, z) - \operatorname{dist}(\phi^{-1}(e^{i\theta}), z)\big) \geq \mathsf{K}^{-1}\beta_j r_j/2.$$

Thus, the first kind of integration with $e^{i\theta}$ in $\phi(\partial D' \cap \mathbb{B}(z, 2r_j))$ is

$$\int_{\phi^{-1}(e^{i\theta}) \in \partial D' \cap \mathbb{B}(z, 2r_j)} h_j \circ \phi^{-1}(e^{i\theta}) \frac{1 - |\phi(w)|^2}{|e^{i\theta} - \phi(w)|^2} \,\mathrm{d}\theta = O\Big(r_j \cdot \frac{\beta_j r_j}{(\beta_j r_j)^2}\Big) = O(\beta_j^{-1}).$$

For the second kind of θ with $\phi^{-1}(e^{i\theta}) \in \partial D' \setminus \partial D$, by [24, p. 65, Corollary 4.7], the harmonic measure for $y \in D$ is absolutely continuous to arc length of ∂D. This implies $G(y) = O(r_j)$ for $y \in \partial D' \setminus \partial D$. Moreover, noting that the distance between $\partial D' \cap \mathbb{B}(z, \beta_j r_j)$ and $\partial D' \setminus \partial D$ is larger than a fixed positive number, we can show using (3.2) that

$$\frac{1 - |\phi(w)|^2}{|e^{i\theta} - \phi(w)|^2} = O(\beta_j r_j).$$

Thus, the second kind of integration with $\phi^{-1}(e^{i\theta}) \in \partial D' \setminus \partial D$ contributes $O(\beta_j r_j^2)$. Summarizing, for every w in $\partial\mathbb{B}(z, \beta_j r_j) \cap D'$, we have

$$G_j(w) = O(\beta_j^{-1}) + O(\beta_j r_j^2).$$

Thus, we can bound $F_j(w) = G_j(w) - \alpha_j/2$ from above by

$$O(\beta_j^{-1}) + O(\beta_j r_j^2) - \alpha_j/2 = \big(O(\beta_j^{-1}) - \alpha_j/4\big) + \big(O(\beta_j r_j^2) - \alpha_j/4\big).$$

Both brackets $(\cdots)$ are negative for large j by our choice of β_j. Hence, we conclude the proof by the maximum principle. $\square$

Proof of Proposition 3.1. Assume, to the contrary, that such a constant C_D does not exist. Then, by Vitali's covering theorem (e.g., [35, Chapter 2]), we can find a point $z \in \partial D$ and a sequence $\{r_j\}_{j\geq 1}$ of positive radii tending to zero such that $\nu_{\partial D}(\mathbb{B}(z, r_j))/r_j \to \infty$. By considering the probability measure $c\nu_{\partial D}$ for a suitable c and applying Lemma 3.3, we can find a positive sequence $\{\beta_j\}_{j\geq 1}$ tending to infinity and a sequence of functions $\{F_j\}_{j\geq 1}$, which are harmonic on D and continuous on ∂D, such that $\beta_j r_j$ tends to zero, $F_j < 0$ on $\overline{D} \setminus \mathbb{B}(z, \beta_j r_j)$, and $\int_{\partial D} F_j \, d\nu_{\partial D} > 0$ for all j.

On the other hand, recall that U_{ν_D} is strictly negative on the component of ∂D where $\nu_{\partial D}$ has positive mass. In particular, $U_{\nu_D}(z) < 0$. By the upper semi-continuity of U_{ν_D}, when $\beta_j r_j$ is sufficiently small, there exists a small positive constant δ such that the function δF_j satisfies

$$\delta F_j + U_{\nu_D} \leq 0 \text{ on } \overline{D} \quad \text{and} \quad \int_{\partial D} \delta F_j \, \mathrm{d}\nu_D > 0.$$

This contradicts Proposition 3.2. $\square$

Now we can state the regularity result for U_{ν_D}. The proof is standard and follows from Proposition 3.1. We provide it here for the convenience of the readers.

Proposition 3.4. *The quasi-potential U_{ν_D} is $1/3$-Hölder continuous.*

Proof. Recall the decomposition $\nu_D = \omega|_{S_D} + \nu_{\partial D}$ and the corresponding potential decomposition $U^\star_{\nu_D} = U^\star_{\omega|_{S_D}} + U^\star_{\nu_{\partial D}}$. It suffices to show that $U^\star_{\nu_{\partial D}}$ is Hölder continuous. Using (2.7), we have

$$U^\star_{\nu_{\partial D}}(z) - U^\star_{\nu_{\partial D}}(w) = \int U^\star_{\nu_{\partial D}} \, \mathrm{d}(\delta_z - \delta_w) = \int (U^\star_{\delta_z} - U^\star_{\delta_w}) \, \mathrm{d}\nu_{\partial D}.$$

By (2.6), we obtain

$$U^\star_{\delta_z}(y) - U^\star_{\delta_w}(y) = \varrho(z,y) - \varrho(w,y) + \log \operatorname{dist}(z,y) - \log \operatorname{dist}(w,y),$$

where ϱ is Lipschitz. Hence, we only need to estimate the integral

$$\int \big(\log \operatorname{dist}(z,y) - \log \operatorname{dist}(w,y) \big) \, \mathrm{d}\nu_{\partial D}(y).$$

Fix a small $a > 0$ to be determined. We only need to consider the case where $\operatorname{dist}(z,w) \leq a$. Let us split the integral into two parts:

$$\Big(\int_{\operatorname{dist}(z,y)>2a} + \int_{\operatorname{dist}(z,y)\leq 2a} \Big) \big(\log \operatorname{dist}(z,y) - \log \operatorname{dist}(w,y) \big) \, \mathrm{d}\nu_{\partial D}(y).$$

If $\operatorname{dist}(z,y) > 2a$, then $\operatorname{dist}(w,y) > a$. Using Proposition 3.1 and the mean value theorem,

$$\begin{aligned} &\Big| \int_{\operatorname{dist}(z,y)>2a} \big(\log \operatorname{dist}(z,y) - \log \operatorname{dist}(w,y) \big) \, \mathrm{d}\nu_{\partial D}(y) \Big| \\ &\leq c'_D \int_{\operatorname{dist}(z,y)>2a} \frac{\operatorname{dist}(z,w)}{\operatorname{dist}(w,y)} \, \mathrm{dLeb}(\partial D) \leq c'_D a^{-1} \cdot \operatorname{dist}(z,w), \end{aligned}$$

where $\operatorname{Leb}(\partial D)$ is the probability measure on ∂D proportional to arc length.

If $\mathrm{dist}(z,y)\leq 2a$, then $\mathrm{dist}(w,y)\leq 3a$. Using Proposition 3.1 again, we have

$$\Big|\int_{\mathrm{dist}(z,y)\leq 2a}\big(\log\mathrm{dist}(z,y)-\log\mathrm{dist}(w,y)\big)\,\mathrm{d}\nu_{\partial D}(y)\Big|$$

$$\leq c''_D\Big|\int_{\mathrm{dist}(z,y)\leq 2a}\log\mathrm{dist}(z,y)\,\mathrm{dLeb}(\partial D)\Big|+c''_D\Big|\int_{\mathrm{dist}(w,y)\leq 3a}\log\mathrm{dist}(w,y)\,\mathrm{dLeb}(\partial D)\Big|,$$

which is $O(c''_D a|\log a|)$. Thus, by taking $a:=\sqrt{\mathrm{dist}(z,w)}$, we complete the proof. □

Remark 3.5. Our boundary assumption on ∂D is also not optimal; as long as ∂D is piecewise $\mathscr{C}^{1+\epsilon}$ and does not contain cusps, U_{ν_D} is γ-Hölder continuous for some $\gamma>0$. To keep this article concise, we do not delve further into this direction.

4. Comparison of the Two Functionals-I

4.1. **Main statement.** In this section, we discuss the upper bound for the functional $\mathcal{E}_m(\mathbf{p})-2\max U^\star_{\delta_\mathbf{p}}$.

Proposition 4.1. *There exists a constant $C>0$ independent of D and m, such that*

$$\sup_{\mathbf{p}\in(X\setminus D)^{(m)}}\big(\mathcal{E}_m(\mathbf{p})-2\max U^\star_{\delta_\mathbf{p}}\big)\leq-\min\mathcal{I}_D+C\frac{\log m}{m}\quad \textit{for all}\quad m>1.$$

We will show that for each $\mathbf{p}=p_1+\cdots+p_m\in(X\setminus D)^{(m)}$, there exists a probability measure $\mu_\mathbf{p}\in\mathcal{M}(X\setminus D)$ such that

$$\mathcal{E}_m(\mathbf{p})-2\max U^\star_{\delta_\mathbf{p}}\leq-\mathcal{I}_D(\mu_\mathbf{p})+C\frac{\log m}{m}. \tag{4.1}$$

This immediately implies Proposition 4.1. Without loss of generality, we may assume that the points $p_1,p_2,\ldots,p_m$ are all distinct, as otherwise the inequality holds trivially.

4.2. **Auxiliary measure $\sigma_\mathbf{p}$.** For a given $\mathbf{p}=p_1+\cdots+p_m\in(X\setminus D)^{(m)}$, recall that $\delta_\mathbf{p}=m^{-1}(\delta_{p_1}+\cdots+\delta_{p_m})$ and that $U^\star_{\delta_\mathbf{p}}$ is its type I ω-potential. Define the set

$$E_\mathbf{p}:=\bigcup_{j=1}^m\mathbb{B}(p_j,m^{-3}),$$

and consider the function

$$V_\mathbf{p}:=\sup_\phi\big\{\phi\text{ is }\omega\text{-subharmonic}:\ \phi\leq U^\star_{\delta_\mathbf{p}}\text{ on }X\setminus E_\mathbf{p}\big\}^*.$$

Note that the boundary $\partial E_\mathbf{p}$ is smooth except at finitely many points. Consequently, $V_\mathbf{p}$ is a continuous ω-subharmonic function satisfying:

$$V_\mathbf{p}=U^\star_{\delta_\mathbf{p}}\quad\text{on }\ X\setminus E_\mathbf{p}\quad\text{and}\quad V_\mathbf{p}>U^\star_{\delta_\mathbf{p}}\quad\text{on }\ E_\mathbf{p},$$

$$\mathrm{dd^c}V_\mathbf{p}=-\omega\quad\text{on }\ X\setminus\partial E_\mathbf{p}.$$

Define the probability measure

$$\sigma_\mathbf{p}:=\mathrm{dd^c}V_\mathbf{p}+\omega,$$

which is supported on $\partial E_\mathbf{p}$. Here, we suppress the dependence of $\sigma_\mathbf{p}$ on m in the notation for simplicity.

Lemma 4.2. *One has* $|V_{\mathbf{p}} - U^\star_{\sigma_{\mathbf{p}}}| \le Cm^{-4}$ *for some* $C > 0$ *independent of* D, m *and* $\mathbf{p}$. *In particular,* $|U^\star_{\sigma_{\mathbf{p}}} - U^\star_{\delta_{\mathbf{p}}}| \le Cm^{-4}$ *on* $X \setminus E_{\mathbf{p}}$.

Proof. By the definition of the ω-potential, we have $V_{\mathbf{p}} - U^\star_{\sigma_{\mathbf{p}}} = \int_X V_{\mathbf{p}}\,\omega$. Using the fact that $V_{\mathbf{p}} = U^\star_{\delta_{\mathbf{p}}}$ on $X \setminus E_{\mathbf{p}}$, we obtain

$$\begin{aligned}\int_X V_{\mathbf{p}}\,\omega &= \int_{E_{\mathbf{p}}} V_{\mathbf{p}}\,\omega + \int_{X\setminus E_{\mathbf{p}}} V_{\mathbf{p}}\,\omega = \int_{E_{\mathbf{p}}} V_{\mathbf{p}}\,\omega + \int_{X\setminus E_{\mathbf{p}}} U^\star_{\delta_{\mathbf{p}}}\,\omega \\ &= \int_{E_{\mathbf{p}}} V_{\mathbf{p}}\,\omega + 0 - \int_{E_{\mathbf{p}}} U^\star_{\delta_{\mathbf{p}}}\,\omega = \int_{E_{\mathbf{p}}} V_{\mathbf{p}}\,\omega - \int_{E_{\mathbf{p}}} U^\star_{\delta_{\mathbf{p}}}\,\omega.\end{aligned}$$

Recall that $U^\star_{\delta_{\mathbf{p}}}$ is uniformly bounded above, independently of $\mathbf{p}$. On $E_{\mathbf{p}}$, the function $V_{\mathbf{p}}$ satisfies $\mathrm{dd^c} V_{\mathbf{p}} = -\omega$ with boundary condition $V_{\mathbf{p}} = U^\star_{\delta_{\mathbf{p}}}$ on $\partial E_{\mathbf{p}}$, which implies $V_{\mathbf{p}} \le c_1$ for some constant $c_1 > 0$ independent of $\mathbf{p}$ and m. Moreover, from formula (2.6) and Lemma 2.2, it follows that for $z \in \overline{E}_{\mathbf{p}}$,

$$U^\star_{\delta_{\mathbf{p}}}(z) \ge \frac{c_2}{m}\sum_{j=1}^m \log \operatorname{dist}(z, p_j), \tag{4.2}$$

where $c_2 > 0$ is independent of $\mathbf{p}$ and m. Since $V_{\mathbf{p}} > U^\star_{\delta_{\mathbf{p}}}$ on $E_{\mathbf{p}}$, we have

$$\begin{aligned}\left|V_{\mathbf{p}}(z) - U^\star_{\sigma_{\mathbf{p}}}(z)\right| = \left|\int_X V_{\mathbf{p}}\,\omega\right| &\le \left|\int_{E_{\mathbf{p}}} V_{\mathbf{p}}\,\omega\right| + \left|\int_{E_{\mathbf{p}}} U^\star_{\delta_{\mathbf{p}}}\,\omega\right| \\ &\le \int_{E_{\mathbf{p}}} \max\left(c_1, |U^\star_{\delta_{\mathbf{p}}}|\right)\omega + \int_{E_{\mathbf{p}}} |U^\star_{\delta_{\mathbf{p}}}|\,\omega \\ &\le c_1 \cdot \mathrm{Area}_\omega(E_{\mathbf{p}}) + 2\int_{E_{\mathbf{p}}} |U^\star_{\delta_{\mathbf{p}}}|\,\omega \\ &\le O(m^{-6}) + \frac{2c_2}{m}\sum_{j=1}^m \int_{E_{\mathbf{p}}} \left|\log \operatorname{dist}(z, p_j)\right|\omega.\end{aligned}$$

We now estimate the last summation term. For each index j, the integral over the region where $\operatorname{dist}(z, p_j) \ge m^{-3}$ is bounded by

$$\mathrm{Area}(E_{\mathbf{p}}) \cdot |\log m^{-3}| = O(m^{-5}\log m).$$

For the disk $\operatorname{dist}(z, p_j) < m^{-3}$, we use the estimate

$$\left|\int_{\mathbb{D}(0,r)} \log|z|\, i\mathrm{d}z \wedge \mathrm{d}\bar{z}\right| = O(r^2|\log r|),$$

which shows this contribution is $O(m^{-6}\log m)$. This completes the proof of the lemma. □

Proposition 4.3. *There exists a constant* $C > 0$ *independent of* D, m *and* $\mathbf{p}$ *such that for all* $m > 1$,

$$\mathcal{E}_m(\mathbf{p}) \le \int_X U^\star_{\sigma_{\mathbf{p}}}\,\mathrm{d}\sigma_{\mathbf{p}} + C\frac{\log m}{m} \tag{4.3}$$

and

$$\left|\max U^\star_{\delta_{\mathbf{p}}} - \max U^\star_{\sigma_{\mathbf{p}}}\right| \le Cm^{-2}.$$

Proof. We use formula (2.6) to rewrite (see (2.13))

$$\mathcal{E}_m(\mathbf{p}) = \frac{1}{m^2}\sum_{j\neq k} G(p_j, p_k) = \frac{1}{m^2}\sum_{j\neq k} U^\star_{\delta_{p_j}}(p_k).$$

Using Lemma 4.2 and that $V_\mathbf{p} = U^\star_{\delta_\mathbf{p}}$ on the support of $\sigma_\mathbf{p}$, we have

$$\int_X U^\star_{\sigma_\mathbf{p}}\,\mathrm{d}\sigma_\mathbf{p} = \int_X U^\star_{\delta_\mathbf{p}}\,\mathrm{d}\sigma_\mathbf{p} + O(m^{-4}) = \int_X U^\star_{\sigma_\mathbf{p}}\,\mathrm{d}\delta_\mathbf{p} + O(m^{-4}) = \frac{1}{m}\sum_{j=1}^m U^\star_{\sigma_\mathbf{p}}(p_j) + O(m^{-4}),$$

where we use (2.7) for the second equality. For each $1 \le k \le m$, put

$$\Psi_k := \frac{1}{m}\sum_{j\neq k} U^\star_{\delta_{p_j}}.$$

Note that $\Psi_1(p_1) + \Psi_2(p_2) + \cdots + \Psi_m(p_m) = m\mathcal{E}_m(\mathbf{p})$. We will show that for each k,

$$\Psi_k(p_k) \le U^\star_{\sigma_\mathbf{p}}(p_k) + O\Big(\frac{\log m}{m}\Big), \tag{4.4}$$

which clearly gives the first assertion. In the open set $E_\mathbf{p}$, denote by Ω_{p_k} the largest simply connected sub-domain of $E_\mathbf{p}$ containing p_k. If $\mathrm{dist}(p_k, p_j) \ge 2m^{-3}$ for all $j \neq k$, then Ω_{p_k} is exactly $\mathbb{B}(p_k, m^{-3})$. Notice that $\Omega_{p_j} = \Omega_{p_k}$ if and only if $\mathrm{dist}(p_j, p_k) < 2m^{-3}$. By definition, $\partial\Omega_{p_k} \subset \partial E_\mathbf{p}$.

Recall that $\mathrm{dd^c} U^\star_{\sigma_\mathbf{p}} = -\omega$ on Ω_{p_k}. So, $\Psi_k - U^\star_{\sigma_\mathbf{p}}$ is subharmonic on Ω_{p_k}. Using maximal modulus principle, we get

$$\Psi_k(p_k) - U^\star_{\sigma_\mathbf{p}}(p_k) < \max_{\partial\Omega_{p_k}}(\Psi_k - U^\star_{\sigma_\mathbf{p}}). \tag{4.5}$$

Now take any point $w \in \partial\Omega_{p_k}$. By Lemma 4.2, we have

$$U^\star_{\sigma_\mathbf{p}}(w) = U^\star_{\delta_\mathbf{p}}(w) + O(m^{-4}) = \frac{1}{m}\sum_{j=1}^m U^\star_{\delta_{p_j}}(w) + O(m^{-4}) = \Psi_k(w) + \frac{1}{m}U^\star_{\delta_{p_k}}(w) + O(m^{-4}).$$

Since $\mathrm{dist}(w, p_k) \ge m^{-3}$, $U^\star_{\delta_{p_1}}(w) \ge -c\log m$ for some $c > 0$. This gives (4.4) and proves the first assertion.

For the second assertion, by Lemma 4.2, it is enough to prove $\big|\max_{\overline{E}_\mathbf{p}} U^\star_{\delta_\mathbf{p}} - \max_{\overline{E}_\mathbf{p}} V_\mathbf{p}\big| = O(m^{-2})$. Equivalently,

$$\max_{\overline{\Omega}_{p_k}} V_\mathbf{p} \le \max_{\overline{\Omega}_{p_k}} U^\star_{\delta_\mathbf{p}} + O(m^{-2}) \quad \text{for every } k.$$

We only show the inequality for $k = 1$. Assume $V_\mathbf{p}|_{\overline{\Omega}_{p_1}}$ attends its maximum at x_1, and $U^\star_{\delta_\mathbf{p}}|_{\overline{\Omega}_{p_1}}$ attends its maximum at x_2. Note that the diameter of Ω_{p_1} is less than $m \cdot 2m^{-3} = 2m^{-2}$. Thus, we can take a smooth local potential φ of ω near Ω_{p_1}, i.e., $\mathrm{dd^c}\varphi = \omega$ on a neighborhood of Ω_{p_1}. The function $V_\mathbf{p} + \varphi$ is harmonic on Ω_{p_1}, whose maximum appears on the boundary, say $x_3 \in \partial\Omega_{p_1}$. Recalling that $V_\mathbf{p} = U^\star_{\delta_\mathbf{p}}$ on $\partial\Omega_{p_1}$, yields

$$V_\mathbf{p}(x_1) + \varphi(x_1) \le V_\mathbf{p}(x_3) + \varphi(x_3) = U^\star_{\delta_\mathbf{p}}(x_3) + \varphi(y_3) \le U^\star_{\delta_\mathbf{p}}(x_2) + \varphi(x_3).$$

The desired inequality follows by the fact $|\varphi(x_1) - \varphi(x_3)| \le \|\varphi\|_{\mathscr{C}^1} \cdot \mathrm{diam}(\Omega_{p_1}) = O(m^{-2})$. The proof of the proposition is complete. $\square$

Remark 4.4. If $\operatorname{dist}(p_j, p_k) \geq 2m^{-3}$ for all $j \neq k$, the set $E_{\mathbf{p}}$ will be the union of disjoint open balls. In the proof of Proposition 4.3 above, Ω_{p_k} will be $\mathbb{B}(p_k, m^{-3})$ and $\Psi_k - U^\star_{\sigma_{\mathbf{p}}}$ will be harmonic on Ω_{p_k}. In (4.5), we will also get a lower bounded estimate:

$$\Psi_k(p_k) - U^\star_{\sigma_{\mathbf{p}}}(p_k) > \min_{\partial\Omega_{p_k}}(\Psi_k - U^\star_{\sigma_{\mathbf{p}}}).$$

This will prove (4.4) replacing "$\leq$" by "$\geq$" and hence give the full inequality of (4.3):

$$\Big|\mathcal{E}_m(\mathbf{p}) - \int_X U^\star_{\sigma_{\mathbf{p}}}\, \mathrm{d}\sigma_{\mathbf{p}}\Big| \leq C\frac{\log m}{m},$$

which will be used in Sec. 6.

By Proposition 4.3, $\sigma_{\mathbf{p}}$ almost satisfies (4.1), except the support requirement. If p_j is too close to the boundary of D, the support of $\sigma_{\mathbf{p}}$ will exceed $X \setminus D$. We need to do one more perturbation to get Proposition 4.1. The idea is similar. We will repeat the process of defining $\sigma_{\mathbf{p}}$, replacing $E_{\mathbf{p}}$ by $E_{\mathbf{p}} \setminus \overline{D}$, $U^\star_{\delta_{\mathbf{p}}}$ by $U^\star_{\sigma_{\mathbf{p}}}$.

4.3. **Construction of $\mu_{\mathbf{p}}$.** Consider the new upper envelop

$$U_{\mathbf{p}} := \sup_\phi \big\{\phi \text{ is } \omega\text{-subharmonic}: \ \phi \leq U^\star_{\sigma_{\mathbf{p}}} \text{ on } X \setminus (E_{\mathbf{p}} \setminus \overline{D})\big\}^*.$$

Note that the boundary $\partial(E_{\mathbf{p}} \setminus \overline{D})$ is smooth expect at finitely many points. Thus, $U_{\mathbf{p}}$ is a continuous ω-subharmonic function satisfying

$$U_{\mathbf{p}} = U^\star_{\sigma_{\mathbf{p}}} \quad \text{on } X \setminus (E_{\mathbf{p}} \setminus \overline{D}) \quad \text{and} \quad U_{\mathbf{p}} \geq U^\star_{\sigma_{\mathbf{p}}} \quad \text{on } E_{\mathbf{p}} \setminus \overline{D},$$

$$\mathrm{dd^c} U_{\mathbf{p}} = -\omega \quad \text{on } X \setminus \partial(E_{\mathbf{p}} \setminus \overline{D}).$$

Define the probability measure

$$\mu_{\mathbf{p}} := \mathrm{dd^c} U_{\mathbf{p}} + \omega,$$

which is supported on $\partial(E_{\mathbf{p}} \setminus \overline{D})$. In particular, it is in $\mathcal{M}(X \setminus D)$.

Lemma 4.5. *One has $|U_{\mathbf{p}} - U^\star_{\mu_{\mathbf{p}}}| \leq Cm^{-5}$ for some $C > 0$ independent of D, m and $\mathbf{p}$. In particular, $|U^\star_{\mu_{\mathbf{p}}} - U^\star_{\sigma_{\mathbf{p}}}| \leq Cm^{-5}$ on $X \setminus (E_{\mathbf{p}} \setminus \overline{D})$.*

Proof. By definition of ω-potential of type I, $U_{\mathbf{p}} - U^\star_{\mu_{\mathbf{p}}} = \int_X U_{\mathbf{p}}\, \omega$. Using the fact that $U_{\mathbf{p}} = U^\star_{\sigma_{\mathbf{p}}}$ on $X \setminus (E_{\mathbf{p}} \setminus \overline{D})$, we obtain

$$\begin{aligned}\int_X U_{\mathbf{p}}\, \omega &= \int_{E_{\mathbf{p}}\setminus\overline{D}} U_{\mathbf{p}}\, \omega + \int_{X\setminus(E_{\mathbf{p}}\setminus\overline{D})} U_{\mathbf{p}}\, \omega = \int_{E_{\mathbf{p}}\setminus\overline{D}} U_{\mathbf{p}}\, \omega + \int_{X\setminus(E_{\mathbf{p}}\setminus\overline{D})} U^\star_{\sigma_{\mathbf{p}}}\, \omega \\ &= \int_{E_{\mathbf{p}}\setminus\overline{D}} U_{\mathbf{p}}\, \omega + 0 - \int_{E_{\mathbf{p}}\setminus\overline{D}} U^\star_{\sigma_{\mathbf{p}}}\, \omega = \int_{E_{\mathbf{p}}\setminus\overline{D}} U_{\mathbf{p}}\, \omega - \int_{E_{\mathbf{p}}\setminus\overline{D}} U^\star_{\sigma_{\mathbf{p}}}\, \omega.\end{aligned}$$

Recall that on $X \setminus \partial E_{\mathbf{p}}$, $V_{\mathbf{p}}$ satisfies $\mathrm{dd^c} V_{\mathbf{p}} = -\omega$ with boundary condition $V_{\mathbf{p}} = U^\star_{\delta_{\mathbf{p}}}$ on $\partial E_{\mathbf{p}}$. By maximum modulus principle, for $z \in X$,

$$V_{\mathbf{p}}(z) \geq \min_{\partial E_{\mathbf{p}}} U^\star_{\delta_{\mathbf{p}}} \geq -3c_2 \log m.$$

by (4.2). Since $U_{\mathbf{p}} \geq U^\star_{\sigma_{\mathbf{p}}}$, it follows from Lemma 4.2 that $U_{\mathbf{p}} \geq -3c_2 \log m - O(m^{-4})$. Thus

$$\Big|\int_X U_{\mathbf{p}}\, \omega\Big| \leq \operatorname{Area}(E_{\mathbf{p}} \setminus \overline{D}) \cdot \big(3c_2 \log m + O(m^{-4})\big) \leq O(m^{-6} \log m).$$

This finishes the proof of the lemma. □

Proposition 4.6. *There exists a constant $C > 0$ independent of D, m and $\mathbf{p}$ such that for all $m > 1$,*

$$\Big|\int_X U^\star_{\sigma_\mathbf{p}}\, \mathrm{d}\sigma_\mathbf{p} - \int_X U^\star_{\mu_\mathbf{p}}\, \mathrm{d}\mu_\mathbf{p}\Big| \le Cm^{-5}$$

and

$$\big|\max U^\star_{\sigma_\mathbf{p}} - \max U^\star_{\mu_\mathbf{p}}\big| \le Cm^{-2}.$$

Proof. Recall that $U_\mathbf{p} = U^\star_{\sigma_\mathbf{p}}$ on $X \setminus (E_\mathbf{p} \setminus \overline{D})$, which contains $\mathrm{supp}(\mu_\mathbf{p})$. Using Lemma 4.5, we have

$$\int_X U^\star_{\mu_\mathbf{p}}\, \mathrm{d}\mu_\mathbf{p} = \int_X U_\mathbf{p}\, \mathrm{d}\mu_\mathbf{p} + O(m^{-5}) = \int_X U^\star_{\sigma_\mathbf{p}}\, \mathrm{d}\mu_\mathbf{p} + O(m^{-5}).$$

Therefore, by (2.7),

$$\begin{aligned}\int_X U^\star_{\sigma_\mathbf{p}}\, \mathrm{d}\sigma_\mathbf{p} - \int_X U^\star_{\mu_\mathbf{p}}\, \mathrm{d}\mu_\mathbf{p} &= \int_X U^\star_{\sigma_\mathbf{p}}\, \mathrm{d}(\sigma_\mathbf{p} - \mu_\mathbf{p}) + O(m^{-5}) \\ &= \int_X (U^\star_{\sigma_\mathbf{p}} - U^\star_{\mu_\mathbf{p}})\, \mathrm{d}\sigma_\mathbf{p} + O(m^{-5}).\end{aligned}$$

Since $\mathrm{supp}(\sigma_\mathbf{p}) \subset X \setminus (E_\mathbf{p} \setminus \overline{D})$, using Lemma 4.5 again, we prove the first assertion.

For the second assertion, by Lemma 4.5, we only need to prove

$$\max_{\overline{E_\mathbf{p}\setminus D}} U_\mathbf{p} \le \max_{\overline{E_\mathbf{p}\setminus D}} U^\star_{\sigma_\mathbf{p}} + O(m^{-2}).$$

We can decompose $\mathbf{E_p} \setminus \overline{D}$ into the union of ℓ ($\ell \le m$) disjoint simply connected open domains $B_1, B_2, \dots, B_\ell$, with diameter bounded by $2m^{-2}$ each. We will show

$$\max_{\overline{B}_k} U_\mathbf{p} \le \max_{\overline{B}_k} U^\star_{\sigma_\mathbf{p}} + O(m^{-2}) \quad \text{for every} \quad k.$$

We present the proof for $k = 1$. Assume $U_\mathbf{p}|_{\overline{B}_1}$ attends its maximum at y_1, and $U^\star_{\sigma_\mathbf{p}}|_{\overline{B}_1}$ attends its maximum at y_2. Take a smooth local potential φ of ω near B_1, i.e., $\mathrm{dd^c}\varphi = \omega$ on a neighborhood of B_1. The function $U_\mathbf{p} + \varphi$ is harmonic on B_1, whose maximum appears on the boundary, say $y_3 \in \partial B_1$. Recalling that $U_\mathbf{p} = U^\star_{\sigma_\mathbf{p}}$ on ∂B_1, yields

$$U_\mathbf{p}(y_1) + \varphi(y_1) \le U_\mathbf{p}(y_3) + \varphi(y_3) = U^\star_{\sigma_\mathbf{p}}(y_3) + \varphi(y_3) \le U^\star_{\sigma_\mathbf{p}}(y_2) + \varphi(y_3).$$

The desired inequality follows by the fact $|\varphi(y_1) - \varphi(y_3)| \le \|\varphi\|_{\mathscr{C}^1} \cdot \mathrm{diam}(B_1) = O(m^{-2})$. The proof of the proposition is complete. □

Proof of Proposition 4.1. Propositions 4.3 and 4.6 give

$$\begin{aligned}\mathcal{E}_m(\mathbf{p}) - 2\max U^\star_{\delta_\mathbf{p}} &\le \int_X U^\star_{\sigma_\mathbf{p}}\, \mathrm{d}\sigma_\mathbf{p} - 2\max U^\star_{\sigma_\mathbf{p}} + O\Big(\frac{\log m}{m}\Big) \\ &\le \int_X U^\star_{\mu_\mathbf{p}}\, \mathrm{d}\mu_\mathbf{p} - 2\max U^\star_{\mu_\mathbf{p}} + O\Big(\frac{\log m}{m}\Big) = -\mathcal{I}_D(\mu_\mathbf{p}) + O\Big(\frac{\log m}{m}\Big).\end{aligned}$$

This gives (4.1) and ends the proof of Proposition 4.1. □

5. Fekete Points of the Hole Event

For proving Theorem 1.1, we also need to investigate the minimum value of the functional $-\mathcal{E}_m(\mathbf{p}) + 2\max U^\star_{\delta_\mathbf{p}}$ defined on $(X \setminus D)^{(m)}$. To achieve this, we adapt the theory of logarithmic potentials with external fields (cf. [44, Chapter I]) on compact Riemann surfaces.

The *ω-capacity* of a closed subset $E \subset X$ is defined as

$$\operatorname{cap}_\omega(E) := \sup_{\mu \in \mathcal{M}(E)} \exp\Big[\int_X U^\star_\mu \,\mathrm{d}\mu\Big],$$

where $U^\star_\nu$ is the ω-potential of type I of ν, see (2.6). If a property holds outside a set of capacity zero, it is said to hold *quasi-everywhere*. It is known that if the ω-capacity of E is zero, then for any other smooth positive $(1,1)$-form ω' on X giving unit area, the ω'-capacity of E is also zero. Additionally, a set of capacity zero is a polar set.

Lemma 5.1. *Let μ be a probability measure on X such that $\int_X U^\star_\mu \,\mathrm{d}\mu$ is finite. If $\operatorname{cap}_\omega(E) = 0$, then $\mu(E) = 0$.*

Proof. Assume, to the contrary, that $\mu(E) > 0$. Then we can take a probability measure $\mu_E := c\mu|_E$ for some appropriate $c > 0$. Since $\int_X U^\star_\mu \,\mathrm{d}\mu > -\infty$, it follows that $\int_X U^\star_{\mu_E} \,\mathrm{d}\mu_E > -\infty$. Thus, $\operatorname{cap}_\omega(E) \geq \exp\big[\int_X U^\star_{\mu_E} \,\mathrm{d}\mu_E\big] > 0$, which is a contradiction. □

Let Q be a real continuous function on X. We define the functional

$$\mathcal{J}^Q(\mu) := -\int_X U^\star_\mu \,\mathrm{d}\mu + 2\int_X Q \,\mathrm{d}\mu, \quad \forall\, \mu \in \mathcal{M}(X \setminus D). \tag{5.1}$$

Here, Q is called an *external field*. The following theorem is analogous to [44, Theorem I.1.3].

Theorem 5.2. *The functional $\mathcal{J}^Q$ admits a unique minimizer $\nu_Q \in \mathcal{M}(X \setminus D)$ with a continuous ω-potential $U^\star_{\nu_Q}$ of type I. Moreover, one has*

(1) $-U^\star_{\nu_Q} + Q \geq \min \mathcal{J}^Q - \int_X Q \,\mathrm{d}\nu_Q$ on $X \setminus D$;
(2) $-U^\star_{\nu_Q} + Q = \min \mathcal{J}^Q - \int_X Q \,\mathrm{d}\nu_Q$ on $\operatorname{supp}(\nu_Q)$.

Proof. The existence of ν_Q follows from the continuity of Q. The uniqueness is based on the fact that $\mu \mapsto -\int_X U^\star_\mu \,\mathrm{d}\mu$ is strictly convex whenever it is finite, as seen in e.g., [58, Proposition 12]. Now, we verify the remaining two statements.

First, we show that (1) holds quasi-everywhere on $X \setminus D$. Suppose, to the contrary, that the set $E := \big\{ -U^\star_{\nu_Q} + Q < \min \mathcal{J}^Q - \int_X Q \,\mathrm{d}\nu_Q \big\} \setminus D$ has positive capacity. Then, for some small $a > 0$,

$$E_a := \Big\{ -U^\star_{\nu_Q} + Q \leq \min \mathcal{J}^Q - \int_X Q \,\mathrm{d}\nu_Q - a \Big\} \setminus D \tag{5.2}$$

also has positive capacity. Hence, there exists a probability measure σ supported on E_a such that $\int_X U^\star_\sigma \,\mathrm{d}\sigma$ is finite. On the other hand, since

$$\int_X (-U^\star_{\nu_Q} + Q) \,\mathrm{d}\nu_Q = -\int_X U^\star_{\nu_Q} \,\mathrm{d}\nu_Q + \int_X Q \,\mathrm{d}\nu_Q = \min \mathcal{J}^Q - \int_X Q \,\mathrm{d}\nu_Q,$$

and $-U^\star_{\nu_Q} + Q$ is lower semicontinuous, there is a closed ball $B \subset X \setminus D$ such that

$$-U^\star_{\nu_Q} + Q > \min \mathcal{J}^Q - \int_X Q \, \mathrm{d}\nu_Q - a/2 \quad \text{on } B, \quad \text{and} \quad \nu_Q(B) > 0. \tag{5.3}$$

Let $c := \nu_Q(B) > 0$. Consider the probability measure $\sigma_\varepsilon := \nu_Q + \varepsilon(c\sigma - \nu_Q|_B)$ on X for a small $\varepsilon > 0$. Using (2.7), we rewrite $\mathcal{J}^Q(\sigma_\varepsilon)$ as

$$\begin{aligned}
&\mathcal{J}^Q(\nu_Q) - 2\int_X U^\star_{\nu_Q} \, \mathrm{d}(\varepsilon c\sigma - \varepsilon\nu_Q|_B) + 2\int_X Q \, \mathrm{d}(\varepsilon c\sigma - \varepsilon\nu_Q|_B) + O(\varepsilon^2) \\
&= \mathcal{J}^Q(\nu_Q) + 2\int_X (-U^\star_{\nu_Q} + Q) \, \mathrm{d}(\varepsilon c\sigma) - 2\int_X (-U^\star_{\nu_Q} + Q) \, \mathrm{d}(\varepsilon\nu_Q|_B) + O(\varepsilon^2) \\
&\leq \mathcal{J}^Q(\nu_Q) - 2 \cdot a \cdot \varepsilon c + 2 \cdot a/2 \cdot \varepsilon c + O(\varepsilon^2) \quad \text{[use (5.2), (5.3)]} \\
&= \mathcal{J}^Q(\nu_Q) - ca\varepsilon + O(\varepsilon^2).
\end{aligned}$$

Thus, for ε small enough, $\mathcal{J}^Q(\sigma_\varepsilon) < \mathcal{J}^Q(\nu_Q)$, which is impossible since ν_Q is the minimizer of $\mathcal{J}^Q$.

Next, we prove (2). Since $\int_X U^\star_{\nu_Q} \, \mathrm{d}\nu_Q$ is finite, by Lemma 5.1, the set E of capacity zero must have ν_Q-measure 0. Therefore, (1) holds ν_Q-almost everywhere. Furthermore, $\int_X (-U^\star_{\nu_Q} + Q) \, \mathrm{d}\nu_Q = \min \mathcal{J}^Q - \int_X Q \, \mathrm{d}\nu_Q$ implies that (2) holds ν_Q-almost everywhere on $\mathrm{supp}(\nu_Q)$. The lower semi-continuity of $-U^\star_{\nu_Q}$ ensures that (2) holds everywhere on $\mathrm{supp}(\nu_Q)$.

Finally, to prove (1), from (2) we see that $U^\star_{\nu_Q}$ is continuous as a function on $\mathrm{supp}(\nu_Q)$. By a classical result [57, p. 54, Theorem III.2], $U^\star_{\nu_Q}$ is continuous as a function on X. Therefore, (1) holds everywhere on $X \setminus D$. □

Theorem 5.3. *One has*

$$U^\star_{\nu_Q} + \min \mathcal{J}^Q - \int_X Q \, \mathrm{d}\nu_Q = \sup_\phi \big\{\phi \textit{ is } \omega\textit{-subharmonic} : \ \phi \leq Q \textit{ on } X \setminus D\big\}^*,$$

where $*$ denotes the upper semicontinuous regularization.

Proof. By Theorem 5.2, $U^\star_{\nu_Q} + \min \mathcal{J}^Q - \int_X Q \, \mathrm{d}\nu_Q$ is an ω-subharmonic function bounded from above by Q on $X \setminus D$. This establishes the direction "$\leq$".

Denote the function on the right-hand side by V, which is also ω-subharmonic. For the probability measure $\nu := \mathrm{dd^c} V + \omega$, we have $\mathrm{dd^c}(V - U^\star_{\nu_Q}) = \nu - \nu_Q$. Thus, $V - U^\star_{\nu_Q}$ is subharmonic on $X \setminus \mathrm{supp}(\nu_Q)$. On the other hand, by Theorem 5.2, $Q = U^\star_{\nu_Q} + \min \mathcal{J}^Q - \int_X Q \, \mathrm{d}\nu_Q$ on $\mathrm{supp}(\nu_Q) \subset X \setminus D$. We see that $V \leq U^\star_{\nu_Q} + \min \mathcal{J}^Q - \int_X Q \, \mathrm{d}\nu_Q$ on $\mathrm{supp}(\nu_Q)$. By the maximum principle, $V - U^\star_{\nu_Q} \leq \min \mathcal{J}^Q - \int_X Q \, \mathrm{d}\nu_Q$ on $X \setminus \mathrm{supp}(\nu_Q)$. This completes the proof of the direction "$\geq$". □

For each $m > 1$ and $\mathbf{p} \in (X \setminus D)^{(m)}$, define (see (2.10), (2.13))

$$\mathcal{K}^Q_m(\mathbf{p}) := -\frac{m}{m-1}\mathcal{E}_m(\mathbf{p}) + 2\int_X Q \, \mathrm{d}\delta_{\mathbf{p}}. \tag{5.4}$$

Let $\mathbf{f}_m = \sum_{j=1}^m f_{m,j}$ be a minimizer of $\mathcal{K}^Q_m$. It may not be unique, but we fix one. We call $\mathscr{F}_m := \{f_{m,1}, \ldots, f_{m,m}\}$ an *m-th Fekete set* and call $f_{m,j}$ *Fekete points*. Clearly, all the points in $\mathscr{F}_m \subset X \setminus D$ are pairwise distinct.

Theorem 5.4. *One has*

$$-U^\star_{\nu_Q}(x)+Q(x)=\min\mathcal{J}^Q-\int_X Q\,\mathrm{d}\nu_Q,\quad \forall x\in\mathscr{F}_m.$$

Proof. By Theorem 5.2, it suffices to show that

$$-U^\star_{\nu_Q}(x)+Q(x)\le\min\mathcal{J}^Q-\int_X Q\,\mathrm{d}\nu_Q \tag{5.5}$$

for every x in $\mathscr{F}_m$. Without loss of generality, we may assume $x=f_{m,1}$. Define the function

$$F(z):=\mathcal{K}^Q_m(z+f_{m,2}+\cdots+f_{m,m})$$

for $z\in X\setminus D$, which attains a minimum value at $z=f_{m,1}$. Let $\mathbf{f}'_m:=f_{m,2}+\cdots+f_{m,m}$. By the definition of $\mathcal{E}_m$, the new function (see (2.6) and (2.13))

$$\widetilde{F}(z):=-\frac{m}{m-1}\frac{2}{m^2}\sum_{j\ge2}G(z,f_{m,j})+\frac{2}{m}Q(z)=\frac{2}{m}\big(-U^\star_{\delta_{\mathbf{f}'_m}}(z)+Q(z)\big)$$

must attain its minimum on $X\setminus D$ at $z=f_{m,1}$. In other words,

$$-U^\star_{\delta_{\mathbf{f}'_m}}(f_{m,1})+Q(f_{m,1})\le-U^\star_{\delta_{\mathbf{f}'_m}}(z)+Q(z),\quad\forall z\in X\setminus D. \tag{5.6}$$

Now, consider the auxiliary function $\phi(z):=U^\star_{\delta_{\mathbf{f}'_m}}(z)-U^\star_{\delta_{\mathbf{f}'_m}}(f_{m,1})+Q(f_{m,1})$ on X. Clearly, ϕ is ω-subharmonic and bounded from above by Q on $X\setminus D$. By Theorem 5.3,

$$\phi(z)\le U^\star_{\nu_Q}(z)+\min\mathcal{J}^Q-\int_X Q\,\mathrm{d}\nu_Q,\quad\forall z\in X.$$

Taking $z=f_{m,1}$, we obtain the desired inequality (5.5). $\square$

Using the theorems above, one can show that $\delta_{\mathbf{f}_m}$ converges weakly to ν_Q. Since this is not required for the current article, we leave the proof to the reader. Now, we show that any two Fekete points cannot be too close.

Proposition 5.5. *If $U^\star_{\nu_Q}$ is γ-Hölder continuous for some $0<\gamma\le1$, then there exists a constant $c_Q>0$ independent of m such that*

$$\operatorname{dist}(f_{m,j_1},f_{m,j_2})\ge c_Q m^{-1/\gamma},\quad \textit{if}\quad j_1\ne j_2.$$

Some arguments in the proof are inspired by [39, Proposition A.2], although they are technically more involved. One new difficulty is finding appropriate local holomorphic coordinate charts on X to apply the Schwarz lemma. Another new difficulty is dealing with the local weight functions of certain holomorphic line bundles. The next two lemmas will be useful for handling these difficulties.

Lemma 5.6. *Let $g:\mathbb{D}\to\mathbb{C}$ be a holomorphic function. Let x,y be two points in $\mathbb{D}$. Assume that $g(y)=0$, $|g(x)|=a$, and $\sup_U|g|<b$ for some constants $a,b>0$, where $U\subset\mathbb{D}$ is a simply connected region containing x. Then*

$$|x-y|\ge\frac{a}{a+b}\cdot\operatorname{dist}(x,\partial U).$$

Proof. Assume, to the contrary, that $|x-y|<\frac{a}{a+b}\cdot\operatorname{dist}(x,\partial U)$. Then $y\in U$, and

$$\operatorname{dist}(y,\partial U)\ge\operatorname{dist}(x,\partial U)-|x-y|\ge\frac{b}{a+b}\cdot\operatorname{dist}(x,\partial U)>\frac{b}{a}\cdot|x-y|.$$

Hence, the disc $D' := \mathbb{D}(y, \frac{b}{a} \cdot |x-y|)$ centered at y with radius $\frac{b}{a} \cdot |x-y|$ is contained in U. By the maximum principle, $\sup_{D'} |g| \le \sup_U |g| < b$. Thus, by the Schwarz lemma, noting that $g(y) = 0$,

$$|g(x)| = |g(y + (x-y))| \le \frac{|x-y|}{\frac{b}{a} \cdot |x-y|} \cdot \sup_{D'} |g| < \frac{a}{b} \cdot b = a,$$

which is a contradiction. $\square$

Lemma 5.7. *Let $\mathcal{L}$ be a holomorphic line bundle on X, and let $U \subset X$ be a simply connected region. Let $\mathfrak{h}_1, \mathfrak{h}_2$ be two smooth metrics of $\mathcal{L}$ over U giving the same curvature. Then, for any holomorphic section $s_1 \in \Gamma(U, \mathcal{L})$ of $\mathcal{L}$ on U, one can find a holomorphic section $s_2 \in \Gamma(U, \mathcal{L})$ such that*

$$\|s_2\|_{\mathfrak{h}_2} = \|s_1\|_{\mathfrak{h}_1} \quad \textit{on } U.$$

Proof. We can define the ratio $\mathfrak{h}_2/\mathfrak{h}_1$ at every point of U and check that it is a positive smooth function on U. Since the curvature of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ coincide, $\log(\mathfrak{h}_2/\mathfrak{h}_1)$ is a harmonic function on U. As U is simply connected, we can write $\log(\mathfrak{h}_2/\mathfrak{h}_1) = \mathrm{Re}(u)$ as the real part of a holomorphic function $u \in \mathscr{O}(U)$. Hence, $s_2 := e^u \cdot s_1$ suffices. $\square$

Proof of Proposition 5.5. **Step 1.** By symmetry, it suffices to show the existence of $c_Q > 0$ and $M_Q \in \mathbb{Z}_+$ such that $\mathrm{dist}(f_{m,1}, f_{m,2}) \ge c_Q m^{-1/\gamma}$ for all $m \ge M_Q$.

Let $\mathbf{f}'_m := f_{m,2} + \cdots + f_{m,m}$. For every $z \in \mathrm{supp}(\nu_Q)$, we have

$$\begin{aligned} U^\star_{\delta_{\mathbf{f}'_m}}(z) - U^\star_{\delta_{\mathbf{f}'_m}}(f_{m,1}) &\le Q(z) - Q(f_{m,1}) \quad \text{[use (5.6)]} \\ &= U^\star_{\nu_Q}(z) - U^\star_{\nu_Q}(f_{m,1}) \quad \text{[by Theorems 5.2, 5.4]}. \end{aligned}$$

Consequently, $U^\star_{\delta_{\mathbf{f}'_m}}(z) - U^\star_{\nu_Q}(z) \le U^\star_{\delta_{\mathbf{f}'_m}}(f_{m,1}) - U^\star_{\nu_Q}(f_{m,1})$ for $z \in \mathrm{supp}(\nu_Q)$. Since $U^\star_{\nu_Q}$ is harmonic outside $\mathrm{supp}(\nu_Q)$, $U^\star_{\delta_{\mathbf{f}'_m}} - U^\star_{\nu_Q}$ is subharmonic on $X \setminus \mathrm{supp}(\nu_Q)$. By the maximum principle,

$$U^\star_{\delta_{\mathbf{f}'_m}}(z) - U^\star_{\delta_{\mathbf{f}'_m}}(f_{m,1}) \le U^\star_{\nu_Q}(z) - U^\star_{\nu_Q}(f_{m,1}) \quad \text{for all } z \in X. \tag{5.7}$$

Since $U^\star_{\nu_Q}$ is γ-Hölder continuous with some constant $A := \|U^\star_{\nu_Q}\|_{\mathscr{C}^\gamma}$, if $\mathrm{dist}(z, f_{m,1}) \le (m-1)^{-1/\gamma}$, the right-hand side of (5.7) is $\le A \cdot (m-1)^{-1}$. Hence, (2.11) implies that

$$\log \left\| \mathbf{1}_{\mathscr{O}(\mathbf{f}'_m)}(z) \right\|_{\mathfrak{h}_{\mathbf{f}'_m}} - \log \left\| \mathbf{1}_{\mathscr{O}(\mathbf{f}'_m)}(f_{m,1}) \right\|_{\mathfrak{h}_{\mathbf{f}'_m}} \le A \cdot (m-1)^{-1},$$

or equivalently,

$$\frac{\prod_{j=2}^m \left\| \mathbf{1}_{\mathscr{O}(f_{m,j})} \right\|_{\mathfrak{h}_j}(z)}{\prod_{j=2}^m \left\| \mathbf{1}_{\mathscr{O}(f_{m,j})} \right\|_{\mathfrak{h}_j}(f_{m,1})} \le e^A, \qquad \forall z \in \mathbf{B}_m := \mathbb{B}\big(f_{m,1}, (m-1)^{-1/\gamma}\big), \tag{5.8}$$

where $\mathfrak{h}_j$ is the metric of $\mathscr{O}(f_{m,j})$ defined as in (2.9).

Step 2. The idea is to show that $f_{m,2}$ cannot be too close to $f_{m,1}$ by exploring (5.8). First, we make some technical preparations which will be helpful later.

For every point $y \in X$, we take a local holomorphic chart $g_y : \mathbb{D}_2 \to V_y$ with $g_y(0) = y$. Since $\{g_y(\mathbb{D})\}_{y \in X}$ covers X, by compactness, we can find finitely many points $y_1, \ldots, y_M$ such that X is covered by the union of $W_j := g_{y_j}(\mathbb{D})$ for $j = 1, \ldots, M$.

Since each $g_{y_j} : \mathbb{D}_2 \to V_{y_j}$ is a biholomorphism, by compactness, both the restrictions $g_{y_j}|_{\overline{\mathbb{D}}}$ and $g_{y_j}^{-1}|_{\overline{W_j}}$ are K_j-Lipschitz for some large $\mathsf{K}_j > 0$ with respect to the Euclidean norm on $\mathbb{D}$ and the given distance on X. Set $\mathsf{K} := \max\{\mathsf{K}_1, \ldots, \mathsf{K}_M\}$.

For each $j = 1, 2, \ldots, m$, we fix a smooth weight function ρ_j on V_{y_j} satisfying the equation $\mathrm{dd^c}\rho_j = \omega$. By compactness, we can take some large constant $\mathsf{K}' > 0$ such that all ρ_j are K'-Lipschitz on W_j for $j = 1, 2, \ldots, M$.

Step 3. By a compactness argument, there exists a uniform radius $r > 0$ such that for any $z \in X$, the small disc $\mathbb{B}(z, r)$ is contained entirely in one of $W_1, \ldots, W_M$. Take a large integer M_Q so that $(m-1)^{-1/\gamma} < r$ for all $m \geq M_Q$. Without loss of generality, we can assume that $\mathbf{B}_m \subset W_1$.

Since W_1 is open, any holomorphic line bundle on W_1 is holomorphically trivial (cf. [23, p. 229]). In particular, for each line bundle $\mathscr{O}(f_{m,j})$, $j = 2, \ldots, m$, we can fix a nowhere vanishing section $e_j \in \Gamma(W_1, \mathscr{O}(f_{m,j}))$ and use it to trivialize $\mathscr{O}(f_{m,j})|_{W_1}$ by identifying every holomorphic section $s \in \Gamma(W_1, \mathscr{O}(f_{m,j}))$ with the holomorphic function $s/e_j \in \mathscr{O}(W_1)$. In accordance with this trivialization, the metric $\mathfrak{h}_j|_{W_1}$ of $\mathscr{O}(f_{m,j})|_{W_1}$ defines a smooth weight function φ_j on W_1 by the law

$$\|s\|_{\mathfrak{h}_j}(z) = |s/e_j|(z) \cdot e^{-\varphi_j(z)}, \qquad \forall z \in W_1.$$

Since the curvature of $\mathfrak{h}_j$ is ω, we have $\mathrm{dd^c}\varphi_j = \omega$.

In this way, each canonical section $\mathbf{1}_{\mathscr{O}(f_{m,j})} \in \Gamma(W_1, \mathscr{O}(f_{m,j}))$ for $j = 2, \ldots, m$ has norm

$$\|\mathbf{1}_{\mathscr{O}(f_{m,j})}\|_{\mathfrak{h}_j}(z) = |h_j(z)| \cdot e^{-\varphi_j(z)}, \qquad \forall z \in W_1,$$

where $h_j := \mathbf{1}_{\mathscr{O}(f_{m,j})}/e_j$ is some holomorphic function. Hence, we can read (5.8) as

$$\frac{\prod_{j=2}^m |h_j(z)| \cdot \prod_{j=2}^m e^{-\varphi_j(z)}}{\prod_{j=2}^m |h_j(f_{m,1})| \cdot \prod_{j=2}^m e^{-\varphi_j(f_{m,1})}} \leq e^A, \qquad \forall z \in \mathbf{B}_m. \tag{5.9}$$

Step 4. For some technical reason to be seen later, we would like to have a uniform Lipschitz bound for each φ_j on W_1. One way to achieve this is by using some singularity decomposition result (see Lemma 2.2) of the Green kernel $G(x, y)$ near the diagonal $\{x = y\} \subset X \times X$ and by choosing appropriate e_j, φ_j.

Nevertheless, here we present an alternative method. To apply Lemma 5.7, for every $j = 2, \ldots, m$, we define another metric $\mathfrak{h}'_j$ of the line bundle $\mathscr{O}(f_{m,j})$ on W_1 by evaluating any holomorphic section $s \in \Gamma(W_1, \mathscr{O}(f_{m,j}))$ with the norm $\|s\|_{\mathfrak{h}'_j}(z) = |s/e_j|(z) \cdot e^{-\rho_1(z)}$ at each point $z \in W_1$. Since $\mathrm{dd^c}\rho_1 = \omega$, the curvature of $\mathfrak{h}'_j$ coincides with that of $\mathfrak{h}_j$ on W_1. By Lemma 5.7, we can find some holomorphic section $\mathbf{1}'_j \in \Gamma(W_1, \mathscr{O}(f_{m,j}))$ with the norm equality $\|\mathbf{1}_{\mathscr{O}(f_{m,j})}\|_{\mathfrak{h}_j} = \|\mathbf{1}'_j\|_{\mathfrak{h}'_j}$ on W_1. Equivalently, we have

$$|h_j(z)| \cdot e^{-\varphi_j(z)} = |\widehat{h}_j(z)| \cdot e^{-\rho_1(z)}, \qquad \forall z \in W_1,$$

where $\widehat{h}_j := \mathbf{1}'_j/e_j \in \mathscr{O}(W_1)$ is the representative holomorphic function of $\mathbf{1}'_j$ in the local trivialization. Thus, we can rewrite (5.9) as

$$\frac{\prod_{j=2}^m \widehat{h}_j(z)}{\prod_{j=2}^m \widehat{h}_j(f_{m,1})} \cdot e^{\sum_{j=2}^m [\rho_1(f_{m,1}) - \rho_1(z)]} \leq e^A, \qquad \forall z \in \mathbf{B}_m. \tag{5.10}$$

Note that by our preparation, the weight function ρ_1 is K'-Lipschitz on W_1, which contains $\mathbf{B}_m$. Hence, we have

$$\sum_{j=2}^m |\rho_1(f_{m,1}) - \rho_1(z)| \leq (m-1) \cdot \mathsf{K}' \cdot (m-1)^{-1/\gamma} \leq \mathsf{K}', \qquad \forall z \in \mathbf{B}_m.$$

By the above two estimates, the holomorphic function

$$H := \prod_{j=2}^{m} \frac{\widehat{h}_j}{\widehat{h}_j(f_{m,1})},$$

which is well-defined on W_1, has absolute norm $\leq e^{A+\mathsf{K}'}$ on $\mathbf{B}_m$.

Step 5. Assume, to the contrary, that $\operatorname{dist}(f_{m,1}, f_{m,2}) < \lambda \cdot (m-1)^{-1/\gamma}$ for some sufficiently small λ. It is clear that $f_{m,2} \in \mathbf{B}_m$, $H(f_{m,1}) = 1$, and $H(f_{m,2}) = 0$ because the canonical section $\mathbf{1}_{\mathscr{O}(f_{m,2})}$ vanishes at $f_{m,2}$ by definition.

Now we are in a position to apply Lemma 5.6 with the following choices:

$$g := H \circ g_{y_1}, \quad x := g_{y_1}^{-1}(f_{m,1}), \quad y := g_{y_1}^{-1}(f_{m,2}), \quad a = 1, \quad b := e^{A+\mathsf{K}'}, \quad U := g_{y_1}^{-1}(\mathbf{B}_m).$$

Since g_{y_1} is K-Lipschitz by our preparation, we have

$$(m-1)^{-1/\gamma} = \operatorname{dist}(f_{m,1}, \partial \mathbf{B}_m) = \operatorname{dist}(g_{y_1}(x), g_{y_1}(\partial U)) \leq \mathsf{K} \cdot \operatorname{dist}(x, \partial U).$$

Thus, Lemma 5.6 guarantees that

$$|x-y| \geq \frac{1}{1+e^{A+\mathsf{K}'}} \cdot \operatorname{dist}(x, \partial U) \geq \frac{1}{1+e^{A+\mathsf{K}'}} \cdot \mathsf{K}^{-1} \cdot (m-1)^{-1/\gamma}.$$

On the other hand, since $g_{y_1}^{-1}$ is also K-Lipschitz, we have

$$|x-y| = |g_{y_1}^{-1}(f_{m,1}) - g_{y_1}^{-1}(f_{m,2})| \leq \mathsf{K} \cdot \operatorname{dist}(f_{m,1}, f_{m,2}).$$

Combining the above two inequalities, we obtain

$$\operatorname{dist}(f_{m,1}, f_{m,2}) \geq \frac{1}{1+e^{A+\mathsf{K}'}} \cdot \mathsf{K}^{-2} \cdot (m-1)^{-1/\gamma}.$$

Therefore, if $\lambda < \frac{\mathsf{K}^{-2}}{1+e^{A+\mathsf{K}'}}$, we get a contradiction. Hence, $c_Q = \frac{\mathsf{K}^{-2}}{1+e^{A+\mathsf{K}'}}$ suffices. □

6. Functional Values with Fekete Points

In this section, we construct a configuration $\mathbf{p} \in (X \setminus D)^{(m)}$ that satisfies

$$-\mathcal{E}_m(\mathbf{p}) + 2 \max U_{\delta_{\mathbf{p}}}^{\star} \leq \min \mathcal{I}_D + O\Big(\frac{\log m}{m}\Big). \tag{6.1}$$

Although ν_D is the minimizer of $\mathcal{I}_D$, the functional $-\mathcal{E}_m(\mathbf{p}) + 2 \max U_{\delta_{\mathbf{p}}}^{\star}$ does not directly relate to ν_D. Instead, it serves as an approximation to $\mathcal{I}_D$, and its explicit properties are not immediately accessible. To address this difficulty, we employ the functionals $\mathcal{J}^Q$ (see (5.1)) and $\mathcal{K}_m^Q$ (see (5.4)), which were introduced earlier. The close relationship between these functionals is established by the following key result.

Proposition 6.1. *For $Q := U_{\nu_D}^{\star}$, one has $\nu_Q = \nu_D$, where ν_Q is defined as in Theorem 5.2.*

Proof. Since $\operatorname{supp}(\nu_D) \subset X \setminus D$, integrating the first inequality in Theorem 5.2 against ν_D, we get

$$-\int U_{\nu_Q}^{\star} \, \mathrm{d}\nu_D + \int U_{\nu_D}^{\star} \, \mathrm{d}\nu_D \geq \min \mathcal{J}^Q - \int U_{\nu_D}^{\star} \, \mathrm{d}\nu_Q.$$

Using (2.7), we obtain $\min \mathcal{J}^Q \leq \int U_{\nu_D}^{\star} \, \mathrm{d}\nu_D$.

On the other hand, Q itself is an ω-subharmonic function and is bounded from above by Q on $X \setminus D$. According to Theorem 5.3, it follows that

$$U^\star_{\nu_D} \le U^\star_{\nu_Q} + \min \mathcal{J}^Q - \int U^\star_{\nu_D}\, \mathrm{d}\nu_Q.$$

Integrating this inequality against ν_D, we get $\int U^\star_{\nu_D}\, \mathrm{d}\nu_D \le \min \mathcal{J}^Q$.

Therefore, $\int U^\star_{\nu_D}\, \mathrm{d}\nu_D = \min \mathcal{J}^Q$. Moreover, direct computation shows that $\mathcal{J}^Q(\nu_D) = \int U^\star_{\nu_D}\, \mathrm{d}\nu_D = \min \mathcal{J}^Q$. By the uniqueness of the minimizer of the functional $\mathcal{J}^Q$ (see Theorem 5.2), we conclude that $\nu_Q = \nu_D$. □

The functional $\mathcal{E}_m(\mathbf{p})$ defined in (2.13) can be viewed as a discrete version of $\int U^\star_\mu\, \mathrm{d}\mu$. Similarly, the functionals $-\mathcal{E}_m(\mathbf{p}) + 2\max U^\star_{\delta_{\mathbf{p}}}$ and $\mathcal{K}^Q_m(\mathbf{p})$ (see (5.4)) are discrete versions of $\mathcal{I}_D(\mu)$ and $\mathcal{J}^Q(\mu)$, respectively. The introduction of $\mathcal{J}^Q$ is advantageous because, within the framework of potential theory, such functionals have been extensively studied; see, for instance, [44]. Using Proposition 6.1, we will find a $\mathbf{p}$ satisfying (6.1) from the Fekete points of $\mathcal{K}^Q_m(\mathbf{p})$.

Hereafter, we fix the external field $Q := U^\star_{\nu_D}$. By Proposition 3.4, Q is $1/3$-Hölder continuous. As a result, Proposition 5.5 ensures that distinct Fekete points $f_{m,1}, \dots, f_{m,m}$ are separated for $m > 1$, with the distance estimate $\operatorname{dist}(f_{m,j_1}, f_{m,j_2}) \ge c_Q m^{-3}$ for $j_1 \ne j_2$ (we still use c_Q for convenience in this section, which depends on D).

For each $m > 1$, similar as in Sect. 4.2, define the set

$$E_m := \bigcup_{j=1}^{m} \mathbb{B}(f_{m,j}, c_Q m^{-5}),$$

and consider the function

$$V_m := \sup_{\phi} \big\{\phi \text{ is } \omega\text{-subharmonic} :\ \phi \le U^\star_{\delta_{\mathbf{f}_m}} \text{ on } \partial E_m \big\}^*,$$

which is a continuous ω-subharmonic function satisfying:

$$V_m = U^\star_{\delta_{\mathbf{f}_m}} \quad \text{on } X \setminus E_m \quad \text{and} \quad V_m > U^\star_{\delta_{\mathbf{f}_m}} \quad \text{on } E_m,$$
$$\mathrm{dd^c} V_m = -\omega \quad \text{on } X \setminus \partial E_m.$$

Define the probability measure

$$\sigma_m := \mathrm{dd^c} V_m + \omega,$$

supported on $\partial E_m = \cup_{j=1}^m \partial\mathbb{B}(f_{m,j}, c_Q m^{-5})$. The same proof as Lemma 4.2 gives

$$|V_m - U^\star_{\sigma_m}| = O(m^{-8}) \quad \text{as} \quad m \to \infty. \tag{6.2}$$

Here, the situation is better than Sec. 4.2, because E_m is the union of disjoint open balls. So, using Proposition 4.3 and Remark 4.4 (modify the proof a bit), we have the following result.

Proposition 6.2. *There exists a constant $C_D > 0$ independent of m such that for all $m > 1$,*

$$\Big|\mathcal{E}_m(\mathbf{f}_m) - \int_X U^\star_{\sigma_m}\, \mathrm{d}\sigma_m\Big| \le C_D \frac{\log m}{m}.$$

Lemma 6.3. *There exists a constant $C_D > 0$ such that, for all $m > 1$, one has*

$$\Big|\int U^\star_{\nu_D}\, \mathrm{d}\sigma_m - \int U^\star_{\nu_D}\, \mathrm{d}\delta_{\mathbf{f}_m}\Big| \le \frac{C_D}{m}.$$

Proof. Since $U^\star_{\nu_D}$ is $1/3$-Hölder continuous, we have

$$\Big|\int U^\star_{\nu_D}\,\mathrm{d}\sigma_m-\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m}\Big|\le\frac{1}{m}\sum_{j=1}^{m}\Big|\int U^\star_{\nu_D}\,\mathrm{d}\sigma_{m,j}-U^\star_{\nu_D}(f_{m,j})\Big|=O(m^{-5/3}),$$

which implies the Lemma. □

The auxiliary measure σ_m will be used to approximate $\delta_{\mathbf{f}_m}$, as $\int U^\star_{\delta_{\mathbf{f}_m}}\,\mathrm{d}\delta_{\mathbf{f}_m}=-\infty$, while $\int U^\star_{\sigma_m}\,\mathrm{d}\sigma_m>-\infty$. In the subsequent analysis, we will need the following *principle of domination* for probability measures.

Lemma 6.4. *Let μ_1,μ_2 be probability measures on X. If $U^\star_{\mu_1}\le U^\star_{\mu_2}$ on $\operatorname{supp}(\mu_2)$, then $\mu_1=\mu_2$.*

Proof. Observe that $U^\star_{\mu_1}-U^\star_{\mu_2}$ is subharmonic on $X\setminus\operatorname{supp}(\mu_2)$. Its maximum appears on the support of μ_2. By hypothesis, $U^\star_{\mu_1}-U^\star_{\mu_2}\le 0$ on $\operatorname{supp}(\mu_2)$, giving $U^\star_{\mu_1}-U^\star_{\mu_2}\le 0$ on X. But $\int U^\star_{\mu_1}\,\omega=\int U^\star_{\mu_2}\,\omega=0$. We conclude that $U^\star_{\mu_1}=U^\star_{\mu_2}$ a.e. on X and thus, $U^\star_{\mu_1}=U^\star_{\mu_2}$ on X. □

Lemma 6.5. *There exists a constant $C_D>0$ independent of m such that*

$$\int U^\star_{\nu_D}\,\mathrm{d}\nu_D-\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m}\ge -C_D\frac{\log m}{m}\quad\textit{for all }\ m>1.$$

Proof. By integrating (5.7) against $\nu_D=\nu_Q$, and using (2.7), we obtain

$$\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}'_m}-U^\star_{\delta_{\mathbf{f}'_m}}(f_{m,1})\le\int U^\star_{\nu_D}\,\mathrm{d}\nu_D-U^\star_{\nu_D}(f_{m,1}).$$

Hence,

$$\int U^\star_{\nu_D}\,\mathrm{d}\nu_D-\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m}\ge U^\star_{\nu_D}(f_{m,1})-U^\star_{\delta_{\mathbf{f}'_m}}(f_{m,1})+\varepsilon_{m,D},\tag{6.3}$$

where

$$\varepsilon_{m,D}:=\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}'_m}-\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m}=\frac{1}{m(m-1)}\sum_{j=2}^{m}U^\star_{\nu_D}(f_{m,j})-\frac{1}{m}U^\star_{\nu_D}(f_{m,1})$$

clearly satisfies $|\varepsilon_{m,D}|\le 2\cdot m^{-1}\cdot\max|U^\star_{\nu_D}|=O(m^{-1})$.

Take a $w\in\partial\mathbb{B}(f_{m,1},c_Qm^{-5})$. Using (2.6) and Lemma 2.2, we can bound $\big|U^\star_{\delta_{\mathbf{f}'_m}}(f_{m,1})-U^\star_{\delta_{\mathbf{f}_m}}(w)\big|$ from above by the sum of

$$\frac{1}{m}\big|\log\operatorname{dist}(w,f_{m,1})\big|+\sum_{j=2}^{m}\Big|\frac{1}{m}\log\operatorname{dist}(w,f_{m,j})-\frac{1}{m-1}\log\operatorname{dist}(f_{m,1},f_{m,j})\Big|\tag{6.4}$$

and

$$\frac{1}{m}\big|\varrho(w,f_{m,1})\big|+\sum_{j=2}^{m}\Big|\frac{1}{m}\varrho(w,f_{m,j})-\frac{1}{m-1}\varrho(f_{m,1},f_{m,j})\Big|.\tag{6.5}$$

Since ϱ is Lipschitz, (6.5) is $O(m^{-1})$. By the monotonicity of the $\log$ function,

$$\begin{aligned}&\big|\log\operatorname{dist}(f_{m,1},f_{m,j})-\log\operatorname{dist}(w,f_{m,j})\big|\le\\&\max\Big(\log\frac{\operatorname{dist}(f_{m,1},f_{m,j})}{\operatorname{dist}(f_{m,1},f_{m,j})-c_Qm^{-5}},\log\frac{\operatorname{dist}(w,f_{m,j})+c_Qm^{-5}}{\operatorname{dist}(w,f_{m,j})}\Big)=O(m^{-2}),\end{aligned}\tag{6.6}$$

because $\operatorname{dist}(f_{m,1}, f_{m,j}) \geq c_Q m^{-3}$. Hence, (6.4) is $O(\log m/m)$. Summarizing, we have

$$\left|U^\star_{\delta_{\mathbf{f}'_m}}(f_{m,1}) - U^\star_{\delta_{\mathbf{f}_m}}(w)\right| = O\Big(\frac{\log m}{m}\Big). \tag{6.7}$$

On the other hand, the $1/3$-Hölder continuity of $U^\star_{\nu_D}$ yields $\left|U^\star_{\nu_D}(f_{m,1}) - U^\star_{\nu_D}(w)\right| = O(m^{-5/3})$. Thus, we conclude that

$$U^\star_{\nu_D}(f_{m,1}) - U^\star_{\delta_{\mathbf{f}'_m}}(f_{m,1}) \geq U^\star_{\nu_D}(w) - U^\star_{\delta_{\mathbf{f}_m}}(w) - O\Big(\frac{\log m}{m}\Big)$$

by the above two estimates. Recall from (6.2) that $\left|U^\star_{\sigma_m}(w) - U^\star_{\mathbf{f}_m}(w)\right| = O(m^{-8})$. Hence, (6.3) imply that

$$\int U^\star_{\nu_D}\,\mathrm{d}\nu_D - \int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m} \geq U^\star_{\nu_D}(w) - U^\star_{\sigma_m}(w) - O\Big(\frac{\log m}{m}\Big) \quad \text{for} \quad w \in \mathbb{B}(f_{m,1}, c_Q m^{-5}). \tag{6.8}$$

By symmetry or an analogous argument, it follows that (6.8) holds for any $w \in \partial\mathbb{B}(f_{m,j}, c_Q m^{-5})$, $j = 1, \dots, m$, i.e., on the support of σ_m.

To prove the desired inequality in the lemma, it is enough to find a point $w \in \operatorname{supp}(\sigma_m)$ such that $U^\star_{\nu_D}(w) - U^\star_{\sigma_m}(w) \geq 0$. Suppose for contradiction, $U^\star_{\nu_D}(w) - U^\star_{\sigma_m}(w) < 0$ for all $w \in \operatorname{supp}(\sigma_m)$. By Lemma 6.4, it is not possible. So we finish the proof of the lemma. □

The following is the main result of this section, giving (6.1).

Proposition 6.6. *There exists a constant $C_D > 0$ such that for all $m > 1$,*

$$-\mathcal{E}_m(\mathbf{f}_m) \leq -\int U^\star_{\nu_D}\,\mathrm{d}\nu_D + C_D\frac{\log m}{m}$$

and

$$\max U^\star_{\delta_{\mathbf{f}_m}} \leq \max U^\star_{\nu_D} + C_D\frac{\log m}{m}.$$

Proof. By symmetry, the inequality (5.6) remains true after replacing $(f_{m,1}, \mathbf{f}'_m)$ by $(f_{m,k}, \sum_{j\neq k} f_{m,j})$ for every $k = 1, 2, \dots, m$. Summing up all these m inequalities, we obtain (use formula (2.6))

$$-\frac{1}{m-1}\sum_{j\neq k} G(f_{m,j}, f_{m,k}) + \sum_{j=1}^m Q(f_j) \leq -\sum_{j=1}^m G(z, f_{m,j}) + mQ(z)$$

for all $z \in X \setminus D$, where $Q = U^\star_{\nu_D}$. Multiplying both sides by $\frac{m-1}{m^2}$, we get

$$-\mathcal{E}_m(\mathbf{f}_m) + \frac{m-1}{m}\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m} \leq -\frac{m-1}{m}U^\star_{\delta_{\mathbf{f}_m}}(z) + \frac{m-1}{m}U^\star_{\nu_D}(z) \quad \text{for} \quad z \in X \setminus D. \tag{6.9}$$

By Lemma 6.4, one can find a point $w \in \operatorname{supp}(\nu_D) \subset X\setminus D$ such that $U^\star_{\delta_{\mathbf{f}_m}}(w) - U^\star_{\nu_D}(w) \leq 0$. So (6.9) implies

$$-\mathcal{E}_m(\mathbf{f}_m) \leq -\frac{m-1}{m}\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m}. \tag{6.10}$$

On the other hand, by (2.8), we have $\int (U^\star_{\sigma_m} - U^\star_{\nu_D})\,\mathrm{d}(\sigma_m - \nu_D) \leq 0$. Equivalently,

$$-\int U^\star_{\sigma_m}\,\mathrm{d}\sigma_m \geq \int U^\star_{\nu_D}\,\mathrm{d}\nu_D - 2\int U^\star_{\nu_D}\,\mathrm{d}\sigma_m.$$

Combining this with Proposition 6.2 and Lemma 6.3, we get

$$-\mathcal{E}_m(\mathbf{f}_m) + O\Big(\frac{\log m}{m}\Big) \geq \int U^\star_{\nu_D}\,\mathrm{d}\nu_D - 2\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m} - O\Big(\frac{1}{m}\Big). \tag{6.11}$$

Taking (6.10) into account, we obtain

$$\int U^\star_{\nu_D}\,\mathrm{d}\nu_D - \int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m} \le -\frac{1}{m}\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m} + C'_D\frac{\log m}{m}. \tag{6.12}$$

Since $U^\star_{\nu_D}$ is bounded, the integral $\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m}$ is finite and depending on D. We conclude the first assertion using (6.10) again.

For the second assertion, it is sufficient to show that $U^\star_{\delta_{\mathbf{f}_m}}(z) \le U^\star_{\nu_D}(z) + C_D\frac{\log m}{m}$ for all $z \in X$. Plugging (6.11) into the left-hand side of (6.9), we get for all $z \in X \setminus D$,

$$-\frac{1}{m}\int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m} + \int U^\star_{\nu_D}\,\mathrm{d}\nu_D - \int U^\star_{\nu_D}\,\mathrm{d}\delta_{\mathbf{f}_m} - O\Big(\frac{\log m}{m}\Big) \le \frac{m-1}{m}\big(U^\star_{\nu_D}(z) - U^\star_{\delta_{\mathbf{f}_m}}(z)\big).$$

By Lemma 6.5 and the boundedness of $U^\star_{\nu_D}$, we deduce that

$$U^\star_{\delta_{\mathbf{f}_m}}(z) \le U^\star_{\nu_D}(z) + C_D\frac{\log m}{m}, \quad \forall z \in X \setminus D.$$

Noting that $U^\star_{\nu_D} - U^\star_{\delta_{\mathbf{f}_m}}$ is harmonic on D, by the maximum principle, the above inequality holds for any $z \in D$ as well. This completes the proof of the proposition. □

7. Constructing Hole Event Sections by Fekete Points

We have obtained a set of Fekete points $\mathcal{F}_m := \{f_{m,1}, \dots, f_{m,m}\} \subset X \setminus D$. When $X = \mathbb{P}^1$, $\mathcal{F}_m$ can be directly realized as the zero set of a degree m homogeneous polynomial in the hole event. However, in the higher genus case, there may not exist points $q_1, \dots, q_g$ in $X \setminus D$ such that (2.5) holds for $p_j = f_{m,j}$, due to the constraints imposed by the Abel-Jacobi equation (2.5). In other words, this m-th Fekete set may not correspond to the zero set of a section in $H_{n,D}$. To address this issue, we construct a valid section s_n in the hole event $H_{n,D}$ whose zero set Z_{s_n} deviates only mildly from $\mathcal{F}_m$. The following lemma plays a crucial role in our approach.

Lemma 7.1. *Let $\mathbb{B}(y,R) \subset X$ be the disc centered at $y \in X$ with radius $R > 0$ with respect to ω_0. Then there exist constants $0 < \zeta, \beta_1, \beta_2, \beta_3 \ll 1$ and an integer $k \gg 1$ such that, for any $\mathbf{Z} \in \mathrm{Jac}(X)$ and any $m \ge 1$ points $p_1, \dots, p_m \in X$, one can find $\mathbf{P}^{[j]} = P^{[j]}_1 + \dots + P^{[j]}_g \in X^{(g)}$ ($1 \le j \le k$) and $\mathbf{Q} = Q_1 + \dots + Q_g \in X^{(g)}$ satisfying the following conditions:*

$$P^{[j]}_\ell, Q_\ell \in \mathbb{H}^{R,\zeta}_{y,p_1,\dots,p_m} := \mathbb{B}(y,R) \setminus \bigcup_{j=1}^m \mathbb{B}(p_j, \zeta/\sqrt{m}) \quad \textit{for all } \ell, j,$$

$$\sum_{j=1}^k \mathcal{A}_g(\mathbf{P}^{[j]}) + \mathcal{A}_g(\mathbf{Q}) = \mathbf{Z}, \tag{7.1}$$

$$\operatorname{dist}\big(\mathbf{Q}, \pi_g(\mathscr{R}_g) \cup \mathscr{W}\big) \ge \beta_1, \tag{7.2}$$

$$\operatorname{dist}(P^{[j_1]}_{\ell_1}, P^{[j_2]}_{\ell_2}) \ge \beta_2 \quad \textit{for all } (j_1,\ell_1) \ne (j_2,\ell_2), \tag{7.3}$$

$$\operatorname{dist}(P^{[j]}_{\ell_1}, Q_{\ell_2}) \ge \beta_3 \quad \textit{for all } j, \ell_1, \ell_2. \tag{7.4}$$

The proof is quite intricate and will be separated into seven steps according to the thinking process. For clarity, we will use $\mathsf{K}_1, \mathsf{K}_2, \mathsf{K}_3, \dots$ instead of merely $O(1)$ to denote uniform positive constants independent of **Z**, m, and $p_1, \dots, p_m$.

Proof. **Step 1.** By choosing $\zeta \ll 1$ sufficiently small, we can ensure that the union of any m closed discs $\bigcup_{j=1}^m \overline{\mathbb{B}}(p_j, \zeta/\sqrt{m})$ does not cover the open annulus $\mathbb{B}(y, R) \setminus \overline{\mathbb{B}}(y, 2R/3)$. Here, we assume the annulus is nonempty by shrinking R if necessary. This follows from an area comparison:

$$\begin{aligned}\sum_{j=1}^m \mathrm{Area}_{\omega_0}\big(\overline{\mathbb{B}}(p_j, \zeta/\sqrt{m})\big) &\le m \cdot \mathsf{K}_1 \cdot (\zeta/\sqrt{m})^2 \\ &= \mathsf{K}_1 \cdot \zeta^2 < \mathrm{Area}_{\omega_0}\big(\mathbb{B}(y, R) \setminus \overline{\mathbb{B}}(y, 2R/3)\big).\end{aligned} \tag{7.5}$$

The existence of a uniform constant $\mathsf{K}_1 > 0$ is guaranteed by the compactness of X. Thus, we can choose g points Q_ℓ in the nonempty open set

$$\big(\mathbb{B}(y, R) \setminus \overline{\mathbb{B}}(y, 2R/3)\big) \setminus \bigcup_{j=1}^m \overline{\mathbb{B}}(p_j, \zeta/\sqrt{m})$$

for $\ell = 1, \dots, g$, such that $\mathbf{Q} := Q_1 + \cdots + Q_g \notin \pi_g(\mathscr{R}_g) \cup \mathscr{W}$, since $\pi_g(\mathscr{R}_g) \cup \mathscr{W}$ is a proper subvariety of $X^{(g)}$. However, it remains to show that we can choose $\beta_1 > 0$ independent of **Z**, m, and $p_1, \dots, p_m$ such that the requirement (7.2) holds.

Step 2. To further refine this argument, we proceed as follows. First, we select g points $Q'_1, \dots, Q'_g$ in $\mathbb{B}(y, R) \setminus \overline{\mathbb{B}}(y, 2R/3)$ such that $\mathbf{Q}' := Q'_1 + \cdots + Q'_g \notin \pi_g(\mathscr{R}_g) \cup \mathscr{W}$. Next, by a continuity argument, we can choose a sufficiently small positive constant $\mathsf{K}_2 \ll 1$ such that $\overline{\mathbb{B}}(Q'_\ell, \mathsf{K}_2) \subset \mathbb{B}(y, R) \setminus \overline{\mathbb{B}}(y, 2R/3)$ for $\ell = 1, \dots, g$ and such that $Q_1 + \cdots + Q_g \notin \pi_g(\mathscr{R}_g) \cup \mathscr{W}$ whenever $Q_1, \dots, Q_g$ are mild perturbations of $Q'_1, \dots, Q'_g$ respectively within distance K_2, i.e., $\mathrm{dist}(Q_\ell, Q'_\ell) \le \mathsf{K}_2$ for $\ell = 1, \dots, g$. By compactness, for all such $Q_1, \dots, Q_g$, we have (7.2) for some uniform constant $\beta_1 > 0$. Finally, we need to ensure that $\bigcup_{j=1}^m \overline{\mathbb{B}}(p_j, \zeta/\sqrt{m})$ does not cover any of $\overline{\mathbb{B}}(Q_\ell, \mathsf{K}_2)$ for $\ell = 1, \dots, g$. This can be achieved by modifying (7.5) as

$$\mathsf{K}_1 \cdot \zeta^2 < \mathrm{Area}_{\omega_0}\big(\overline{\mathbb{B}}(Q'_\ell, \mathsf{K}_2)\big) \quad \text{for every} \quad \ell = 1, \dots, g. \tag{7.6}$$

Thus, (7.2) is obtained.

Step 3. We will choose all the remaining points $P_\bullet^{[\bullet]}$ in $\mathbb{B}(y, R/2)$ so that (7.4) automatically holds for sufficiently small $\beta_3 \ll 1$. Now the only remaining obstacles are (7.1) and (7.3).

Fix a local holomorphic coordinate chart $\varphi : \mathbb{D} \hookrightarrow \mathbb{B}(y, R/2)$ with

$$\|\mathrm{d}\varphi\| > \mathsf{K}_3 > 0 \tag{7.7}$$

with respect to ω_0 and the Euclidean metric on $\mathbb{D}$. Recall the construction (2.1) of the Jacobian variety defined by the holomorphic 1-forms $\phi_1, \dots, \phi_g$. In the local chart, read

$$\varphi^*\phi_1(z) = \omega_1(z)\,\mathrm{d}z, \quad \dots, \quad \varphi^*\phi_g(z) = \omega_g(z)\,\mathrm{d}z,$$

where z is the standard coordinate of the unit disc $\mathbb{D}$. By the standard theory of Riemann surfaces (e.g., [36, p. 87]), the equation $\det(\omega_\ell(z_j))_{1\le \ell, j\le g} = 0$ defines a proper subvariety of $\mathbb{D}^g$ with coordinates $(z_1, \dots, z_g)^{\mathsf{T}}$. Fix a point $(z'_1, \dots, z'_g)^{\mathsf{T}} \in \mathbb{D}^g$ outside this subvariety, i.e., $\det(\omega_\ell(z'_j))_{1\le \ell, j\le g} \ne 0$. Therefore, the fixed matrix $M := (\omega_\ell(z'_j))_{1\le \ell, j\le g}$ is invertible, and the g fixed points $z'_1, \dots, z'_g$ are pairwise distinct.

Step 4. Fix a small positive constant

$$\delta := \min\Big\{\frac{|z'_{\ell_1} - z'_{\ell_2}|}{3}, 1 - |z'_\ell| : 1 \leq \ell_1 < \ell_2 \leq g,\ \ell = 1, \ldots, g\Big\} > 0, \tag{7.8}$$

which is clearly independent of **Z**, m, and $p_1, \ldots, p_m$. Define a holomorphic map (see (2.2))

$$F : \mathbb{D}_\delta^g \to \mathrm{Jac}(X), \quad (z_1, \ldots, z_g)^\intercal \mapsto \mathcal{A}\big(\varphi(z'_1 + z_1)\big) + \cdots + \mathcal{A}\big(\varphi(z'_g + z_g)\big).$$

Then the differential $\mathrm{d}F$ of F at the origin $(0, \ldots, 0)^\intercal \in \mathbb{D}^g$ has the Jacobian matrix M.

Select another system $(\widetilde{\phi}_1, \ldots, \widetilde{\phi}_g)^\intercal := M^{-1} \cdot (\phi_1, \ldots, \phi_g)^\intercal$ of $\mathbb{C}$-linearly independent holomorphic 1-forms on X. The Jacobian variety $\widetilde{\mathrm{Jac}}(X)$ associated with $(\widetilde{\phi}_1, \ldots, \widetilde{\phi}_g)$ is $\widetilde{\mathrm{Jac}}(X) := \mathbb{C}^g / M^{-1} \cdot \Lambda$ by (2.1). The corresponding Abel-Jacobi map $\widetilde{\mathcal{A}} : X \to \widetilde{\mathrm{Jac}}(X)$ defined as

$$\widetilde{\mathcal{A}}(x) := \Big(\int_{p_\star}^x \widetilde{\phi}_1, \ldots, \int_{p_\star}^x \widetilde{\phi}_g\Big)^\intercal \pmod{M^{-1} \cdot \Lambda}, \quad \forall x \in X,$$

satisfies $\widetilde{\mathcal{A}} = M^{-1} \cdot \mathcal{A}$. Here, we also denote M^{-1} as the isomorphism from $\mathrm{Jac}(X) = \mathbb{C}^g/\Lambda$ to $\widetilde{\mathrm{Jac}}(X) = \mathbb{C}^g / M^{-1} \cdot \Lambda$.

Define another holomorphic map

$$\widetilde{F} : \mathbb{D}_\delta^g \to \widetilde{\mathrm{Jac}}(X), \quad (z_1, \ldots, z_g)^\intercal \mapsto \widetilde{\mathcal{A}}\big(\varphi(z'_1 + z_1)\big) + \cdots + \widetilde{\mathcal{A}}\big(\varphi(z'_g + z_g)\big). \tag{7.9}$$

It is clear that $\widetilde{F} = M^{-1} \cdot F$. Once we identify both the tangent spaces of $\mathbb{D}_\delta^g$ and $\widetilde{\mathrm{Jac}}(X)$ at any point as $\mathbb{C}^g$, the differential $\mathrm{d}\widetilde{F} : \mathbb{C}^g \to \mathbb{C}^g$ of $\widetilde{F}$ at $(0, \ldots, 0) \in \mathbb{D}_\delta^g$ is the identity map.

Step 5. We plan to choose

$$P_\ell^{[j]} := \varphi(z'_\ell + z_\ell^{[j]}) \quad \text{for some} \quad z_\ell^{[j]} \in \mathbb{D}_\delta \quad \text{for all} \quad j, \ell. \tag{7.10}$$

Let $\widetilde{G} := \widetilde{F} - \widetilde{F}(0, \ldots, 0)$ be the translation of $\widetilde{F}$ so that $\widetilde{G}$ maps the origin of $\mathbb{D}_\delta^g$ to that of $\widetilde{\mathrm{Jac}}(X)$. First, we transform the equation (7.1) equivalently to

$$\sum_{j=1}^k G(z_1^{[j]}, \ldots, z_g^{[j]}) = M^{-1} \cdot \big(\mathbf{Z} - \mathcal{A}_g(\mathbf{Q})\big) - k\widetilde{F}(0, \ldots, 0) =: [\widetilde{\mathbf{Z}}] \in \widetilde{\mathrm{Jac}}(X) \tag{7.11}$$

for some $\widetilde{\mathbf{Z}} \in \mathbb{C}^g$ in a fundamental domain of $\widetilde{\mathrm{Jac}}(X) = \mathbb{C}^g / M^{-1} \cdot \Lambda$. Note that the Euclidean norm of $\widetilde{\mathbf{Z}}$ is uniformly bounded from above by some finite constant K_4.

Step 6. The key point is that the differential $\mathrm{d}\widetilde{G} : \mathbb{C}^g \to \mathbb{C}^g$ at the origin of $\mathbb{D}_\delta^g$ is the identity map. By the inverse function theorem, there exists a small neighborhood U of the origin in $\widetilde{\mathrm{Jac}}(X)$ on which $\widetilde{G}$ is invertible. We denote the inverse by $\widetilde{G}^{-1}$. The idea is to show, by means of volume comparison, that for some large integer k and for sufficiently small $0 < \zeta \ll 1$, we can choose certain k points $\widetilde{\mathbf{P}}^{[j]} \in \mathbb{C}^g$ near the origin for $j = 1, \ldots, k$, such that $\sum_{j=1}^k \widetilde{\mathbf{P}}^{[j]} = 0$, and such that

$$(z_1^{[j]}, \ldots, z_g^{[j]}) := \widetilde{G}^{-1}\big([k^{-1} \cdot \widetilde{\mathbf{Z}} + \widetilde{\mathbf{P}}^{[j]}]\big) \tag{7.12}$$

meets the requirement (7.3).

Now, we fill in some technical details. For convenience, we regard U as an open neighborhood of the origin of $\mathbb{C}^g$. It is clear that the differential $\mathrm{d}\widetilde{G}^{-1} : \mathbb{C}^g \to \mathbb{C}^g$ at the origin is the identity map. Hence, we can fix a sufficiently small constant $\epsilon > 0$ such that for every $\mathbf{w}$ in the polydisc $\mathbb{D}^g_\epsilon \subset U$, we have

$$\widetilde{G}^{-1}(\mathbf{w}) = \mathbf{w} + O(\|\mathbf{w}\|^2), \tag{7.13}$$

where the error term $O(\|\mathbf{w}\|^2)$ has Euclidean norm $\leq \mathsf{K}_5 \cdot \|\mathbf{w}\|^2$.

Denote the finite square lattice

$$\mathcal{W}_{t,\rho} := \big\{a\rho + b\rho \cdot i \in \mathbb{C} : a, b \in \mathbb{Z} \cap [1, t]\big\}$$

for some $t \in \mathbb{Z}_+$ and $\rho > 0$ to be determined. Let $\mathfrak{I} : \mathbb{C} \to \mathbb{C}^g$ be the diagonal map, sending z to $(z, z, \ldots, z)$. We will take $k = 2t^2$, and take the aforementioned points $\widetilde{\mathbf{P}}^{[j]} \in \mathbb{C}^g$ for $j = 1, \ldots, t^2$ as mild perturbations of the t^2 points in $\mathfrak{I}(\mathcal{W}_{t,\rho})$, and for $j = 1 + t^2, \ldots, 2t^2$ as mild perturbations of the t^2 points in $-\mathfrak{I}(\mathcal{W}_{t,\rho})$. Here, perturbations mean that the differences are within $\mathbb{D}^g_{\rho/3}$. In practice, we will take

$$\widetilde{\mathbf{P}}^{[j+t^2]} = -\widetilde{\mathbf{P}}^{[j]} \quad \text{for} \quad j = 1, \ldots, t^2. \tag{7.14}$$

First of all, we need to ensure that all possible $\{k^{-1} \cdot \widetilde{\mathbf{Z}} + \widetilde{\mathbf{P}}^{[j]}\}_{j=1}^k$ chosen in this way are contained in $\mathbb{D}^g_\epsilon$. For this purpose, we require that

$$\|k^{-1} \cdot \widetilde{\mathbf{Z}}\| = \|2^{-1} \cdot t^{-2} \cdot \widetilde{\mathbf{Z}}\| \leq \epsilon/100 \quad \text{and} \quad t \cdot \rho \leq \epsilon/2. \tag{7.15}$$

Next, we verify (7.14) for each j. The only risk is that one of the g coordinates of $\widetilde{G}^{-1}(k^{-1} \cdot \widetilde{\mathbf{Z}} + \widetilde{\mathbf{P}}^{[j]})$ or $\widetilde{G}^{-1}(k^{-1} \cdot \widetilde{\mathbf{Z}} + \widetilde{\mathbf{P}}^{[j+t^2]}) = \widetilde{G}^{-1}(k^{-1} \cdot \widetilde{\mathbf{Z}} - \widetilde{\mathbf{P}}^{[j]})$ lies in the set $\mathscr{B} := \varphi^{-1}\big(\cup_{j=1}^m \mathbb{B}(p_j, \zeta/\sqrt{m})\big)$. Note that such "bad" choices of $\widetilde{\mathbf{P}}^{[j]}$ are either in

$$\bigcup_{j=0}^{g} \Big(\widetilde{G}(\mathbb{D}^j_\delta \times \mathscr{B} \times \mathbb{D}^{g-1-j}_\delta) - k^{-1} \cdot \widetilde{\mathbf{Z}}\Big) \quad \text{or in} \quad \bigcup_{j=0}^{g} \Big(k^{-1} \cdot \widetilde{\mathbf{Z}} - \widetilde{G}(\mathbb{D}^j_\delta \times \mathscr{B} \times \mathbb{D}^{g-1-j}_\delta)\Big),$$

whose total Euclidean volume is $\leq \mathsf{K}_6 \cdot \zeta^2$ (see (7.7)). Since all possible choices of $\widetilde{\mathbf{P}}^{[j]}$ have a Euclidean volume equal to that of $\mathbb{D}^g_{\rho/3}$, as long as

$$\mathsf{K}_6 \cdot \zeta^2 < \mathrm{Vol}(\mathbb{D}^g_{\rho/3}) = \mathsf{K}_7 \cdot \rho^{2g}, \tag{7.16}$$

we can select $\widetilde{\mathbf{P}}^{[j]}$ not to be "bad". Thus, (7.1) is satisfied due to (7.11), (7.12), and (7.14).

Step 7. Finally, we check the requirement (7.3). By the definition (7.12), for any $\ell = 1, \ldots, g$, and for any two distinct $j_1, j_2 \in \{1, \ldots, k\}$, using (7.13) we obtain that

$$|z_\ell^{[j_1]} - z_\ell^{[j_2]}| \geq (\rho - \rho/3 - \rho/3) - 2 \cdot \mathsf{K}_5 \cdot g\epsilon^2 = \rho/3 - 2g\mathsf{K}_5 \cdot \epsilon^2. \tag{7.17}$$

Here, $g\epsilon^2$ stands for an upper bound for both $\|k^{-1} \cdot \widetilde{\mathbf{Z}} + \widetilde{\mathbf{P}}^{[j_1]}\|^2$ and $\|k^{-1} \cdot \widetilde{\mathbf{Z}} + \widetilde{\mathbf{P}}^{[j_2]}\|^2$.

Under the restriction (7.15), we can still ensure that

$$\rho/3 - 2g\mathsf{K}_5 \cdot \epsilon^2 \geq \rho/6. \tag{7.18}$$

Indeed, (7.15) follows from

$$2^{-1} \cdot t^{-2} \cdot \mathsf{K}_4 \leq \epsilon/100, \qquad t \cdot \rho \leq \epsilon/2. \tag{7.19}$$

By possibly shrinking $\epsilon > 0$, we can check that by selecting

$$t := \big[\sqrt{50\mathsf{K}_4/\epsilon}\,\big] + 1, \qquad \rho := 12g\mathsf{K}_5\epsilon^2 > 0,$$

both the requirements (7.18) and (7.19) are met. Thus, (7.17) implies that

$$|z_\ell^{[j_1]} - z_\ell^{[j_2]}| \geq \rho/6 > 0, \quad \text{hence} \quad \mathrm{dist}(P_\ell^{[j_1]}, P_\ell^{[j_2]}) \geq \mathsf{K}_8 \cdot \rho/6 > 0.$$

On the other hand, for any $1 \leq j_1, j_2 \leq k$, and for any two distinct $\ell_1, \ell_2 \in \{1, \ldots, g\}$, by the construction (7.8) and (7.10), we have

$$\left|(z'_{\ell_1} + z^{[j_1]}_{\ell_1}) - (z'_{\ell_2} + z^{[j_2]}_{\ell_2})\right| \geq \delta, \quad \text{hence} \quad \operatorname{dist}(P^{[j_1]}_{\ell_1}, P^{[j_2]}_{\ell_2}) \geq \mathsf{K}_8 \cdot \delta > 0.$$

In summary, we can take $\beta_2 := \min\{\mathsf{K}_8 \cdot \rho/6,\ \mathsf{K}_8 \cdot \delta\}$, and demand $\zeta > 0$ to satisfy (7.6) and (7.16), ensuring that all the requirements are fulfilled. □

Using Lemma 7.1, we can construct a valid section in $H_{n,D}$, whose zeros are mostly coming from the m-th Fekete set.

Proposition 7.2. *There exists an n_0 depending on D such that for every $n \geq n_0$, one can find a holomorphic section $s_n \in H_{n,D}$ with zero set $Z_{s_n} = \{p_1, \ldots, p_m, q_1, \ldots, q_g\}$, satisfying*

$$p_j = f_{m,j} \quad \textit{for all } \ j \leq m - k_D g, \tag{7.20}$$

$$\operatorname{dist}(p_{j_1}, p_{j_2}) \geq c_D m^{-3} \quad \textit{for all } \ j_1 \neq j_2, \tag{7.21}$$

$$\operatorname{dist}(p_j, q_l) \geq c_D/\sqrt{m} \quad \textit{for all } \ j, l, \tag{7.22}$$

$$\operatorname{dist}(q_l, D) \geq c_D \quad \textit{for all } \ l, \tag{7.23}$$

$$\operatorname{dist}\left(q_1 + \cdots + q_g, \pi_g(\mathscr{R}_g) \cup \mathscr{W}\right) \geq c_D, \tag{7.24}$$

where $k_D \in \mathbb{Z}_+$ and $c_D > 0$ are independent of n.

Proof. We apply Lemma 7.1 to an open disc $\mathbb{B}(y, R) \Subset X \setminus \overline{D}$ to obtain $k_D := k \in \mathbb{N}^*$. Now, starting with the data

$$p_j := f_{m,j} \quad \text{for } j \leq m - kg, \quad \mathbf{Z} := \mathcal{A}_n(\mathcal{L}^n) - \sum_{j=1}^{m-kg} \mathcal{A}(f_{m,j}),$$

we obtain the remaining $kg + g$ points

$$p_{m-kg+1} + \cdots + p_m := \mathbf{P}^{[1]} + \cdots + \mathbf{P}^{[k]} \quad \text{and} \quad q_1 + \cdots + q_g := \mathbf{Q}.$$

These n points are the zeros of some holomorphic section s_n due to equation (7.1). The desired distance inequalities follow from the definition of $\mathbb{H}^{R,\zeta}_{y,p_1,\ldots,p_m}$ and (7.2), (7.3), (7.4). □

The zeros of s_n also provide an element in $\mathcal{S}_{m,D}$ (see (2.15)) for all sufficiently large m. The last inequality of the corollary implies that $p_1 + \cdots + p_m \notin \mathscr{H}_m$, i.e., the choice of $q_1 + \cdots + q_g$ is unique.

8. Comparison of the Two Functionals-II

In this section, we aim to approximate the minimum of the functional $\mathcal{I}_D$ by $-\mathcal{E}_m(\mathbf{p}) + 2\max U^\star_{\delta_\mathbf{p}}$, and to demonstrate that such configurations $\mathbf{p} \in X^{(m)}$ occupy a sufficiently large volume.

Proposition 8.1. *There exist constants $m_0 \in \mathbb{Z}_+$ and $C_D > 0$, both depending on D, such that for every integer $m \geq m_0$, there exists an open subset $\Omega_{m,D} \subset \mathcal{S}_{m,D} \subset X^{(m)}$ (see (2.15)) with volume $\operatorname{Vol}(\Omega_{m,D}) \geq m^{-12m}$, satisfying*

$$-\mathcal{E}_m(\mathbf{p}) + \frac{2(m+1)}{m}\mathcal{F}_m(\mathbf{p}) \leq \min \mathcal{I}_D + C_D \frac{\log m}{m}, \qquad \forall\, \mathbf{p} \in \Omega_{m,D}. \tag{8.1}$$

Take any $\mathbf{p} := p_1 + \cdots + p_m \in \mathcal{S}_{m,D}$. We require the following two lemmas, which ensure the stability of the functional values $-\mathcal{E}_m(\mathbf{p}) + 2\max U^\star_{\delta_\mathbf{p}}$ under mild perturbations of $\mathbf{p}$.

Lemma 8.2. *For any $\mathbf{p}' := p_1' + \cdots + p_m'$ satisfying $p_j' \in \mathbb{B}(p_j, m^{-5})$, the following holds:*

$$\left|\mathcal{E}_m(\mathbf{p}) - \mathcal{E}_m(\mathbf{p}')\right| \leq \frac{C}{m} \quad \textit{and} \quad \left|\max U^\star_{\delta_\mathbf{p}} - \max U^\star_{\delta_{\mathbf{p}'}}\right| \leq C\frac{\log m}{m}$$

for some constant $C > 0$ independent of $\mathbf{p}$ and m.

Proof. By (2.13) and Lemma 2.2, the difference $\left|\mathcal{E}_m(\mathbf{p}) - \mathcal{E}_m(\mathbf{p}')\right|$ is bounded above by

$$\frac{1}{m^2}\sum_{j\neq\ell}\left|\log\operatorname{dist}(p_j,p_\ell) - \log\operatorname{dist}(p_j',p_\ell')\right| + \frac{1}{m^2}\sum_{j\neq\ell}\left|\varrho(p_j,p_\ell) - \varrho(p_j',p_\ell')\right|.$$

Since ϱ is Lipschitz, the second term is $O(m^{-5})$. For the first term, using the monotonicity of the $\log$ function, each term $\left|\log\operatorname{dist}(p_j,p_\ell) - \log\operatorname{dist}(p_j',p_\ell')\right|$ is bounded by

$$\max\Big(\log\frac{\operatorname{dist}(p_j,p_\ell)}{\operatorname{dist}(p_j,p_\ell) - 2m^{-5}}, \log\frac{\operatorname{dist}(p_j,p_\ell) + 2m^{-5}}{\operatorname{dist}(p_j,p_\ell)}\Big) = O(m^{-1}),$$

because $\operatorname{dist}(p_j,p_\ell) \geq m^{-4}$ for $j \neq \ell$ from the definition of $\mathcal{S}_{m,D}$. Therefore, the summation of $m(m-1)$ terms, divided by m^2, is also $O(m^{-1})$, which establishes the first assertion.

For the second assertion, we let

$$K_j := \mathbb{B}(p_j, m^{-5}) \cup \mathbb{B}(p_j', m^{-5}) \quad \text{for each } j \quad \text{and} \quad \Omega := \cup_{j=1}^m K_j. \tag{8.2}$$

It is enough to show the following three estimates:

$$\left|U^\star_{\delta_\mathbf{p}}(w) - U^\star_{\delta_{\mathbf{p}'}}(w)\right| = O(\log m/m) \quad \text{if} \quad w \notin \Omega; \tag{8.3}$$

$$\max_{X\setminus\Omega} U^\star_{\delta_\mathbf{p}} \geq \sup_\Omega U^\star_{\delta_\mathbf{p}} - O(m^{-1}); \tag{8.4}$$

$$\max_{X\setminus\Omega} U^\star_{\delta_{\mathbf{p}'}} \geq \sup_\Omega U^\star_{\delta_{\mathbf{p}'}} - O(m^{-1}). \tag{8.5}$$

By (2.6) and Lemma 2.2, the left-hand side of (8.3) is bounded from above by

$$\frac{1}{m}\sum_{j=1}^m\left|\log\operatorname{dist}(w,p_j) - \log\operatorname{dist}(w,p_j')\right| + \frac{1}{m}\sum_{j=1}^m\left|\varrho(w,p_j) - \varrho(w,p_j')\right|.$$

Since ϱ is Lipschitz, the second term is $O(m^{-5})$. Note that for all but at most one $j = j_0$, we have $\operatorname{dist}(w,p_j) \geq \frac{1}{3}m^{-4}$, since $\operatorname{dist}(p_j,p_\ell) \geq m^{-4}$ for $j \neq \ell$. For the first term, using the monotonicity of the $\log$ function, each $|\cdots|$ for $j \neq j_0$ is bounded from above by

$$\max\Big(\log\frac{\operatorname{dist}(w,p_j)}{\operatorname{dist}(w,p_j) - m^{-5}}, \log\frac{\operatorname{dist}(w,p_j) + m^{-5}}{\operatorname{dist}(w,p_j)}\Big) = O(m^{-1}),$$

because $\operatorname{dist}(w,p_j) \geq \frac{1}{3}m^{-4}$ for $j \neq j_0$. For $j = j_0$, since $\operatorname{dist}(w,p_{j_0}) \geq m^{-5}$ and $\operatorname{dist}(w,p_{j_0}') \geq m^{-5}$, the term $\left|\log\operatorname{dist}(w,p_{j_0}) - \log\operatorname{dist}(w,p_{j_0}')\right|$ is bounded by

$$\left|\log\operatorname{dist}(w,p_{j_0})\right| + \left|\log\operatorname{dist}(w,p_{j_0}')\right| \leq 2\max\left\{|\log m^{-5}|, O(1)\right\} = O(\log m).$$

Thus, the summation of m terms, divided by m, is also $O(\log m/m)$. This proves (8.3).

To prove (8.4), observe that K_j's are disjoint simply connected domains. Thus, it is enough to show that for every j,

$$\max_{\partial K_j} U^\star_{\delta_{\mathbf{p}}} \geq \sup_{K_j} U^\star_{\delta_{\mathbf{p}}} - O(m^{-1}). \tag{8.6}$$

We give the proof for $j = 1$. If the maximum of $U^\star_{\delta_{\mathbf{p}}}|_{\overline{K}_1}$ appears on ∂K_1, (8.6) holds trivially. Now we assume $x \in K_1$ is a maximum of $U^\star_{\delta_{\mathbf{p}}}|_{\overline{K}_1}$ and $y \in \partial K_1$ is a maximum of $U^\star_{\delta_{\mathbf{p}}}|_{\partial K_1}$. Note that the diameter of K_1 is at most $3m^{-5}$. Thus, we can find a local potential φ of ω, i.e., $\mathrm{dd^c}\varphi = \omega$ on a neighborhood of K_1. Then $U^\star_{\delta_{\mathbf{p}}} + \varphi$ is subharmonic on K_1, whose maximum on $\overline{K}_1$ appears on the boundary, say $z \in \partial K_1$. We have

$$U^\star_{\delta_{\mathbf{p}}}(x) + \varphi(x) \leq U^\star_{\delta_{\mathbf{p}}}(z) + \varphi(z) \leq U^\star_{\delta_{\mathbf{p}}}(y) + \varphi(z).$$

Then (8.6) follows by the fact that $|\varphi(x) - \varphi(z)| \leq \|\varphi\| \mathrm{diam}(K_1) = O(m^{-5})$. Hence, we conclude (8.4). The proof of (8.5) is analogous. □

Lemma 8.3. *Fix $k_D \in \mathbb{N}$ and $c_D > 0$. For any $\mathbf{p}' := p'_1 + \cdots + p'_m \in X^{(m)}$ satisfying $\mathrm{dist}(p'_{j_1}, p'_{j_2}) \geq c_D m^{-3}$ for $j_1 \neq j_2$, and with $p'_j = p_j$ for $1 \leq j \leq m - k_D g$, one has*

$$\big|\mathcal{E}_m(\mathbf{p}) - \mathcal{E}_m(\mathbf{p}')\big| \leq C\frac{\log m}{m} \quad \text{and} \quad \big|\max U^\star_{\delta_{\mathbf{p}}} - \max U^\star_{\delta_{\mathbf{p}'}}\big| \leq C\frac{\log m}{m}$$

for some constant $C > 0$ independent of $\mathbf{p}$ and m.

Proof. Since $\mathbf{p}$ and $\mathbf{p}'$ differ only in the choice of $k_D g$ points, by (2.13) and Lemma 2.2, we can bound $\big|\mathcal{E}_m(\mathbf{p}) - \mathcal{E}_m(\mathbf{p}')\big|$ from above by

$$\frac{1}{m^2}\sum_{j=1}^{m}\sum_{\substack{\ell=m-k_D g+1,\\ \ell\neq j}}^{m}\Big(\big|\log\mathrm{dist}(p_j,p_\ell) - \log\mathrm{dist}(p'_j,p'_\ell)\big| + \big|\varrho(p_j,p_\ell) - \varrho(p'_j,p'_\ell)\big|\Big). \tag{8.7}$$

By the assumptions, each $\big|\log\mathrm{dist}(\cdot,\cdot)\big|$ is $O(\log m)$. Using the boundedness of ϱ, we conclude that (8.7) is $k_D O(\log m/m)$. This proves the first assertion.

For the second assertion, we define K_j's and Ω as in (8.2) and going to show (8.3), (8.4), (8.5). By (2.6), we can bound $\big|U^\star_{\delta_{\mathbf{p}}}(w) - U^\star_{\delta_{\mathbf{p}'}}(w)\big|$ by

$$\frac{1}{m}\sum_{j=m-k_D g+1}^{m}\big|\log\mathrm{dist}(w,p_j) - \log\mathrm{dist}(w,p'_j)\big| + \frac{1}{m}\sum_{j=m-k_D g+1}^{m}\big|\varrho(w,p_j) - \varrho(w,p'_j)\big|.$$

If $w \notin \Omega$, the first term is $k_D O(\log m/m)$, and the second term is $k_D O(m^{-1})$. This proves (8.3). The proof of (8.4) and (8.5) are a bit different from Lemma 8.2 because K_j may not be simply connected here. For the K_j's not simply connected, we show (8.6), replacing K_j by $\mathbb{B}(p_j, m^{-5})$ and $\mathbb{B}(p'_j, m^{-5})$ respectively. The remaining arguments are similar. □

Now we are ready to prove Proposition 8.1. The idea is to fix a set $\{f_{m,1}, \ldots, f_{m,m}\}$ of Fekete points for every large m, and then to apply Proposition 7.2 to show that certain mild perturbations $\mathbf{p} \in X^{(m)}$ of $f_{m,1} + \cdots + f_{m,m}$, which correspond to zero sets of some sections in $H_{n,D}$, constitute a "large" volume, and that their functional values $-\mathcal{E}_m(\mathbf{p}) + 2\max U^\star_{\delta_{\mathbf{p}}}$ are close to $\min \mathcal{I}_D$.

Proof of Proposition 8.1. Take an m-th Fekete set $\{f_{m,1},\dots,f_{m,m}\}$ for every $m \geq 1$. Applying Proposition 7.2 with the same notation, we obtain a holomorphic section $s_n \in H_{n,D}$ whose zero set reads in two separated parts as $\mathbf{p} := p_1 + \cdots + p_m$ and $\mathbf{q} := q_1 + \cdots + q_g$. Define

$$\Omega_{m,D} := \otimes_{j=1}^{m}\big(\mathbb{B}(p_j, m^{-5}) \setminus D\big) \subset X^{(m)}.$$

Since D has a smooth boundary and half of the unit disc has Euclidean area $\pi/2 > 1$, the volume of $\mathbb{B}(p_j, m^{-5}) \setminus D$ is greater than m^{-11} for sufficiently large m, regardless of where p_j is in the compact complement $X \setminus D$. Thus, the volume of $\Omega_{m,D}$ is greater than m^{-12m} for $m \gg 1$.

Now we check that $\Omega_{m,D} \subset \mathcal{S}_{m,D}$. Take any point $\mathbf{p}' := p_1' + \cdots + p_m' \in \Omega_{m,D}$, i.e., $p_j' \in \mathbb{B}(p_j, m^{-5})$. Let $\mathbf{q}' := q_1' + \cdots + q_g'$ be a solution to the Abel-Jacobi equation $\mathcal{A}_m(\mathbf{p}') + \mathcal{A}_g(\mathbf{q}') = \mathcal{A}_n(\mathcal{L}^n)$. We need to show that $\operatorname{dist}(p_{j_1}', p_{j_2}') \geq m^{-4}$ for all $j_1 \neq j_2$, $\operatorname{dist}(p_j', q_l') \geq 1/m$ for all j, l, and $q_l' \in X \setminus D$ for all l. The first inequality follows directly from (7.21) when $m \gg 1$. For the second and third assertions, since $\mathcal{A}_1$ is holomorphic, for a fixed Euclidean distance in $\operatorname{Jac}(X)$, we have

$$\begin{aligned}\operatorname{dist}\big(\mathcal{A}_m(\mathbf{p}), \mathcal{A}_m(\mathbf{p}')\big) &= \operatorname{dist}\Big(\sum_{j=1}^{m} \mathcal{A}_1(p_j), \sum_{j=1}^{m} \mathcal{A}_1(p_j')\Big)\\ &\leq \sum_{j=1}^{m} \operatorname{dist}\big(\mathcal{A}_1(p_j), \mathcal{A}_1(p_j')\big) \leq \sum_{j=1}^{m} O(1)\operatorname{dist}(p_j, p_j') = O(m^{-4}).\end{aligned}$$

Consequently, $\operatorname{dist}\big(\mathcal{A}_g(\mathbf{q}), \mathcal{A}_g(\mathbf{q}')\big) \leq O(m^{-4})$. Recalling (7.24) that $\operatorname{dist}\big(\mathbf{q}, \pi_g(\mathscr{R}_g) \cup \mathscr{W}\big)$ is bounded below by a constant, by the inverse function theorem and the compactness argument, $\mathcal{A}_g$ is a biholomorphism in a small ball centered at $\mathbf{q}$ with some uniform radius. For sufficiently large m, $\mathcal{A}_g(\mathbf{q}')$ is in the image of such a ball under $\mathcal{A}_g$. Thus, $\mathbf{q}'$ is unique, and we can read $\mathbf{q}'$ as $q_1' + \cdots + q_g'$ in a suitable order so that $\operatorname{dist}(q_l, q_l') \leq O(m^{-4})$. Whence $\operatorname{dist}(p_j', q_l') \geq 1/m$ by (7.22) and $q_l' \in X \setminus D$ by (7.23).

It remains to verify (8.1). Now we apply Lemma 8.3 to the two points $\mathbf{p}$ and $\mathbf{f}_m := f_{m,1} + \cdots + f_{m,m}$, which clearly meet the criterion by (7.20). Taking also Proposition 6.6 into consideration, we see that

$$-\mathcal{E}_m(\mathbf{p}) + 2\max U^{\star}_{\delta_{\mathbf{p}}} \leq \min \mathcal{I}_D + O(\log m/m).$$

Combining this with Lemma 8.2 yields

$$-\mathcal{E}_m(\mathbf{p}') + 2\max U^{\star}_{\delta_{\mathbf{p}'}} \leq \min \mathcal{I}_D + O(\log m/m), \qquad \forall\, \mathbf{p}' \in \Omega_{m,D}.$$

Thus, we can conclude the proof by Lemma 2.4. □

9. Decay of Hole Probabilities

In this section, we present the proof of Theorem 1.1 for the case where $g \geq 1$, which is more intricate than the case $g = 0$ discussed in Remark 9.2.

Proof of Theorem 1.1. By applying (2.12) and Proposition 2.3, we can express

$$\mathbf{P}_n(H_{n,D}) = \binom{n}{m}^{-1} \int_{\mathcal{R}_{m,D}} \mathscr{A}_m^*(V_n^{\mathrm{FS}})$$

$$= \binom{n}{m}^{-1} \int_{\mathbf{p}\in\mathcal{R}_{m,D}} C_n \exp\Big[m^2\Big(\mathcal{E}_m(\mathbf{p}) - \frac{2(m+1)}{m}\mathcal{F}_m(\mathbf{p})\Big)\Big]\kappa_n(\mathbf{p}).$$

Taking the logarithm yields

$$\log \mathbf{P}_n(H_{n,D}) \le \log C_n + m^2 \sup_{\mathbf{p}\in(X\setminus D)^{(m)}} \Big(\mathcal{E}_m(\mathbf{p}) - \frac{2(m+1)}{m}\mathcal{F}_m(\mathbf{p})\Big) + \log\int_{X^{(m)}} \kappa_n. \tag{9.1}$$

Lemma 2.4 implies that as $m\to\infty$, $\mathcal{F}_m(\mathbf{p}) \ge \max U^\star_{\delta_\mathbf{p}} - O(\log m/m)$. From the uniform upper bound of $\max U^\star_{\delta_\mathbf{p}}$ for all $\mathbf{p}$, we obtain

$$\sup_{\mathbf{p}\in(X\setminus D)^{(m)}} \Big(\mathcal{E}_m(\mathbf{p}) - \frac{2(m+1)}{m}\mathcal{F}_m(\mathbf{p})\Big) \le O\Big(\frac{\log m}{m}\Big) + \sup_{\mathbf{p}\in(X\setminus D)^{(m)}} \big(\mathcal{E}_m(\mathbf{p}) - 2\max U^\star_{\delta_\mathbf{p}}\big).$$

The last term is $-\min\mathcal{I}_D + O(\log m/m)$ as per Proposition 4.1.

To estimate the integral of κ_n, we use Proposition 2.3 to get

$$\log\int_{X^{(m)}}\kappa_n \le \log\big(e^{O(m)}\mathrm{diam}(X)^{2mg}\big) = O(m\log m).$$

Therefore, (9.1) gives

$$\log \mathbf{P}_n(H_{n,D}) \le \log C_n - m^2\min\mathcal{I}_D + Cm\log m \tag{9.2}$$

for some $C>0$ independent of D and m. By Lemma 9.1 below, we conclude

$$\log \mathbf{P}_n(H_{n,D}) \le -m^2\min\mathcal{I}_D + C'm\log m.$$

For the lower bound, using Proposition 8.1 to define $\Omega_{m,D}$, we have

$$\begin{aligned}\mathbf{P}_n(H_{n,D}) &\ge \binom{n}{m}^{-1}\int_{\Omega_{m,D}} \mathscr{A}_m^*(V_n^{\mathrm{FS}})\\ &= \binom{n}{m}^{-1} \int_{\mathbf{p}\in\Omega_{m,D}} C_n \exp\Big[m^2\Big(\mathcal{E}_m(\mathbf{p}) - \frac{2(m+1)}{m}\mathcal{F}_m(\mathbf{p})\Big)\Big]\kappa_n(\mathbf{p}).\end{aligned}$$

From (8.1), it follows that

$$\log \mathbf{P}_n(H_{n,D}) \ge -\log\binom{n}{m} + \log C_n - m^2\min\mathcal{I}_D - C_D m\log m + \log\int_{\mathbf{p}\in\Omega_{m,D}}\kappa_n. \tag{9.3}$$

Finally, to estimate the integral of κ_n over $\Omega_{m,D}$, we note that $\Omega_{m,D}\subset\mathcal{S}_{m,D}$, and for any point $\mathbf{p} = p_1+\cdots+p_m \in \mathcal{S}_{m,D}$ and $\mathbf{q} := \mathcal{B}_m(\mathbf{p}) = q_1+\cdots+q_g$, we have

$$\prod_{j=1}^m\prod_{l=1}^g \mathrm{dist}(p_j,q_l)^2 \ge m^{-2mg}.$$

Using $\xi(\mathbf{q}) > c$ in Proposition 2.3 and the volume estimate from Proposition 8.1, the logarithm of the integral is at least

$$-O(m) + \log\int_{\mathbf{p}\in\Omega_{m,D}} m^{-2mg}\, i\mathrm{d}z_1\wedge\mathrm{d}\overline{z}_1\wedge\cdots\wedge i\mathrm{d}z_m\wedge\mathrm{d}\overline{z}_m \ge -O(m\log m).$$

Therefore, (9.3) gives

$$\log \mathbf{P}_n(H_{n,D}) \ge \log C_n - m^2\min\mathcal{I}_D - C_D m\log m - O(m\log m). \tag{9.4}$$

Consequently, by Lemma 9.1 below, there exists a $C'_D > 0$ independent of m such that

$$\log \mathbf{P}_n(H_{n,D}) \geq -m^2 \min \mathcal{I}_D - C'_D m \log m.$$

This concludes the proof of Theorem 1.1. $\square$

Some useful estimates for C_n may be found in the literature. For the sake of completeness, we provide a proof here.

Lemma 9.1. *There exists a constant $c > 0$, independent of n and D, such that*

$$-c\, n \log n \leq \log C_n \leq c\, n \log n.$$

Proof. Since $\mathbf{P}_n(H_{n,\varnothing}) = 1$ and $\min \mathcal{I}_\varnothing = 0$, applying (9.2) to the case $D = \varnothing$ yields

$$\log C_n \geq m^2 \min \mathcal{I}_\varnothing - O(m \log m) = -O(m \log m).$$

On the other hand, applying (9.4) to the case $D = \varnothing$, we get

$$\log C_n \leq m^2 \min \mathcal{I}_\varnothing + O(m \log m) = O(m \log m).$$

completing the proof. $\square$

Remark 9.2. For the case when $g = 0$, i.e., $X = \mathbb{P}^1$, the proof follows a similar strategy but is significantly simplified. By using a density formula for the zeros (see [58, Proposition 3]), we only need to consider the functional $\mathcal{E}_m(\mathbf{p}) - \frac{2(m+1)}{m}\mathcal{F}_m(\mathbf{p})$ with $m = n$, where $\mathcal{F}_m$ is defined by $\mathcal{F}_m(\mathbf{p}) := \log \left\| e^{U^\star_{\delta_{\mathbf{p}}}} \right\|_{L^{2m}(\omega_0)}$. The absence of $\mathbf{q}$ simplifies the analysis, as there are no additional variables to account for.

10. Asymptotic Behavior of Hole Probabilities

To establish Theorem 1.2, it suffices to demonstrate the inequality for a fixed point x_0 and all sufficiently small radii r. The full theorem will then follow by a standard compactness argument.

Fix a point $x_0 \in X$. For a radius $r > 0$, we denote $\mathcal{I}_r := \mathcal{I}_{\mathbb{B}(x_0,r)}$ and $\nu_r := \nu_{\mathbb{B}(x_0,r)}$ for brevity. This section is dedicated to proving the following proposition, which, in conjunction with Theorem 1.1, implies Theorem 1.2.

Proposition 10.1. *There exist constants $C > c > 0$, independent of r, such that*

$$cr^4 \leq \min \mathcal{I}_r \leq Cr^4 \quad \textit{for every} \quad 0 < r \ll 1.$$

We will break down the proof into several lemmas.

Lemma 10.2. *For any probability measure μ supported on $X \setminus \mathbb{B}(x_0, r)$, one has $\mathcal{I}_r(\mu) \geq cr^4$ for some $c > 0$ independent of r and μ.*

Proof 1. Consider a local coordinate function $\phi : V \to \mathbb{C}$ with $\phi(x_0) = 0$, where V contains $\overline{\mathbb{B}}(x_0, r_0)$ for some small $r_0 > 0$. By compactness, there exists a sufficiently small $\epsilon_0 > 0$ such that both ω_0 and ω are greater than $\epsilon_0^2 \cdot \phi^* \mathrm{dd^c}|z|^2$ on $\overline{\mathbb{B}}(x_0, r_0)$, where z is the coordinate on $\mathbb{C}$.

For any probability measure μ supported on $X \setminus \mathbb{B}(x_0, r)$ with $r < r_0$, the ω-potential U_μ of μ satisfies the equation $\mathrm{dd^c} U_\mu = -\omega$ on $\mathbb{B}(x_0, r)$. Note that for any $r \in [0, r_0]$, the range $\phi(\overline{\mathbb{B}}(x, r))$ contains $\overline{\mathbb{D}}(0, \epsilon_1 \cdot r) \subset \mathbb{C}$ for some $\epsilon_1 = \mathsf{const} \cdot \epsilon_0$, where const denotes a positive constant independent of r and μ, whose specific value may vary depending on the context.

We can verify that

$$\widehat{U}_\mu := U_\mu \circ \phi^{-1} : \overline{\mathbb{D}}(0, \epsilon_1 \cdot r_0) \to \mathbb{R}_{\leq 0}$$

is strictly subharmonic:

$$-\mathrm{dd^c}\widehat{U}_\mu = -(\phi^{-1})^*\mathrm{dd^c}U_\mu = (\phi^{-1})^*\omega \geq (\phi^{-1})^*(\epsilon_0^2 \cdot \phi^*\mathrm{dd^c}|z|^2) = \epsilon_0^2 \cdot \mathrm{dd^c}|z|^2.$$

Using Jensen's formula (cf. e.g., [40, 43]), for any $0 < t \leq \epsilon_1 \cdot r_0$, we have

$$\widehat{U}_\mu(0) - \frac{1}{2\pi}\int_{|z|=t} \widehat{U}_\mu \,\mathrm{d}\theta = -2\int_{x=0}^{t} \frac{\mathrm{d}x}{x}\int_{\mathbb{D}_x} \mathrm{dd^c}\widehat{U}_\mu \geq 2\int_{x=0}^{t} \frac{\mathrm{d}x}{x}\int_{\mathbb{D}_x} \epsilon_0^2 \cdot \mathrm{dd^c}|z|^2.$$

Discarding the first term $\widehat{U}_\mu(0) \leq 0$, we obtain

$$-\frac{1}{2\pi}\int_{|z|=t} \widehat{U}_\mu \,\mathrm{d}\theta \geq 2\int_{x=0}^{t} \frac{\mathrm{d}x}{x}\int_{\mathbb{D}_x} \epsilon_0^2 \cdot \mathrm{dd^c}|z|^2 = \mathsf{const} \cdot \epsilon_0^2 \cdot t^2$$

for any $0 < t \leq \epsilon_1 \cdot r_0$. Thus,

$$-\int_{\mathbb{D}_r} \widehat{U}_\mu \,\mathrm{dd^c}|z|^2 = -\mathsf{const} \cdot \int_{t=0}^{r} t\,\mathrm{d}t \int_{|z|=t} \widehat{U}_\mu \,\mathrm{d}\theta \geq -\mathsf{const} \cdot \epsilon_0^2 \cdot r^4$$

for any $0 < r \leq \epsilon_1 \cdot r_0$. Since $U_\mu \leq 0$, we can bound $\mathcal{I}_r(\mu) \geq -\int_{\mathbb{B}(x,r)} U_\mu \cdot \omega$ from below by

$$-\int_{\phi(\mathbb{B}(x,r))} U_\mu \circ \phi^{-1} \cdot (\phi^{-1})^*\omega \geq -\int_{\mathbb{D}(0,\epsilon_1 \cdot r)} \widehat{U}_\mu \cdot \epsilon_0^2 \,\mathrm{dd^c}|z|^2 \geq \mathsf{const} \cdot \epsilon_0^4 \cdot r^4$$

for any $r \in (0, r_0)$. □

Proof 2. In the definition of $\mathcal{I}_r$, U_μ is non-positive. It suffices to show that $-\int_X U_\mu\,\omega \geq cr^4$. We will show that for any $\omega_{\mathbb{D}}$-subharmonic function U_r with $U_r \leq 0$ on $\mathbb{D}$ and $\mathrm{dd^c}U_r = -\omega_{\mathbb{D}}$ on $\mathbb{D}_r$, one has $-\int_{\mathbb{D}_r} U_r\,\omega_{\mathbb{D}} \geq cr^4$, where $\omega_{\mathbb{D}} := f(z)\,i\mathrm{d}z \wedge \mathrm{d}\overline{z}$ is some fixed Kähler form on $\mathbb{D}$. This clearly gives the desired result on X.

Suppose, for contradiction, there exists a sequence (U_j, r_j) with $\mathrm{dd^c}U_j = -\omega_{\mathbb{D}}$, $U_j \leq 0$, and $\lim_{j\to\infty} r_j = 0$, such that

$$\lim_{j\to\infty} r_j^{-4}\int_{\mathbb{D}(0,r_j)} U_j\,\omega_{\mathbb{D}} = 0.$$

Consider the dilation map $\tau_r : \mathbb{D} \to \mathbb{D}(0,r), z \mapsto rz$. Define

$$\Phi_j := -r_j^{-2}\tau_{r_j}^* U_j.$$

Then, on $\mathbb{D}$, for large j, we have $\Phi_j \geq 0$ and

$$\mathrm{dd^c}\Phi_j = r_j^{-2}\tau_{r_j}^*(-\mathrm{dd^c}U_j) = r_j^{-2}\tau_{r_j}^*\omega_{\mathbb{D}} = f(r_j z)\,i\mathrm{d}z \wedge \mathrm{d}\overline{z} \geq c\,i\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

for some $c > 0$ independent of j. But,

$$\frac{1}{c}\int_{\mathbb{D}} \Phi_j\,i\mathrm{d}z \wedge \mathrm{d}\overline{z} \leq \int_{\mathbb{D}} \Phi_j\,r_j^{-2}\tau_{r_j}^*\omega_{\mathbb{D}} = \int_{\mathbb{D}} r_j^{-2}(\tau_{r_j})_*(\Phi_j)\,\omega_{\mathbb{D}} = -r_j^{-4}\int_{\mathbb{D}(0,r_j)} U_j\,\omega_{\mathbb{D}}.$$

This indicates that the L^1-norm of Φ_j converges to 0. However, this is not possible because $\mathrm{dd^c}\Phi_j$ does not converge to 0. Hence, we reach a contradiction. □

To prove the upper bound, we construct a local quasi-subharmonic function first.

Lemma 10.3. *Let $\omega_{\mathbb{D}}$ be a Kähler form on $\mathbb{D}$. There exist an integer $\ell > 2$ and a constant $\alpha > 0$, such that for every small $0 < r \ll 1$, there exists a continuous $\omega_{\mathbb{D}}$-subharmonic function Φ_r on $\mathbb{D}$, such that $\Phi_r = 0$ on $\mathbb{D} \setminus \mathbb{D}_{\ell r}$, $-\alpha r^2 \leq \Phi_r \leq 0$ on $\mathbb{D}_{\ell r}$, and $\mathrm{dd^c}\Phi_r = -\omega_{\mathbb{D}}$ on $\mathbb{D}_r$.*

Proof. Take two positive constants $\beta_1, \beta_2 > 0$ such that

$$\beta_1 \, \mathrm{dd^c}|z|^2 \leq \omega_{\mathbb{D}} \leq \beta_2 \, \mathrm{dd^c}|z|^2$$

on $\mathbb{D}_{1/2}$. Let ℓ be a large positive integer to be determined. Define an auxiliary function

$$\Phi_r(z) := -\beta_1(|z| - \ell r)^2$$

on $\mathbb{D}_{\ell r} \setminus \mathbb{D}_r$, where $r \in (0, 1/(2\ell)]$. Direct computation shows that

$$\mathrm{dd^c}\Phi_r = -\beta_1 \, \mathrm{dd^c}|z|^2 + 2\beta_1 \ell r \, \mathrm{dd^c}|z| \geq -\beta_1 \, \mathrm{dd^c}|z|^2 \geq -\omega_{\mathbb{D}}$$

on $\mathbb{D}_{\ell r} \setminus \mathbb{D}_r \subset \mathbb{D}_{1/2}$. Next, we trivially extend $\Phi_r = 0$ outside $\mathbb{D}_{\ell r}$, and we extend Φ_r inside $\mathbb{D}_r$ as the unique solution to the elliptic equation $\mathrm{dd^c}\Phi_r = -\omega_{\mathbb{D}}$ with the boundary condition $\Phi_r|_{\partial \mathbb{D}_r} = -\beta_1(\ell - 1)^2 r^2$.

Now we show that $\Phi_r \leq 0$ on $\mathbb{D}_r$. To do this, we introduce a comparison function Ψ_r which solves the equation $\mathrm{dd^c}\Psi_r = -\beta_2 \, \mathrm{dd^c}|z|^2$ with the same boundary condition $\Psi_r|_{\partial \mathbb{D}_r} = -\beta_1(\ell - 1)^2 r^2$. Noting that

$$\mathrm{dd^c}(\Phi_r - \Psi_r) = -\omega_{\mathbb{D}} + \beta_2 \, \mathrm{dd^c}|z|^2 \geq 0$$

with the vanishing boundary values $(\Phi_r - \Psi_r)|_{\partial \mathbb{D}_r} = 0$, by the maximum principle, $\Phi_r - \Psi_r \leq 0$ on $\mathbb{D}_r$. Thus, we only need to check that $\Psi_r \leq 0$. Indeed, it is easy to see that

$$\Psi(z) = -\beta_1(\ell - 1)^2 r^2 + \beta_2(r^2 - |z|^2) \leq (-\beta_1(\ell - 1)^2 + \beta_2) \cdot r^2 \leq 0$$

for $z \in \mathbb{D}_r$ as long as $\ell \geq \sqrt{\beta_2/\beta_1} + 1$.

It is clear that Φ_r is a continuous $\omega_{\mathbb{D}}$-subharmonic function and satisfies all the desired properties with $\alpha = \beta_1(\ell - 1)^2$. □

Lemma 10.4. *For every small $0 < r \ll 1$, there exists a probability measure μ supported outside $\mathbb{B}(x_0, r)$ such that $\mathcal{I}_r(\mu) \leq C r^4$ for some $C > 0$ independent of r.*

Proof. Using Lemma 10.3, we can find an ω-subharmonic function Φ_r on X such that $\Phi_r = 0$ on $X \setminus \mathbb{B}(x_0, \ell r)$, $-\alpha r^2 \leq \Phi_r \leq 0$ on $\mathbb{B}(x_0, \ell r)$, and $\mathrm{dd^c}\Phi_r = -\omega$ on $\mathbb{B}(x_0, r)$, where ℓ and α are independent of r. Let $\mu := \mathrm{dd^c}\Phi_r + \omega$. Clearly, $\mathrm{supp}(\mu) \subset X \setminus \mathbb{B}(x_0, r)$. Observe that Φ_r is the ω-potential of type M of μ. Furthermore, since $\mu = \omega$ outside $\overline{\mathbb{B}}(x_0, \ell r)$, the mass of μ on $\overline{\mathbb{B}}(x_0, \ell r)$ is equal to $\omega(\overline{\mathbb{B}}(x_0, \ell r))$. Hence,

$$\mathcal{I}_r(\mu) = -\int_{\overline{\mathbb{B}}(x_0, \ell r)} \Phi_r \, \mathrm{d}\mu - \int_{\overline{\mathbb{B}}(x_0, \ell r)} \Phi_r \, \omega \leq \alpha r^2 \cdot \int_{\overline{\mathbb{B}}(x_0, \ell r)} (\mathrm{d}\mu + \omega) = O(r^4).$$

This completes the proof. □

11. Open Problems

We emphasize that our approach for Theorem 1.1 is not directly applicable in higher dimensions, as it relies heavily on the Abel-Jacobi theory. Additionally, from a physical perspective, conformal field theory (CFT) exhibits particularly rich structures in real dimension 2, which do not extend to higher dimensions. Our approach implicitly utilizes some of these CFT structures in real dimension 2 (cf. [2]). Consequently, the study of hole probabilities in higher dimensions remains a challenging and open research direction.

Even on Riemann surfaces, many questions remain unresolved. For instance, in the Ginibre point process [28], which describes the distribution of the eigenvalues of an $n \times n$ random matrix with i.i.d. complex Gaussian entries, the probability $\mathbf{P}_{n,r}$ of observing a hole in the disk centered at 0 with radius $r < 1$ admits the following asymptotic expansion:

$$\mathbf{P}_{n,r} = \exp\left[C_1 n^2 + C_2 n \log n + C_3 n + C_4 \sqrt{n} + o(\sqrt{n})\right], \qquad \text{as} \quad n \to +\infty, \tag{11.1}$$

where $C_1 = -\frac{r^4}{4}$, $C_2 = -\frac{r^2}{2}$, and C_3, C_4 are explicit constants depending on r (cf. [1, 22]).

Comparing the hole probabilities for zeros of random holomorphic sections (see (1.2) and (1.5)) with those of the Ginibre point process, we are naturally led to the following questions:

Question 11.1. Does a similar expansion formula as (11.1) exist for the hole probabilities $\mathbf{P}_n(H_{n,D})$ in Theorem 1.1?

Question 11.2. If such a formula exists, can the coefficient C_2' in the expansion be characterized?

However, even in the simplest case where $(X, \mathcal{L}, \omega) = (\mathbb{P}^1, \mathscr{O}(1), \omega_{\mathrm{FS}})$, one encounters significant computational challenges (cf. [60]), which may require techniques from analytic number theory for resolution.

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References

[1] G. Akemann, M. J. Phillips, and L. Shifrin. Gap probabilities in non-Hermitian random matrix theory. *J. Math. Phys.*, 50(6):063504, 32, 2009.

[2] Luis Alvarez-Gaumé, Jean-Benoît Bost, Gregory Moore, Philip Nelson, and Cumrun Vafa. Bosonization on higher genus Riemann surfaces. *Comm. Math. Phys.*, 112(3):503–552, 1987.

[3] Turgay Bayraktar. Equidistribution of zeros of random holomorphic sections. *Indiana Univ. Math. J.*, 65(5):1759–1793, 2016.

[4] Turgay Bayraktar, Dan Coman, Hendrik Herrmann, and George Marinescu. A survey on zeros of random holomorphic sections. *Dolomites Res. Notes Approx.*, 11:1–19, 2018.

[5] Turgay Bayraktar, Dan Coman, and George Marinescu. Universality results for zeros of random holomorphic sections. *Trans. Amer. Math. Soc.*, 373(6):3765–3791, 2020.

[6] Turgay Bayraktar, Dan Coman, George Marinescu, and Viêt-Anh Nguyên. Zeros of random holomorphic sections of big line bundles with continuous metrics. `arXiv:2404.08116`, 2024.

[7] Pavel Bleher and Xiaojun Di. Correlations between zeros of a random polynomial. *J. Statist. Phys.*, 88(1-2):269–305, 1997.

[8] Pavel Bleher, Bernard Shiffman, and Steve Zelditch. Universality and scaling of correlations between zeros on complex manifolds. *Invent. Math.*, 142(2):351–395, 2000.

[9] André Bloch and George Polyá. On the roots of certain algebraic equations. *Proc. London Math. Soc.*, 33:102–114 (1932), 1932.

[10] Thomas Bloom. Orthogonal polynomials in $\mathbb{C}^n$. *Indiana Univ. Math. J.*, 46(2):427–452, 1997.

[11] Thomas Bloom and Duncan Dauvergne. Asymptotic zero distribution of random orthogonal polynomials. *Ann. Probab.*, 47(5):3202–3230, 2019.

[12] Thomas Bloom and Norman Levenberg. Random polynomials and pluripotential-theoretic extremal functions. *Potential Anal.*, 42(2):311–334, 2015.

[13] Thomas Bloom and Bernard Shiffman. Zeros of random polynomials on $\mathbb{C}^m$. *Math. Res. Lett.*, 14(3):469–479, 2007.

[14] Jeremiah Buckley, Alon Nishry, Ron Peled, and Mikhail Sodin. Hole probability for zeroes of Gaussian Taylor series with finite radii of convergence. *Probab. Theory Related Fields*, 171(1-2):377–430, 2018.

[15] Dan Coman, George Marinescu, and Viêt-Anh Nguyên. Holomorphic sections of line bundles vanishing along subvarieties. *Indiana Univ. Math. J.*, 73(2):493–536, 2024.

[16] Dan Coman, George Marinescu, and Viêt-Anh Nguyên. Restricted spaces of holomorphic sections vanishing along subvarieties. *Math. Z.*, 310(1):Paper No. 9, 2025.

[17] Tien-Cuong Dinh, Subhroshekhar Ghosh, and Hao Wu. Hole event for random holomorphic sections on compact riemann surfaces. `arXiv:2402.11672`, 2024.

[18] Tien-Cuong Dinh, Xiaonan Ma, and George Marinescu. Equidistribution and convergence speed for zeros of holomorphic sections of singular Hermitian line bundles. *J. Funct. Anal.*, 271(11):3082–3110, 2016.

[19] Tien-Cuong Dinh, Xiaonan Ma, and Viêt-Anh Nguyên. Equidistribution speed for Fekete points associated with an ample line bundle. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(3):545–578, 2017.

[20] Tien-Cuong Dinh and Nessim Sibony. Distribution des valeurs de transformations méromorphes et applications. *Comment. Math. Helv.*, 81(1):221–258, 2006.

[21] Alexander Drewitz, Bingxiao Liu, and George Marinescu. Large deviations for zeros of holomorphic sections on punctured riemann surfaces. `arXiv:2109.09156`*, to appear in Michigan Math. J.*, 2021.

[22] P. J. Forrester. Some statistical properties of the eigenvalues of complex random matrices. *Phys. Lett. A*, 169(1-2):21–24, 1992.

[23] Otto Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1981. Translated from the German by Bruce Gilligan.

[24] John B. Garnett and Donald E. Marshall. *Harmonic measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2005.

[25] Subhroshekhar Ghosh and Alon Nishry. Point processes, hole events, and large deviations: random complex zeros and Coulomb gases. *Constr. Approx.*, 48(1):101–136, 2018.

[26] Subhroshekhar Ghosh and Alon Nishry. Gaussian complex zeros on the hole event: the emergence of a forbidden region. *Comm. Pure Appl. Math.*, 72(1):3–62, 2019.

[27] Subhroshekhar Ghosh and Ofer Zeitouni. Large deviations for zeros of random polynomials with i.i.d. exponential coefficients. *Int. Math. Res. Not. IMRN*, 2016(5):1308–1347, 2016.

[28] Jean Ginibre. Statistical ensembles of complex, quaternion, and real matrices. *J. Mathematical Phys.*, 6:440–449, 1965.

[29] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[30] Robert C. Gunning. *Lectures on Riemann surfaces, Jacobi varieties*. Mathematical Notes, No. 12. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1972.
[31] Manjunath Krishnapur. Overcrowding estimates for zeroes of planar and hyperbolic Gaussian analytic functions. *J. Stat. Phys.*, 124(6):1399–1423, 2006.
[32] Andriy O. Kuryliak and Oleh B. Skaskiv. Analytic Gaussian functions in the unit disc: probability of zeros absence. *Mat. Stud.*, 59(1):29–45, 2023.
[33] Serge Lang. *Introduction to Arakelov theory*. Springer-Verlag, New York, 1988.
[34] George Marinescu and Duc-Viet Vu. Bergman kernel functions associated to measures supported on totally real submanifolds. *J. Reine Angew. Math.*, 810:217–251, 2024.
[35] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.
[36] Raghavan Narasimhan. *Compact Riemann surfaces*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1992.
[37] Alon Nishry. Asymptotics of the hole probability for zeros of random entire functions. *Int. Math. Res. Not. IMRN*, 2010(15):2925–2946, 2010.
[38] Alon Nishry. Hole probability for entire functions represented by Gaussian Taylor series. *J. Anal. Math.*, 118(2):493–507, 2012.
[39] Alon Nishry and Aron Wennman. The forbidden region for random zeros: appearance of quadrature domains. *Comm. Pure Appl. Math.*, 77(3):1766–1849, 2024.
[40] Junjiro Noguchi and Jörg Winkelmann. *Nevanlinna theory in several complex variables and Diophantine approximation*, volume 350 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Tokyo, 2014.
[41] S. Nonnenmacher and A. Voros. Chaotic eigenfunctions in phase space. *J. Statist. Phys.*, 92(3-4):431–518, 1998.
[42] Yuval Peres and Bálint Virág. Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process. *Acta Math.*, 194(1):1–35, 2005.
[43] Min Ru. *Nevanlinna theory and its relation to Diophantine approximation*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, [2021] ©2021.
[44] Edward B. Saff and Vilmos Totik. *Logarithmic potentials with external fields*, volume 316 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1997. Appendix B by Thomas Bloom.
[45] Guokuan Shao. Equidistribution of zeros of random holomorphic sections for moderate measures. *Math. Z.*, 283(3-4):791–806, 2016.
[46] Larry A. Shepp and Robert J. Vanderbei. The complex zeros of random polynomials. *Trans. Amer. Math. Soc.*, 347(11):4365–4384, 1995.
[47] Bernard Shiffman. Asymptotic expansion of the variance of random zeros on complex manifolds. *J. Geom. Anal.*, 31(8):8607–8631, 2021.
[48] Bernard Shiffman and Steve Zelditch. Distribution of zeros of random and quantum chaotic sections of positive line bundles. *Comm. Math. Phys.*, 200(3):661–683, 1999.
[49] Bernard Shiffman and Steve Zelditch. Equilibrium distribution of zeros of random polynomials. *Int. Math. Res. Not.*, 2003(1):25–49, 2003.
[50] Bernard Shiffman and Steve Zelditch. Number variance of random zeros on complex manifolds. *Geom. Funct. Anal.*, 18(4):1422–1475, 2008.
[51] Bernard Shiffman and Steve Zelditch. Number variance of random zeros on complex manifolds, II: smooth statistics. *Pure Appl. Math. Q.*, 6(4):1145–1167, 2010.
[52] Bernard Shiffman, Steve Zelditch, and Qi Zhong. Random zeros of complex manifolds: conditional expectations. *J. Inst. Math. Jussieu*, 10(3):753–783, 2011.
[53] Bernard Shiffman, Steve Zelditch, and Scott Zrebiec. Overcrowding and hole probabilities for random zeros on complex manifolds. *Indiana Univ. Math. J.*, 57(5):1977–1997, 2008.
[54] Yum-Tong Siu. Some recent transcendental techniques in algebraic and complex geometry. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 439–448. Higher Ed. Press, Beijing, 2002.

[55] Mikhail Sodin and Boris Tsirelson. Random complex zeroes. III. Decay of the hole probability. *Israel J. Math.*, 147:371–379, 2005.
[56] Isaac Todhunter. *A history of the mathematical theory of probability*. Stechert, New York, 1931.
[57] Masatsugu Tsuji. *Potential theory in modern function theory*. Chelsea Publishing Co., New York, 1975. Reprinting of the 1959 original.
[58] Ofer Zeitouni and Steve Zelditch. Large deviations of empirical measures of zeros of random polynomials. *Int. Math. Res. Not. IMRN*, 2010(20):3935–3992, 2010.
[59] Steve Zelditch. Large deviations of empirical measures of zeros on Riemann surfaces. *Int. Math. Res. Not. IMRN*, 2013(3):592–664, 2013.
[60] Junyan Zhu. Hole probabilities of $\mathrm{SU}(m+1)$ Gaussian random polynomials. *Anal. PDE*, 7(8):1923–1968, 2014.
[61] Scott Zrebiec. The zeros of flat Gaussian random holomorphic functions on $\mathbb{C}^n$, and hole probability. *Michigan Math. J.*, 55(2):269–284, 2007.

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