Received: 8 June 2024 | Revised: 3 April 2025 | Accepted: 30 April 2025

DOI: 10.1112/jlms.70177

RESEARCH ARTICLE

Journal of the London Mathematical Society

Entire curves generating all shapes of Nevanlinna currents

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Funding information
National Key R&D Program of China, Grant/Award Numbers: 2021YFA1003100, 2023YFA1010500; National Natural Science Foundation of China, Grant/Award Numbers: 12288201, 12471081

Abstract

First, we show that every complex torus $\mathbb{T}$ contains some entire curve $g : \mathbb{C} \to \mathbb{T}$ such that the concentric holomorphic discs $\{g\,|_{\overline{\mathbb{D}}_r}\}_{r>0}$ can generate all the Nevanlinna/Ahlfors currents on $\mathbb{T}$ at cohomological level. This confirms an anticipation of Sibony. Developing further our new method, we can construct some twisted entire curve $f : \mathbb{C} \to \mathbb{CP}^1 \times E$ in the product of the rational curve $\mathbb{CP}^1$ and an elliptic curve E, such that, concerning Siu's decomposition, demanding any cardinality $|J| \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ and that $\mathcal{T}_{\text{diff}}$ is trivial ($|J| \geqslant 1$) or not ($|J| \geqslant 0$), we can always find a sequence of concentric holomorphic discs $\{f\,|_{\overline{\mathbb{D}}_{r_j}}\}_{j\geqslant 1}$ to generate a Nevanlinna/Ahlfors current $\mathcal{T} = \mathcal{T}_{\text{alg}} + \mathcal{T}_{\text{diff}}$ with the singular part $\mathcal{T}_{\text{alg}} = \sum_{j\in J} \lambda_j \cdot [\mathsf{C}_j]$ in the desired shape. This fulfills the missing case where $|J| = 0$ in the previous work of Huynh–Xie. By a result of Duval, each C_j must be rational or elliptic. We will show that there is no *a priori* restriction on the numbers of rational and elliptic components in the support of $\mathcal{T}_{\text{alg}}$, thus answering a question of Yau and Zhou. Moreover, we will show that the positive coefficients $\{\lambda_j\}_{j\in J}$ can be arbitrary as long as the total mass of $\mathcal{T}_{\text{alg}}$ is less than or equal to 1.

MSC 2020
32A22, 32C30, 32Q56

1 | INTRODUCTION

One central problem in complex geometry is the famous Green–Griffiths conjecture [12], which stipulates that for every complex projective variety X of general type, all entire curves $f : \mathbb{C} \to X$ shall be factored through certain proper algebraic subvariety $Y \subsetneq X$. Some interest also comes from the analogy between the value distribution of entire curves in Nevanlinna theory and the locations of rational points in Diophantine geometry (cf. [18, 25, 26]), best illustrated as "geometry governs arithmetic".

In his celebrated solution [20] to the Green–Griffiths conjecture for complex projective surfaces satisfying a Chern number inequality $c_1^2 > c_2$, McQuillan introduced the so called *Nevanlinna currents* to capture asymptotic behaviors of entire curves. Later, simplified analogues called *Ahlfors currents* were employed by Duval [7] to obtain a deep, quantitative Brody Lemma [1]. One important corollary is a characterization of complex hyperbolicity in terms of linear isoperimetric inequality for holomorphic discs, which was anticipated by Gromov [14, p. 152 (c)]. In value distribution theory and complex dynamics, Nevanlinna/Ahlfors currents also found important applications, see for instance [4, 5, 10, 16].

Now, let us recall some definitions. Given a compact complex manifold X with a Hermitian form ω. Let $\{F_j : \overline{\mathbb{D}}(z_j, R_j) \to X\}_{j \geqslant 1}$ be a sequence of nonconstant holomorphic discs, smooth up to the boundary, where each $\overline{\mathbb{D}}(z_j, R_j) \subset \mathbb{C}$ is the closed disc centered at z_j with radius R_j. We can associate each F_j with a positive current $\widehat{\mathcal{N}}_{F_j}$ of bidimension (1,1), which evaluates every smooth (1,1)-form ϕ on X by a Jensen-type formula

$$\langle \widehat{\mathcal{N}}_{F_j}, \phi \rangle = \int_0^{R_j} \frac{dt}{t} \int_{\mathbb{D}(z_j, t)} F_j^* \phi.$$

Write $\mathfrak{A}_{F_j}(z_j, R_j) := \langle \widehat{\mathcal{N}}_{F_j}, \omega \rangle$. Consider the sequence of normalized currents

$$\left\{ \mathcal{N}_{F_j} := \frac{1}{\mathfrak{A}_{F_j}(z_j, R_j)} \cdot \widehat{\mathcal{N}}_{F_j} \right\}_{j \geqslant 1}$$

of mass 1. By Banach–Alaoglu's theorem, certain subsequence converges in weak topology to some positive current $\mathcal{N}$. If *a priori* $\{F_j\}_{j \geqslant 1}$ satisfies the *length–area condition*

$$\lim_{j \to \infty} \frac{\mathfrak{L}_{F_j}(z_j, R_j)}{\mathfrak{A}_{F_j}(z_j, R_j)} = 0, \tag{1.1}$$

where

$$\mathfrak{L}_{F_j}(z_j, R_j) := \int_0^{R_j} \mathrm{Length}_\omega \left(F_j(\partial \mathbb{D}(z_j, t)) \right) \frac{dt}{t},$$

then one can check that $\mathcal{N}$ is in fact closed (cf. [21]), and it is called a Nevanlinna current. When $z = 0$, we abbreviate $\mathfrak{L}_F(z, r), \mathfrak{A}_F(z, r)$ as $\mathfrak{L}_F(r), \mathfrak{A}_F(r)$.

In particular, given an entire curve $f : \mathbb{C} \to X$, Ahlfors Lemma (cf. [15, p. 55]) ensures that we can always select some increasing radii $\{r_j\}_{j \geqslant 1}$ tending to infinity such that $\{f|_{\overline{\mathbb{D}}_{r_j}}\}_{j \geqslant 1}$ enjoys (1.1), where $\overline{\mathbb{D}}_{r_j} := \{z \in \mathbb{C} : |z| \leqslant r_j\}$. Thus, we receive some Nevanlinna current associated with f.

The definition of Ahlfors currents is likewise and simpler. Again we start with a sequence $\{F_j : D_j \to X\}_{j\geqslant 1}$ of nonconstant holomorphic discs, smooth up to the boundary, satisfying another length–area condition that

$$\lim_{j\to\infty} \frac{\mathrm{Length}_\omega(F_j(\partial D_j))}{\mathrm{Area}_\omega(F_j(D_j))} = 0. \tag{1.2}$$

From the sequence of normalized currents

$$\left\{ \mathcal{A}_j := \frac{1}{\mathrm{Area}_\omega(F_j(D_j))} \cdot (F_j)_*[D_j] \right\}_{j\geqslant 1}$$

of mass 1, by compactness and diagonal argument, after passing to some subsequence, in the limit we receive some positive current of bidimension $(1, 1)$, which is actually closed by (1.2).

Specifically, given an entire curve $f : \mathbb{C} \to X$, by Ahlfors Lemma (cf. [9, p. 7]), one can find some increasing radii $\{r_j\}_{j\geqslant 1}$ tending to infinity such that $\{f|_{\overline{\mathbb{D}}_{r_j}}\}_{j\geqslant 1}$ satisfies the length–area condition (1.2). Thus, f produces some Ahlfors current.

One advantage of Nevanlinna currents over Ahlfors currents is that, when $f : \mathbb{C} \to X$ is algebraically nondegenerate, its associated Nevanlinna currents are always in the nef class (cf. [21]).

Nevanlinna/Ahlfors curents associated with an entire curve $f : \mathbb{C} \to X$ can be regarded as certain noncompact Poincaré dualities of $f(\mathbb{C})$. Hence, several classical results in value distribution theory can be reformulated in terms of intersections of the corresponding cohomology classes. For instance, the First Main Theorem of Nevanlinna theory can be expressed as an inequality between the algebraic intersection and the geometric intersection (cf. [10]). Therefore, it would be natural and fundamental to ask

Question 1.1. Are all Nevanlinna/Ahlfors currents associated with the same entire curve cohomologically equivalent?

Meanwhile, in a simplied proof [2] of McQuillan's result [20], a key trick of Brunella is to use Siu's decomposition theorem [23] to write every positive closed Nevanlinna current $\mathcal{N}$ as $\mathcal{N}_{\mathrm{alg}} + \mathcal{N}_{\mathrm{diff}}$, where the singular part $\mathcal{N}_{\mathrm{alg}} = \sum_{j\in J} \lambda_j \cdot [\mathsf{C}_j]$ is some positive linear combination ($\lambda_j > 0$; $J \subset \mathbb{Z}_+$, could be empty) of currents of integration on distinct irreducible algebraic curves C_j, and where the diffuse part $\mathcal{N}_{\mathrm{diff}}$ is a positive closed $(1, 1)$-current having zero Lelong number along any algebraic curve. The idea was to handle the two parts $\mathcal{N}_{\mathrm{alg}}, \mathcal{N}_{\mathrm{diff}}$ separately by distinct techniques from algebraic geometry and complex dynamics, respectively. To see the necessity, he raised

Conjecture 1.2 [2], p. 200. *Some entire curve shall produce a Nevanlinna current $\mathcal{N} = \mathcal{N}_{\mathrm{alg}} + \mathcal{N}_{\mathrm{diff}}$ with both nontrivial singular part $\mathcal{N}_{\mathrm{alg}} \neq 0$ and nontrivial diffuse part $\mathcal{N}_{\mathrm{diff}} \neq 0$.*

Question 1.1 (answer: no) and Conjecture 1.2 (answer: yes) were solved by Huynh and the second named author [17] by constructing explicit exotic examples.

Theorem 1.3 [17]. *There exists an entire curve $f : \mathbb{C} \to X$ such that, given any cardinality $|J| \in \mathbb{Z}_+ \cup \{\infty\}$ and any a priori requirement that $\mathcal{T}_{\mathrm{diff}}$ is trivial or not, by taking certain increasing radii*

$\{r_j\}_{j\geqslant 1}$ tending to infinity, the sequence of holomorphic discs $\{f|_{\overline{\mathbb{D}}_{r_j}}\}_{j\geqslant 1}$ yields a Nevanlinna/Ahlfors current $\mathcal{T} = \mathcal{T}_{\mathrm{alg}} + \mathcal{T}_{\mathrm{diff}}$ with the desired shape $\mathcal{T}_{\mathrm{alg}} = \sum_{j\in J} \lambda_j \cdot [\mathsf{C}_j]$ in Siu's decomposition.

Here X is some $\mathbb{CP}^1$-bundle over an elliptic curve. The construction of f is *ad hoc*. For instance, it relies crucially on the rotational symmetry $\sqrt{-1}\cdot\Gamma = \Gamma$ of the Gaussian integer lattice $\Gamma = \mathbb{Z}[\sqrt{-1}]$ ([17, Lemma 3.2]). Moreover, it has the feature that all the obtained Nevanlinna/Ahlfors currents charge positive mass on certain unique elliptic curve C_∞ in X, hence failing to embrace the purely diffuse case where $|J| = 0$. In hindsight, the existence of such f reflects the fact that X is an Oka manifold [11].

After receiving an early manuscript of [17], Sibony asked

Question 1.4 (Sibony, Private Communication, 2021). Show that for $Y = \mathbb{CP}^n$, or complex torus, all Ahlfors currents can be obtained by single entire curve $f : \mathbb{C} \to Y$.

Using holomorphic discs $\{f|_{\overline{\mathbb{D}(z,r)}}\}_{z\in\mathbb{C}, r>0}$ not necessarily centered at the origin, Sibony's Question 1.4 has been confirmed by the second named author. In fact, such phenomenon holds for a large class of compact complex manifolds with certain weak Oka property [27, Theorem A]. For testing the Oka principle [11], we may also extend Sibony's Question 1.4 as follows.

Conjecture 1.5 [27]. *Let X be a compact Oka manifold. There should be some entire curve $g : \mathbb{C} \to X$ such that $\{g|_{\overline{\mathbb{D}}_r}\}_{r>0}$ generates all Nevanlinna/Ahlfors currents on X.*

In this paper, we develop further a technique introduced in [27, section 3] to verify Conjecture 1.5 for any complex torus, hence answering Sibony's Question 1.4 using only concentric holomorphic discs.

Theorem A. *There exists an entire curve $\mathbf{F} : \mathbb{C} \to \mathbb{T}$ into any given complex torus $\mathbb{T}$, such that the concentric holomorphic discs $\{\mathbf{F}|_{\overline{\mathbb{D}}_R}\}_{R>0}$ can generate all the classes of Nevanlinna/Ahlfors currents on $\mathbb{T}$, up to cohomological equivalence.*

Here, the cohomological equivalence is in the sense of de Rham cohomology group or Hodge cohomology group, see the definitions at the beginning of Section 2. It foreshadows striking holomorphic flexibility of entire curves in Oka geometry. The construction of such entire curves $\mathbf{F}$ is based on a sophisticated induction process. The key ingredients are Hodge theory on complex tori, the spectral theorem for Hermitian matrices, and delicate area-growth estimates.

Recall a result [6] of Duval that each irreducible component C_j in the singular part $\mathcal{T}_{\mathrm{alg}} = \sum_{j\in J} \lambda_j \cdot [\mathsf{C}_j]$ of an Nevanlinna/Ahlfors current $\mathcal{T} = \mathcal{T}_{\mathrm{alg}} + \mathcal{T}_{\mathrm{diff}}$ must be rational or elliptic. Hence, we can rewrite $\mathcal{T}_{\mathrm{alg}}$ as

$$\sum_{j\in J_{\mathsf{E}}} c_j^{\mathsf{E}} \cdot [\mathsf{E}_j] + \sum_{k\in J_{\mathsf{R}}} c_k^{\mathsf{R}} \cdot [\mathsf{R}_k] \tag{1.3}$$

for some distinct irreducible elliptic curves E_j and some distinct rational curves R_k, and for some strictly positive constants $c_j^{\mathsf{E}}, c_k^{\mathsf{R}}$ ($J_{\mathsf{E}}, J_{\mathsf{R}}$ are countable sets, could be empty). Thus, we can classify Nevanlinna/Ahlfors currents in different shapes by the data

$$\big(|J_{\mathsf{E}}| \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\},\ |J_{\mathsf{R}}| \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\},\ \mathcal{T}_{\mathrm{diff}} \text{ is trivial/nontrivial}\big). \tag{1.4}$$

The next question is very natural, and was raised by Shing-Tung Yau and Xiangyu Zhou independently to the second named author around the same time (January 2021).

Question 1.6 (Yau, Zhou). Is there any a priori restriction on the numbers $|J_{\mathsf{E}}|$, $|J_{\mathsf{R}}|$ of elliptic and rational components in the singular parts of Nevanlinna/Ahlfors currents?

Let $E = \mathbb{C}/\Gamma$ be an elliptic curve, where $\Gamma \subset \mathbb{C}$ is some lattice. Set $X := \mathbb{CP}^1 \times E$. Denote by $\pi_1 : X \to \mathbb{CP}^1$, $\pi_2 : X \to E$ the projections onto the first and the second factors, respectively. Let ω_{FS} be the Fubini–Study form on $\mathbb{CP}^1$ giving unit area $\int_{\mathbb{CP}^1} \omega_{\mathrm{FS}} = 1$. Let ω_E be the Hermitian form on E induced from the standard Euclidean form $\omega_{\mathbb{C}} := \frac{i}{2}\, dz \wedge d\bar{z}$ on $\mathbb{C}$, with area $\int_E \omega_E =: \varrho$. Here is a strengthening of Theorem 1.3.

Theorem B. *There exists an entire curve $\mathbf{F} : \mathbb{C} \to X$ producing all shapes (1.4) of Nevanlinna/Ahlfors currents by $\{\mathbf{F}|_{\overline{\mathbb{D}}_R}\}_{R>0}$.*

In particular, this answers Yau and Zhou's Question 1.6 completely. In fact, we can show that, for any singular current T of the shape (1.3), where $\mathsf{E}_j = \pi_1^{-1}(x'_j)$, $\mathsf{R}_k = \pi_2^{-1}(y'_k)$ are fibers of π_1, π_2 for arbitrary distinct points $\{x'_j\}_{j=1}^{|J_{\mathsf{E}}|} \subset \mathbb{CP}^1$, $\{y'_k\}_{k=1}^{|J_{\mathsf{R}}|} \subset E$, and where the coefficients $c_j^{\mathsf{E}}, c_k^{\mathsf{R}} > 0$ satisfy that the total mass $\varrho \cdot \sum_{j \in J_{\mathsf{E}}} c_j^{\mathsf{E}} + \sum_{k \in J_{\mathsf{R}}} c_k^{\mathsf{R}}$ of T is less than or equal to 1, then there exists some sequence of increasing radii $\{R_s\}_{s \geqslant 1}$ tending to infinity such that $\{\mathbf{F}|_{\overline{\mathbb{D}}_{R_s}}\}_{s \geqslant 1}$ generates a Nevanlinna/Ahlfors current $\mathcal{T} = \mathcal{T}_{\mathrm{alg}} + \mathcal{T}_{\mathrm{diff}}$ having the aimed singular part $\mathcal{T}_{\mathrm{alg}} = T$. This is a surprising result! Previously we would hesitate to imagine such tremendous holomorphic flexibility of entire curves.

The proof of Theorem B is quite challenging. First of all, we need to design an intriguing model of entire curves with parameters. Next, we have to determine the parameters properly, see Proposition 3.2. One natural idea is to make use of the Brouwer fixed point theorem. As a matter of fact, that was our original approach, which was indeed successful though involved. Nevertheless, we managed to obtain quite explicit formulae about these parameters by using a trick, see (3.18), (3.19), in spirit of Ahlfors' covering surface theory (cf. [8, 22]). In general, our strategy could work as well for any compact Oka manifold Y containing infinitely many rational and elliptic curves, respectively, by constructing a "good" auxiliary holomorphic map $\mathbb{C}^n \to Y$ for some $n \geqslant 1$. We believe that such an approach will be a key toward Conjecture 1.5.

Notation. Throughout this paper, the symbols $\lesssim$ and $\gtrsim$ stand for inequalities up to a positive multiplicative constant. The dependence of these constants on certain parameters, or lack thereof, will be clear from the context. The symbols $\gg$ and $\ll$ mean much larger and much smaller, respectively.

2 | TWISTED ENTIRE CURVES IN COMPLEX TORI PRODUCING ALL NEVANLINNA CURRENTS

In this section, we will construct an entire curve $\mathbf{F} : \mathbb{C} \to \mathbb{T}$ for Theorem A. The key ingredient is Hodge theory (cf. [13, 24]) on complex tori, as we present now.

Let X be a compact Kähler manifold of complex dimension n. The *de Rham cohomology group* $H^l(X, \mathbb{C})$ for every $0 \leqslant l \leqslant 2n$ is the quotient of the $\mathbb{C}$-linear space of closed l-forms (resp., l-currents) by the subspace of exact l-forms (resp., l-currents). We can also define the real group $H^l(X,\mathbb{R})$ using real forms or currents. The *Hodge cohomology group* $H^{p,q}(X,\mathbb{C})$ for each $0 \leqslant p, q \leqslant n$ is the subspace of $H^{p+q}(X,\mathbb{C})$ generated by the class of closed (p,q)-forms (resp., currents). Hodge decomposition states that

$$H^l(X,\mathbb{C}) = \oplus_{p+q=l} H^{p,q}(X,\mathbb{C}).$$

The Poincaré duality gives a perfect pairing

$$H^{p,q}(X,\mathbb{C}) \times H^{n-p,n-q}(X,\mathbb{C}) \longrightarrow \mathbb{C}, \quad ([\phi_1],[\phi_2]) \mapsto \int_X \phi_1 \wedge \phi_2.$$

When $p = q$, define

$$H^{p,p}(X,\mathbb{R}) := H^{p,p}(X,\mathbb{C}) \cap H^{2p}(X,\mathbb{R}).$$

It is clear that all Nevanlinna/Ahlfors currents on X are in $H^{n-1,n-1}(X,\mathbb{R})$.

Let ω be the Hermitian form on $\mathbb{T}$ induced by the standard Euclidean form $\omega_{\mathbb{C}^n} := \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z}_j$ through the canonical projection $\Pi : \mathbb{C}^n \to \mathbb{T} = \mathbb{C}^n/\Lambda$. The complex vector space $H^{n-1,n-1}(\mathbb{T},\mathbb{C})$ admits a $\mathbb{C}$-linear basis (cf. [13, p. 301])

$$\left\{dz_I \wedge d\overline{z}_J \,:\, I, J \subset \{1,\dots,n\}, |I| = |J| = n-1\right\}$$

induced from $\mathbb{C}$-coefficient $(n-1,n-1)$-forms on $\mathbb{C}^n$, where for $I = \{i_1,\dots,i_{n-1}\}$ and $J = \{j_1,\dots,j_{n-1}\}$, we abbreviate $dz_I := dz_{i_1} \wedge \cdots \wedge dz_{i_{n-1}}$ and $d\overline{z}_J := d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_{n-1}}$.

For any $\vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{C}^n \setminus \{0\}, \vec{c} \in \mathbb{C}^n$, we will show that the affine entire curve $F_{v,c}(z) = \Pi(z \cdot \vec{v} + \vec{c})$ from $\mathbb{C}$ to $\mathbb{T}$ produces a unique class of Nevanlinna/Ahlfors current, denoted by $\mathcal{T}_{[\vec{v}]}$, up to cohomological equivalence in $H^{n-1,n-1}(\mathbb{T},\mathbb{R})$. To see that $\mathcal{T}_{[\vec{v}]}$ is unique, independent of $\vec{c}$, and only depends on the equivalent class $[\vec{v}] \in \mathbb{CP}^{n-1}$, we need the following

Observation. For any (1,1)-class ϕ in $H^{1,1}(\mathbb{T},\mathbb{C})$ induced by the (1,1)-form $\sum_{k,\ell=1}^n \xi_{k,\ell}\, dz_k \wedge d\overline{z}_\ell$ on $\mathbb{C}^n$, where $\xi_{k,\ell} \in \mathbb{C}$, one has a neat formula

$$\langle \mathcal{T}_{[\vec{v}]}, \phi\rangle = -2i \sum_{k,\ell=1}^n v_k\, \xi_{k,\ell}\, \overline{v}_\ell / \|\vec{v}\|^2_{\mathbb{C}^n}, \tag{2.1}$$

where $\|\cdot\|_{\mathbb{C}^n}$ stands for the standard Euclidean norm on $\mathbb{C}^n$. Hence, the uniqueness of $\mathcal{T}_{[\vec{v}]}$ in $H^{n-1,n-1}(\mathbb{T},\mathbb{C})$ follows from the Poincaré duality.

Proof. For any disc $\mathbb{D}_R$ with radius $R > 0$, we compute directly

$$\langle (F_{v,c}) * [\mathbb{D}_R], \phi\rangle = \int_{\mathbb{D}_R} F^*_{v,c}\, \phi = \sum_{k,\ell=1}^n \int_{\mathbb{D}_R} \xi_{k,\ell}\, d(v_k z) \wedge d\overline{(v_\ell z)} = -2i\pi R^2 \sum_{k,\ell=1}^n v_k\, \xi_{k,\ell}\, \bar{v}_\ell,$$

$$\mathrm{Area}_\omega\left(F_{v,c}(\mathbb{D}_R)\right) = \int_{\mathbb{D}_R} F^*_{v,c}\, \omega = \sum_{j=1}^n \int_{\mathbb{D}_R} \frac{i}{2}\, d(v_j z) \wedge d\overline{(v_j z)} = \pi R^2\, \|\vec{v}\|^2_{\mathbb{C}^n}.$$

Hence (2.1) follows for Ahlfors currents. Similarly, we can show (2.1) for Nevanlinna currents. □

In the linear space of bidimension $(1,1)$-currents on $\mathbb{T}$, let C be the convex cone generated by $\{\mathcal{T}_{[\vec{v}]}\}_{[\vec{v}]\in\mathbb{CP}^{n-1}}$. Let $C_1 \subset C$ be the subset consisting of mass 1 currents with respect to ω, that is, $C_1 := \{T \in C : \langle T, \omega\rangle = 1\}$. One can check that $\mathcal{T}_{[\vec{v}]} \in C_1$ for every $[\vec{v}] \in \mathbb{CP}^{n-1}$.

Proposition 2.1. *Any Nevanlinna/Ahlfors current $\mathcal{T}$ on $\mathbb{T}$ can be represented by an element T' in C_1 in the same cohomology class $[T'] = [\mathcal{T}]$. Moreover, T' can be chosen in the shape*

$$T' = \sum_{s=1}^{n} \beta_s \cdot \mathcal{T}_{[\vec{v}_s]}$$

for some orthonormal basis $\{\vec{v}_s\}_{s=1}^n$ of $\mathbb{C}^n$ and some nonnegative coefficients $\beta_1, \dots, \beta_n$ with $\sum_{s=1}^n \beta_s =$ 1. Both $\{\vec{v}_s\}_{s=1}^n$ and $\{\beta_s\}_{s=1}^n$ depend on $\mathcal{T}$.

Proof. For any two vectors $\vec{u} := (u_1, u_2 \dots, u_n), \vec{w} := (w_1, w_2, \dots, w_n)$ in $\mathbb{C}^n$, denote by $\phi_{\vec{u},\vec{w}}$ the (1,1)-form on $\mathbb{T}$ induced by $\frac{i}{2} \sum_{k=1}^n u_k dz_k \wedge \sum_{\ell=1}^n \overline{w}_\ell d\overline{z}_\ell$ from $\mathbb{C}^n$. Consider the bilinear form

$$\mathcal{B}(\vec{u}, \vec{w}) := \langle \mathcal{T}, \phi_{\vec{u},\vec{w}}\rangle,$$

which is clearly semi-positive and Hermitian. Hence, by the spectral theorem (cf., e.g., [19, p. 582, Theorem 6.4]), we can find some orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ of $\mathbb{C}^n$, such that $\mathcal{B}$ reads as

$$\mathcal{B}(\vec{u}, \vec{w}) = \sum_{s=1}^{n} \beta_s \langle \vec{u}, \vec{v}_s\rangle \cdot \overline{\langle \vec{w}, \vec{v}_s\rangle} \qquad (\forall\, \vec{u},\, \vec{w} \in \mathbb{C}^n)$$

for some nonnegative numbers $\beta_1, \dots, \beta_n \geqslant 0$. Moreover, (2.1) verifies that

$$\sum_{s=1}^{n} \beta_s \langle \vec{u}, \vec{v}_s\rangle \cdot \overline{\langle \vec{w}, \vec{v}_s\rangle} = \langle T', \phi_{\vec{u},\vec{w}}\rangle,$$

where $T' := \sum_{s=1}^n \beta_s \cdot \mathcal{T}_{[\vec{v}_s]}$. Thus, $\mathcal{T}$ is cohomologous to T' by Hodge theory.

Finally, noting that that ω can be rewritten as $\sum_{s=1}^n \phi_{\vec{v}_s,\vec{v}_s}$, and that $\mathcal{T}(\omega) = 1$ by the definition of Nevanlinna/Ahlfors currents, using (2.1) again, we receive

$$1 = \langle \mathcal{T}, \omega\rangle = \sum_{s=1}^{n} \langle \mathcal{T}, \phi_{\vec{v}_s,\vec{v}_s}\rangle = \sum_{s=1}^{n} \beta_s.$$

Hence, we conclude the proof. □

Runge's approximation theorem. *Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected. Then any holomorphic function f defined on a neighborhood of K can be approximated uniformly on K by a sequence of polynomials.*

The following two consequences of Runge's approximation theorem will be used later.

Lemma 2.2. *For every $x \in \partial\mathbb{D}_R$ and a small open set U containing x, there exists an entire function φ depending on R, x, U, such that $|\varphi(x)| > 1$ and $|\varphi| < 1$ on $\overline{\mathbb{D}}_R \setminus U$.*

Proof. By [27, Key Lemma], we can find some holomorphic function $\widehat{\varphi}$ defined on a neighborhood of $\overline{\mathbb{D}}_R$ such that $|\widehat{\varphi}(x)| > 1$ and $|\widehat{\varphi}| < 1$ on $\overline{\mathbb{D}}_R \setminus U$. Now, Runge's approximation theorem concludes the proof. □

Remark 2.3. Here is an explicit elegant construction communicated to us by Fusheng Deng. For some sufficiently small $\epsilon > 0$, the entire function $\varphi(z) := \exp(z/x - 1 + \epsilon)$ satisfies the desired properties.

Lemma 2.4. *Let $\{D_j\}_{1\leqslant j\leqslant s}$ be a family of disjoint close discs in $\mathbb{C} \setminus \overline{\mathbb{D}}_R$ for some $R > 0$. Then for any entire function f, for any prescribed values $z_1, \ldots, z_s \in \mathbb{C}$, there exists some entire function g, such that $|g - f| < \delta$ on $\overline{\mathbb{D}}_R$, and that $|g - z_j| < \delta$ on D_j for every $j = 1, \ldots, s$.*

Proof. Apply Runge's approximation theorem to $K = \overline{\mathbb{D}}_R \bigcup \left(\cup_{j=1}^{s} D_j \right)$. □

From now on, we fix

$$\text{a countable subset } \{\mathcal{T}_{\mathfrak{m}}\}_{\mathfrak{m}\geqslant 1} \text{ of } \mathcal{C}_1, \tag{2.2}$$

which generates, in the finite-dimensional cohomology group $H^{n-1,n-1}(\mathbb{T}, \mathbb{R})$, dense image with respect to that of $\mathcal{C}_1$. We also fix a bijection

$$(\sigma, \tau) : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+ \times \mathbb{Z}_+,$$

so that $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+$ is surjective and every fiber $\sigma^{-1}(\mathfrak{m})$ contains infinitely many integers for $\mathfrak{m} \geqslant 1$. Our desired entire curve $\mathbf{F} = \Pi \circ \widehat{\mathbf{F}} : \mathbb{C} \to \mathbb{T}$ will be obtained by some lifting $\widehat{\mathbf{F}} : \mathbb{C} \to \mathbb{C}^n$, which is the limit of a sequence of entire curves $\widehat{F}_t : \mathbb{C} \to \mathbb{C}^n$ designed by the following sophisticated algorithm.

Step 0. Set $R_0 := 2n$. Let $\widehat{F}_0 : \mathbb{C} \to \mathbb{C}^n$ be any entire curve. Choose $\epsilon_0 := 1$.

In **Step** $t - 1$ for $t \geqslant 1$, we record certain entire curve $\widehat{F}_{t-1}$ on $\mathbb{C}^n$, a large radius R_{t-1} and a small error bound ϵ_{t-1}.

Step t for $t \geqslant 1$. Read $\mathfrak{m} = \sigma(t)$. By Proposition 2.1, we can rewrite $\mathcal{T}_{\mathfrak{m}}$ in (2.2) as $\sum_{j=1}^{n} \beta_{t,j} \cdot \mathcal{T}_{[\vec{v}_{t,j}]}$ for some nonnegative constants $\beta_{t,1}, \ldots, \beta_{t,n}$ and some orthonormal basis $\{\vec{v}_{t,s}\}_{s=1}^{n}$ of $\mathbb{C}^n$. Set $R_t := 2R_{t-1}$.

We can sparsely select n distinct points $x_{t,1}, \ldots, x_{t,n}$ on $\partial\mathbb{D}_{R_t}$ such that the closed unit discs $\{\overline{\mathbb{D}}(x_{t,j}, 1)\}_{j=1}^{n}$ are pairwise disjoint. By Lemma 2.2, for the point $x_{t,1}$ and its neighborhood $\mathbb{D}(x_{t,1}, 1)$, we can find some entire function $\varphi_{t,1}$ such that

$$|\varphi_{t,1}(x_{t,1})| > 1 \quad \text{and} \quad |\varphi_{t,1}| < 1 \quad \text{on} \quad \overline{\mathbb{D}}_{R_t} \setminus \mathbb{D}(x_{t,1}, 1). \tag{2.3}$$

As $x_{t,1}, \ldots, x_{t,n}$ are in the same circle around the origin, by composing $\varphi_{k,1}$ with some rotations, we can also define $\varphi_{t,j}(z) := \varphi_{t,1}(e^{i\theta_{t,j}} \cdot z)$, for $j = 2, \ldots, n$, with likewise property

$$|\varphi_{t,j}(x_{t,j})| > 1 \quad \text{and} \quad |\varphi_{t,j}| < 1 \quad \text{on} \quad \overline{\mathbb{D}}_{R_t} \setminus \mathbb{D}(x_{t,j}, 1).$$

Here, each angle $\theta_{t,j} \in [0, 2\pi)$ is uniquely determined by $e^{i\theta_{t,j}} \cdot x_{t,j} = x_{t,1}$.

Next, we define an entire curve

$$\widehat{F}_t := \widehat{F}_{t-1} + \sum_{j=1}^{n} \frac{\sqrt{\beta_{t,j}}}{\alpha_t} \cdot \varphi_{t,j}^{\alpha_t} \cdot \vec{v}_{t,j} \tag{2.4}$$

in $\mathbb{C}^n$, where the integer exponent $\alpha_t \gg 1$ is chosen to meet the following sophisticated requirements.

(♠). The entire curve $F_t := \Pi\circ\widehat{F}_t$ satisfies the length–area estimate

$$\frac{\mathrm{Length}_\omega(F_t(\mathbb{D}_{R_t}))}{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}))} < 2^{-t}, \qquad \frac{\mathfrak{L}_{F_t}(R_t)}{\mathfrak{A}_{F_t}(R_t)} < 2^{-t}. \tag{2.5}$$

(♡). The two normalized currents associated with the holomorphic disc $F_t|_{\overline{\mathbb{D}}_{R_t}}$ are near to the Nevanlinna/Ahlfors current $\mathcal{T}_{\mathfrak{m}}$ in the sense that, for all $1 \leqslant k, \ell \leqslant n$,

$$\Big|\frac{1}{\mathfrak{A}_{F_t}(R_t)} \int_0^{R_t} \frac{dr}{r} \int_{\mathbb{D}_r} F_t^*(dz_k \wedge d\bar{z}_\ell) - \langle \mathcal{T}_{\mathfrak{m}}, dz_k \wedge d\bar{z}_\ell \rangle\Big| < 2^{-t}, \tag{2.6}$$

$$\Big|\frac{1}{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}))} \int_{\mathbb{D}_{R_t}} F_t^*(dz_k \wedge d\bar{z}_\ell) - \langle \mathcal{T}_{\mathfrak{m}}, dz_k \wedge d\bar{z}_\ell \rangle\Big| < 2^{-t}. \tag{2.7}$$

(♣). On $\mathbb{D}_{R_{t-1}+1}$, $\widehat{F}_t$ is very near to $\widehat{F}_{t-1}$, that is,

$$\|\widehat{F}_t(z) - \widehat{F}_{t-1}(z)\|_{\mathbb{C}^n} < 2^{-t} \cdot \epsilon_{t-1} \quad \text{for } z \in \mathbb{D}_{R_{t-1}+1}.$$

The main technical difficulty in our construction is to show the existence of $\alpha_t \gg 1$ satisfying the conditions (♠), (♡), (♣). This will be reached in Proposition 2.5 eventually. For the moment, assuming the existence, we can finish the **Step** t by selecting a small positive error bound $\epsilon_t < \epsilon_{t-1}/2$, such that, if $\widehat{\mathbf{F}}$ is some mild holomorphic perturbation of $\widehat{F}_t$ with $|\widehat{\mathbf{F}} - \widehat{F}_t| < \epsilon_t$ on $\mathbb{D}_{R_t+1}$, after replacing F_t by the new entire curve $\mathbf{F} := \Pi\circ\widehat{\mathbf{F}}$ on the left-hand sides and enlarging 2^{-t} to 2^{-t+1} on the right-hand sides,

(◇). all the corresponding inequalities (2.5), (2.6), (2.7) maintain.

The existence of such ϵ_t can be proved by a *reductio ad absurdum* argument, as the $\mathscr{C}^0$-convergence of a sequence of holomorphic functions on a larger open set $\mathbb{D}_{R_t+1} \supset \overline{\mathbb{D}}_{R_t}$ can guarantee the $\mathscr{C}^1$-convergence on the smaller compact set $\overline{\mathbb{D}}_{R_t}$.

Proof of Theorem A. The condition (♣) guarantees that $\lim_{t\to\infty} \widehat{F}_t$ exists, which denoted by $\widehat{\mathbf{F}} : \mathbb{C} \to \mathbb{C}^n$. By our construction, on each $\mathbb{D}_{R_t+1}$ for $t \geqslant 1$, there holds

$$|\widehat{\mathbf{F}} - \widehat{F}_t| \leqslant \sum_{j\geqslant t} |\widehat{F}_{j+1} - \widehat{F}_j| < \sum_{j\geqslant t} \frac{\epsilon_j}{2^{j+1}} < \sum_{j\geqslant t} \frac{\epsilon_t}{2^{j+1}} < \epsilon_t.$$

Thus, the condition (◇) ensures that the entire curve $\mathbf{F} = \Pi\circ\widehat{\mathbf{F}} : \mathbb{C} \to \mathbb{T}$ produces concentric holomorphic discs $\{\mathbf{F}|_{\overline{\mathbb{D}}_{R_t}}\}_{t\geqslant 1}$ satisfying the length–area conditions for obtaining Nevanlinna/Ahlfors currents, and that for every $t \geqslant 1$, (2.6) and (2.7) hold after replacing F_t by $\mathbf{F}$. Therefore, for any

$\mathcal{T}_{\mathfrak{m}}$ in (2.2), the sequence of infinite holomorphic discs $\{\mathbf{F}\,|_{\overline{\mathbb{D}}_{R_t}}\}_{t\in\sigma^{-1}(\mathfrak{m})}$ can generate, after passing to certain subsequence, some Nevanlinna and Ahlfors currents cohomologous to $\mathcal{T}_{\mathfrak{m}}$. Finally, by the density of $\{\mathcal{T}_{\mathfrak{m}}\}_{\mathfrak{m}\geqslant 1}$ in $\mathcal{C}_1$ as coholomogy classes in $H^{n-1,n-1}(\mathbb{T},\mathbb{R})$, we conclude the proof. □

The remaining task of this section is to show

Proposition 2.5. *In each Step $t \geqslant 1$, one can find some large integer exponent $\alpha_t \gg 1$, such that the requirements* (♠), (♡), (♣) *hold true.*

We fix some notation first. In coordinate systems, write $\widehat{F}_t = (\widehat{f}_{t,1}, \dots, \widehat{f}_{t,n})$, $\vec{v}_{t,j} = (v_{t,j,1}, \dots, v_{t,j,n})$ for every $1 \leqslant j \leqslant n$. Put

$$\Phi_{t,s} := \sum_{j=1}^{n} \frac{\sqrt{\beta_{t,j}}}{\alpha_t} \cdot \varphi_{t,j}^{\alpha_t} \cdot v_{t,j,s}, \qquad 1 \leqslant s \leqslant n,$$

so that $\widehat{f}_{t,s} = \widehat{f}_{t-1,s} + \Phi_{t,s}$. Set $M := \max_{1\leqslant j\leqslant n} \max_{\overline{\mathbb{D}}_{R_t}} |\widehat{f}'_{t-1,j}| \geqslant 0$, and

$$m := \max_{\overline{\mathbb{D}}(x_{t,1},1)\cap\overline{\mathbb{D}}_{R_t}} |\varphi_{t,1}| > 1, \quad m' := \max_{\overline{\mathbb{D}}_{R_t}} |\varphi'_{t,1}| > 0, \quad m_0 := \max_{\overline{\mathbb{D}}_{R_t}\setminus\mathbb{D}(x_{t,1},1)} |\varphi_{t,1}| < 1.$$

Here, we omit the indices t on the left-hand sides for convenience. As every $\varphi_{t,j}$ for $j = 2, \dots, n$ is obtained by composing $\varphi_{t,1}$ with some rotation, we receive

$$\max_{\overline{\mathbb{D}}(x_{t,j},1)\cap\overline{\mathbb{D}}_{R_t}} |\varphi_{t,j}| = m, \qquad \max_{\overline{\mathbb{D}}_{R_t}} |\varphi'_{t,j}| = m', \qquad \max_{\overline{\mathbb{D}}_{R_t}\setminus\mathbb{D}(x_{t,j},1)} |\varphi_{t,j}| = m_0.$$

Proof of the first inequality of (2.5). We can compute

$$\mathrm{Length}_{\omega}\Big(F_t(\partial\mathbb{D}_{R_t})\Big) = \int_{\partial\mathbb{D}_{R_t}} |\widehat{F}_t^*\omega|^{1/2} \quad \text{and} \quad \mathrm{Area}_{\omega}\Big(F_t(\mathbb{D}_{R_t})\Big) = \int_{\mathbb{D}_{R_t}} \widehat{F}_t^*\omega,$$

where

$$\widehat{F}_t^*\omega = \sum_{s=1}^{n} \frac{i}{2}\, \mathrm{d}\widehat{f}_{t,s}(z) \wedge \mathrm{d}\overline{\widehat{f}_{t,s}(z)} = \sum_{s=1}^{n} |\widehat{f}'_{t,s}(z)|^2 \cdot \frac{i}{2}\, dz \wedge d\bar{z}.$$

To get an upper bound for $\mathrm{Length}_{\omega}\big(F_t(\partial\mathbb{D}_{R_t})\big)$, we need to bound every

$$|\widehat{f}'_{t,s}| = |\widehat{f}'_{t-1,s} + \Phi'_{t,s}| \leqslant M + |\Phi'_{t,s}|, \qquad 1 \leqslant s \leqslant n$$

on $\overline{\mathbb{D}}_{R_t}$. Note that (recall $\|v_{t,j}\|_{\mathbb{C}^n} = 1$)

$$|\Phi'_{t,s}| \leqslant \sum_{j=1}^{n} \sqrt{\beta_{t,j}} \cdot |\varphi_{t,j}|^{\alpha_t - 1} \cdot |\varphi'_{t,j}| \cdot |v_{t,j,s}| \leqslant \sum_{j=1}^{n} \sqrt{\beta_{t,j}} \cdot m^{\alpha_t - 1} \cdot m' \cdot 1. \tag{2.8}$$

Hence, we receive

$$\operatorname{Length}_\omega\Big(F_t(\partial\mathbb{D}_{R_t})\Big) < \int_{\partial\mathbb{D}_{R_t}} \left(M + \sum_{j=1}^n \sqrt{\beta_{t,j}} \cdot m^{\alpha_t-1} m' \right) |dz| \lesssim R_t \cdot m^{\alpha_t-1} m'. \tag{2.9}$$

Here, the implicit multiplicative does not depend on α_t.

Now we estimate $\operatorname{Area}_\omega\big(F_t(\mathbb{D}_{R_t})\big)$. In each nonempty open set

$$\mathbb{D}_{R_t} \cap \big\{1 < m^{3/4} < |\varphi_{t,j}| < m\big\} \cap \big\{\varphi'_{t,j} \neq 0\big\}, \qquad j = 1, \dots, n,$$

we take a closed ball $B_{t,j}$. Clearly, $B_{t,j}$ is contained in $\mathbb{D}(x_{t,j}, 1)$. By the compactness of $B_{t,j}$, we can find some $c_{t,j} > 0$ such that $|\varphi'_{t,j}| \geqslant c_{t,j}$ on $B_{t,j}$. By triangle inequality, for every $z \in B_{t,j}$, for sufficiently large $\alpha_t \gg 1$, we have

$$\begin{aligned} |\Phi'_{t,s}(z)| &\geqslant \sqrt{\beta_{t,j}} \cdot |\varphi_{t,j}|^{\alpha_t-1} |\varphi'_{t,j}| \cdot |v_{t,j,s}| - \sum_{k=1,k\neq j}^n \left(\sqrt{\beta_{t,k}} \cdot |\varphi_{t,k}|^{\alpha_t-1} |\varphi'_{t,k}| \cdot |v_{t,k,s}| \right) \\ &\geqslant \sqrt{\beta_{t,j}} \cdot \left(m^{3/4}\right)^{\alpha_t-1} c_{t,j} \cdot |v_{t,j,s}| - 1, \end{aligned} \tag{2.10}$$

as $|\varphi_{t,k}| \leqslant m_0 < 1$ on $B_{t,j} \subset \mathbb{D}(x_{t,j}, 1) \subset \overline{\mathbb{D}}_{R_t} \setminus \mathbb{D}(x_{t,k}, 1)$ for $k \neq j$.

Recall that $\sum_{j=1}^n \beta_{t,j} = 1$ and that $\sum_{s=1}^n |v_{t,j,s}|^2 = 1$ for every $j = 1, \dots, n$. We can take some j_0 and s_0 with $\beta_{t,j_0} \geqslant 1/n$ and $|v_{t,j_0,s_0}| \geqslant 1/\sqrt{n}$. Hence, for $z \in B_{t,j_0}$ and for $\alpha_t \gg 1$, we can continue to estimate (2.10) as $|\Phi'_{t,s_0}(z)| \geqslant (2n)^{-1} \left(m^{3/4}\right)^{\alpha_t-1} c_{t,j_0}$ as $m > 1$. Thus, we get

$$|\widehat{f}'_{t,s_0}| = |\widehat{f}'_{t-1,s_0} + \Phi'_{t,s_0}| \geqslant |\Phi'_{t,s_0}| - M \geqslant (4n)^{-1} \left(m^{3/4}\right)^{\alpha_t-1} c_{t,j_0}$$

on B_{t,j_0}, again assuming $\alpha_t \gg 1$. Consequently,

$$\operatorname{Area}_\omega\Big(F_t(\mathbb{D}_{R_t})\Big) \geqslant \operatorname{Area}_\omega\Big(F_t(B_{t,j_0})\Big) \geqslant \int_{B_{t,j_0}} |\widehat{f}'_{t,s_0}(z)|^2 \, \frac{i}{2}\, dz \wedge d\bar{z} \gtrsim \left(m^{3/2}\right)^{\alpha_t-1}. \tag{2.11}$$

Here, the implicit multiplicative depends on B_{t,j_0}, and does not depend on α_t.

By (2.9) and (2.11), we conclude that

$$\frac{\operatorname{Length}_\omega(F_t(\partial\mathbb{D}_{R_t}))}{\operatorname{Area}_\omega(F_t(\mathbb{D}_{R_t}))} \lesssim \frac{R_t \cdot m^{\alpha_t-1} m'}{\left(m^{3/2}\right)^{\alpha_t-1}} = \frac{R_t \cdot m'}{m^{\alpha_t/2-1/2}}.$$

Letting $\alpha_t \gg 1$, we finish the proof. □

Proof of the second inequality of (2.5). Recall the definition

$$\mathfrak{L}_{F_t}(R_t) := \int_0^{R_t} \operatorname{Length}_\omega(F_t(\partial\mathbb{D}_r)) \, \frac{dr}{r} \quad \text{and} \quad \mathfrak{A}_{F_t}(R_t) := \int_0^{R_t} \operatorname{Area}_\omega(F_t(\mathbb{D}_r)) \, \frac{dr}{r}.$$

By the same argument as (2.9), for every $r \leqslant R_t$, we can show that

$$\operatorname{Length}_\omega(F_t(\partial\mathbb{D}_r)) \lesssim r \cdot m^{\alpha_t-1} m',$$

where the implicit multiplicative is independent of α_t. Hence,

$$\mathfrak{L}_{F_t}(R_t) \lesssim \int_0^{R_t} r \cdot m^{\alpha_t-1} m' \,\frac{dr}{r} \leqslant R_t \cdot m^{\alpha_t-1} m'.$$

To bound $\mathfrak{A}_{F_t}(R_t)$ from below, we consider $R_t^- := \max\{|z| : z \in B_{t,j_0}\} < R_t$. For all $r \in [R_t^-, R_t]$, (2.11) gives $\mathrm{Area}_\omega(F_t(\mathbb{D}_r)) \gtrsim (m^{3/2})^{\alpha_t-1}$. Hence,

$$\mathfrak{A}_{F_t}(R_t) \geqslant \int_{R_t^-}^{R_t} \mathrm{Area}_\omega(F_t(\mathbb{D}_r)) \,\frac{dr}{r} \gtrsim \int_{R_t^-}^{R_t} (m^{3/2})^{\alpha_t-1} \,\frac{dr}{r} \gtrsim (m^{3/2})^{\alpha_t-1},$$

where the implicit multiplicative is independent of α_t.

Thus, we conclude that

$$\frac{\mathfrak{L}_{F_t}(R_t)}{\mathfrak{A}_{F_t}(R_t)} \lesssim \frac{R_t \cdot m^{\alpha_t-1} m'}{(m^{3/2})^{\alpha_t-1}} = \frac{R_t \cdot m'}{m^{\alpha_t/2-1/2}}.$$

By taking sufficiently large $\alpha_t \gg 1$, we finish the proof. □

Proof of (2.7). Fix $k, \ell \in \{1, \dots, n\}$. By (2.1), we have

$$\langle \mathcal{T}_{\mathfrak{m}}, dz_k \wedge d\bar{z}_\ell \rangle = \sum_{j=1}^{n} \beta_{t,j} \cdot \langle \mathcal{T}_{[\vec{v}_{t,j}]}, dz_k \wedge d\bar{z}_\ell \rangle = -2i \sum_{j=1}^{n} \beta_{t,j} \cdot v_{t,j,k} \bar{v}_{t,j,\ell}.$$

Now we compute

$$\widehat{F}_t^*(dz_k \wedge d\bar{z}_\ell) = d\widehat{f}_{t,k}(z) \wedge d\overline{\widehat{f}_{t,\ell}(z)} = \widehat{f}'_{t,k}(z) \overline{\widehat{f}'_{t,\ell}(z)} \, dz \wedge d\bar{z}.$$

Decompose

$$\widehat{f}'_{t,k}(z) \cdot \overline{\widehat{f}'_{t,\ell}(z)} = (\widehat{f}'_{t-1,k} + \Phi'_{t,k}) \cdot \overline{(\widehat{f}'_{t-1,\ell} + \Phi'_{t,\ell})} = \Theta_1(z) + \Theta_2(z) + \Theta_3(z),$$

where the dominant term is

$$\Theta_1(z) := \Phi'_{t,k}(z) \cdot \overline{\Phi'_{t,\ell}(z)} = \left(\sum_{j=1}^{n} \sqrt{\beta_{t,j}} \cdot \varphi_{t,j}^{\alpha_t-1} \varphi'_{t,j} \cdot v_{t,j,k} \right) \cdot \left(\sum_{s=1}^{n} \sqrt{\beta_{t,s}} \cdot \overline{\varphi}_{t,s}^{\alpha_t-1} \overline{\varphi'_{t,s}} \cdot \overline{v}_{t,s,\ell} \right),$$

and where the remaining terms are

$$\Theta_2(z) := \widehat{f}'_{t-1,k}(z) \cdot \overline{\widehat{f}'_{t-1,\ell}(z)}, \quad \Theta_3(z) := \widehat{f}'_{t-1,k}(z) \cdot \overline{\Phi'_{t,\ell}(z)} + \Phi'_{t,k}(z) \cdot \overline{\widehat{f}'_{t-1,\ell}(z)}.$$

By our construction, if $j \neq s$, then $|\varphi_{t,j}|^{\alpha_t-1} |\varphi_{t,s}|^{\alpha_t-1} \leqslant m^{\alpha_t-1}$ on $\mathbb{D}_{R_t}$. Noting that $|\beta_{t,j}| \leqslant 1$, $|v_{t,j,k}| \leqslant 1$, we have

$$\Theta_1 = \sum_{j=1}^{n} \beta_{t,j} \cdot v_{t,j,k} \bar{v}_{t,j,\ell} \cdot |\varphi_{t,j}|^{2\alpha_t-2} |\varphi'_{t,j}|^2 + O(m^{\alpha_t-1} m'^2).$$

By the same reasoning, we have $|\Theta_2| = O(M^2)$, $|\Theta_3| = O(Mm^{\alpha_t-1}m')$. Denote

$$\mathscr{A}_{t,j} := \int_{\mathbb{D}_{R_t}} |\varphi_{t,j}|^{2\alpha_t-2}|\varphi'_{t,j}|^2(z)\, dz \wedge d\bar{z}, \qquad j = 1, \ldots, n.$$

Observe that each $\mathscr{A}_{t,j} = \mathscr{A}_{t,1}$, as $\varphi_{t,j}$ is obtained by composing $\varphi_{t,1}$ with some rotation.

Recall (2.11) that $\mathrm{Area}_\omega\big(F_t(\mathbb{D}_{R_t})\big) \gtrsim (m^{3/2})^{\alpha_t-1}$. By comparing the exponents of m, we obtain that

$$\frac{1}{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}))}\left| \int_{\mathbb{D}_{R_t}} \widehat{F}_t^*(dz_k \wedge d\bar{z}_\ell) - \sum_{j=1}^n \beta_{t,j} \cdot v_{t,j,k} \cdot \bar{v}_{t,j,\ell} \cdot \mathscr{A}_{t,j} \right| \lesssim \frac{m^{\alpha_t-1}}{(m^{3/2})^{\alpha_t-1}},$$

where the implicit multiplicative is independent of α_t. Using that all $\mathscr{A}_{t,j}$'s are equal, we receive that

$$\frac{1}{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}))}\left| \int_{\mathbb{D}_{R_t}} \widehat{F}_t^*(dz_k \wedge d\bar{z}_\ell) - \mathscr{A}_{t,1} \sum_{j=1}^n \beta_{t,j} \cdot v_{t,j,k} \cdot \bar{v}_{t,j,\ell} \right| \longrightarrow 0 \tag{2.12}$$

as α_t tends to infinity. Hence, to prove (2.7), it remains to show that

$$\mathscr{A}_{t,1}/\mathrm{Area}_\omega\Big(F_t(\mathbb{D}_{R_t})\Big) = -2i + o(1) \quad \text{as} \quad \alpha_t \to \infty. \tag{2.13}$$

By letting $k = \ell$ running through $1, \ldots, n$, we deduce from (2.12) that

$$\frac{1}{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}))}\left| \int_{\mathbb{D}_{R_t}} \widehat{F}_t^*\left(\frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j\right) - \frac{i}{2}\mathscr{A}_{t,1} \sum_{j=1}^n \beta_{t,j} \cdot \sum_{k=1}^n |v_{t,j,k}|^2 \right| \longrightarrow 0.$$

Equivalently,

$$\frac{1}{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}))}\left| \mathrm{Area}_\omega\Big(F_t(\mathbb{D}_{R_t})\Big) - \frac{i}{2}\mathscr{A}_{t,1} \sum_{j=1}^n \beta_{t,j} \cdot \|\vec{v}_{t,j}\|^2_{\mathbb{C}^n} \right| \longrightarrow 0.$$

Thus, (2.13) follows from $\sum_{j=1}^n \beta_{t,j} = 1$ and $\|\vec{v}_{t,j}\|_{\mathbb{C}^n} = 1$. This concludes the proof. □

Proof of (2.6). For every $j = 1, \ldots, n$, we define

$$\mathscr{B}_{t,j} := \int_0^{R_t} \frac{dr}{r} \int_{\mathbb{D}_r} |\varphi_{t,j}|^{2\alpha_t-2}|\varphi'_{t,j}|^2\, dz \wedge d\bar{z} = \int_0^{R_t} \frac{dr}{r} \int_{\mathbb{D}_r} |\varphi_{t,1}|^{2\alpha_t-2}|\varphi'_{t,1}|^2\, dz \wedge d\bar{z}.$$

Repeating the same argument, by comparing the exponents of m, for $\alpha_t \to \infty$, we have

$$\frac{1}{\mathfrak{A}_{F_t}(R_t)}\left| \int_0^{R_t} \frac{dr}{r} \int_{\mathbb{D}_r} \widehat{F}_t^*(dz_k \wedge d\bar{z}_\ell) - \mathscr{B}_{t,1} \sum_{j=1}^n \beta_{t,j} \cdot v_{t,j,k} \cdot \bar{v}_{t,j,\ell} \right| \longrightarrow 0.$$

Now we only need to show that $\mathscr{B}_{t,1}/\mathfrak{A}_{F_t}(R_t) = -2i + o(1)$ as $\alpha_t \to \infty$. This can be done by the same argument as in the preceding proof. □

Proof of Proposition 2.5. Summarizing, (♠) and (♡) are proved already. The requirement (♣) is easily satisfied, as on $\mathbb{D}_{R_{t-1}+1}$, by letting α_t tend to infinity, we have

$$\|\widehat{F}_t - \widehat{F}_{t-1}\|_{\mathbb{C}^n} = \left\| \sum_{j=1}^{n} \frac{\sqrt{\beta_{t,j}}}{\alpha_t} \cdot \varphi_{t,j}^{\alpha_t} \cdot \vec{v}_{t,j} \right\|_{\mathbb{C}^n} \longrightarrow 0$$

as $|\varphi_{t,j}| \leqslant m_0 < 1$. □

3 | EXOTIC ENTIRE CURVES PRODUCING ALL SHAPES OF NEVANLINNA CURRENTS

3.1 | An algorithm for constructing exotic entire curves

Let $\pi : \mathbb{C} \to E = \mathbb{C}/\Gamma$ be the canonical projection. The *Weierstrass ℘-function* associated with Γ is given by

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

It is meromorphic on $\mathbb{C}$, doubly periodic with respect to Γ. Hence, it induces a holomorphic map, still denoted by $\wp$, from $\mathbb{C}$ to $\mathbb{CP}^1$, and a holomorphic map $\widetilde{\wp} : E \to \mathbb{CP}^1$ with $\wp = \widetilde{\wp} \circ \pi$. Let $\Pi := (\wp, \pi) : \mathbb{C}^2 \to X = \mathbb{CP}^1 \times E$ be the total projection. Our desired entire curve $\mathbf{F}$ for Theorem B will be obtained as $\mathbf{F} := \Pi \circ \widehat{\mathbf{F}}$, where $\widehat{\mathbf{F}}$ is the limit of some sequence of entire curves $\{\widehat{F}_t : \mathbb{C} \to \mathbb{C}^2\}_{t \geqslant 1}$. The concentric holomorphic discs $\{\mathbf{F}|_{\overline{\mathbb{D}}_R}\}_{R>0}$ will produce all shapes of Nevanlinna/Ahlfors currents on X. Fix a reference Hermitian form $\omega := \pi_1^* \omega_{\mathrm{FS}} + \pi_2^* \omega_E$ on X.

Take a dense sequence $\{x_j\}_{j=1}^{\infty}$ of distinct points in $\mathbb{CP}^1$, and another dense sequence $\{y_k\}_{k=1}^{\infty}$ of distinct points in E.

Enumerate the countable set

$$\left\{ \left[(P,Q); (A_j)_{j=1}^{P}; (B_k)_{k=1}^{Q} \right] : (P,Q) \in \mathbb{Z}_{\geqslant 0}^2 \setminus \{(0,0)\}, A_j, B_k \in \mathbb{Q}_+, \sum_{j=1}^{P} A_j + \sum_{k=1}^{Q} B_k = 1 \right\}$$

as $\left\{ \left[(P_{\mathfrak{m}}, Q_{\mathfrak{m}}); \mathbf{A}_{\mathfrak{m}}; \mathbf{B}_{\mathfrak{m}} \right] \right\}_{\mathfrak{m} \geqslant 1}$. The value $P_{\mathfrak{m}}$ (resp., $Q_{\mathfrak{m}}$) counts the number of irreducible elliptic (resp., rational) components in the singular part of certain Nevanlinna/Ahlfors current $\mathcal{T}$ to be obtained, while $\mathbf{A}_{\mathfrak{m}} \in \mathbb{Q}_+^{P_{\mathfrak{m}}}$ (resp., $\mathbf{B}_{\mathfrak{m}} \in \mathbb{Q}_+^{Q_{\mathfrak{m}}}$) records the masses of $\mathcal{T}$ on these elliptic (resp., rational) curves.

As in Section 2, we fix a bijection

$$(\sigma, \tau) : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+ \times \mathbb{Z}_+.$$

Step 0: Set $R_0 := 2, \epsilon_0 := 1$. Let $\widehat{F}_0 : \mathbb{C} \to \mathbb{C}^2$ be any entire curve.

In **Step** $t-1$ for $t \geqslant 1$, we record an entire curve $\widehat{F}_{t-1} = (\widehat{g}_{t-1}, \widehat{h}_{t-1})$ in $\mathbb{C}^2$, some large radius $R_{t-1} \gg 1$, and a small error bound $0 < \epsilon_{t-1} \ll 1$.

Step t: Read $\mathfrak{m} = \sigma(t)$. Set

$$R_t := \max\{2R_{t-1}, P_{\mathfrak{m}} + Q_{\mathfrak{m}}\}.$$

We choose $P_{\mathfrak{m}} + Q_{\mathfrak{m}}$ distinct points $\{\widehat{x}_{t,j}\}_{j=1}^{P_{\mathfrak{m}}}$, $\{\widehat{y}_{t,k}\}_{k=1}^{Q_{\mathfrak{m}}}$ in $\partial\mathbb{D}_{R_t}$, such that the closed unit discs $\{\overline{\mathbb{D}}(\widehat{x}_{t,j},1)\}_{j=1}^{P_{\mathfrak{m}}}$, $\{\overline{\mathbb{D}}(\widehat{y}_{t,k},1)\}_{k=1}^{Q_{\mathfrak{m}}}$ are pairwise disjoint. Without loss of generality, we may assume that $P_{\mathfrak{m}} \geqslant 1$. By Lemma 2.2, for the point $\widehat{x}_{t,1}$ and its neighborhood $\mathbb{D}(\widehat{x}_{t,1},1)$, we can find some entire function $\varphi_{t,1}$ such that

$$|\varphi_{t,1}(\widehat{x}_{t,1})| > 1 \quad \text{and} \quad |\varphi_{t,1}| < 1 \quad \text{on} \quad \overline{\mathbb{D}}_{R_t} \setminus \mathbb{D}(\widehat{x}_{t,1},1). \tag{3.1}$$

As $\widehat{x}_{t,1}, \widehat{x}_{t,2}, \dots, \widehat{x}_{t,P_{\mathfrak{m}}}$ are in the same circle centered at the origin, by composing $\varphi_{t,1}$ with some rotations, we can also define $\varphi_{t,j}(z) := \varphi_{t,1}(e^{i\theta_{t,j}} \cdot z)$ with likewise property, where each angle $\theta_{t,j} \in [0,2\pi)$ is uniquely determined by $e^{i\theta_{t,j}} \cdot \widehat{x}_{t,j} = \widehat{x}_{t,1}$ for $j = 2, \dots, P_{\mathfrak{m}}$. Similarly, we can obtain $\psi_{t,k}$ for $1 \leqslant k \leqslant Q_{\mathfrak{m}}$ by composing $\varphi_{t,1}$ with some rotations, such that

$$|\psi_{t,k}(\widehat{y}_{t,k})| > 1, \quad |\psi_{t,k}| < 1 \quad \text{on} \quad \overline{\mathbb{D}}_{R_t} \setminus \mathbb{D}(\widehat{y}_{t,k},1). \tag{3.2}$$

Let $\delta_t \in (0, 2^{-t})$ be a small constant satisfying

$$\mathrm{dist}_{\omega_{\mathrm{FS}}}(x_{j_1}, x_{j_2}) > 6\delta_t, \quad \mathrm{dist}_{\omega_E}(y_{k_1}, y_{k_2}) > 6\delta_t,$$

for all $1 \leqslant j_1 \neq j_2 \leqslant P_{\mathfrak{m}}, 1 \leqslant k_1 \neq k_2 \leqslant Q_{\mathfrak{m}}$.

As $\wp : \mathbb{C} \to \mathbb{CP}^1$ and $\pi : \mathbb{C} \to E$ are Lipschitz and surjective, applying Lemma 2.4, we can find auxiliary entire functions $g^\star_{t-1}, h^\star_{t-1}$, such that on $\overline{\mathbb{D}}_{R_{t-1}+1}$

$$|\widehat{g}_{t-1} - g^\star_{t-1}| < \epsilon_{t-1} \cdot 2^{-t-2}, \qquad |\widehat{h}_{t-1} - h^\star_{t-1}| < \epsilon_{t-1} \cdot 2^{-t-2}, \tag{3.3}$$

$$\mathrm{dist}_{\omega_{\mathrm{FS}}}\left(\wp\circ\widehat{g}_{t-1}, \wp\circ g^\star_{t-1}\right) < \delta_t, \qquad \mathrm{dist}_{\omega_E}\left(\pi\circ\widehat{h}_{t-1}, \pi\circ h^\star_{t-1}\right) < \delta_t,$$

and such that for every $1 \leqslant j \leqslant P_{\mathfrak{m}}, 1 \leqslant k \leqslant Q_{\mathfrak{m}}$,

$$\mathrm{dist}_{\omega_{\mathrm{FS}}}\left(x_j, \wp\circ g^\star_{t-1}\right) < \delta_t \quad \text{on} \quad \overline{\mathbb{D}}(\widehat{x}_{t,j},1), \tag{3.4}$$

$$\mathrm{dist}_{\omega_E}\left(y_k, \pi\circ h^\star_{t-1}\right) < \delta_t \quad \text{on} \quad \overline{\mathbb{D}}(\widehat{y}_{t,k},1). \tag{3.5}$$

Now we define

$$\widehat{g}_t := g^\star_{t-1} + \sum_{k=1}^{Q_{\mathfrak{m}}} q_{t,k} \cdot \psi_{t,k}^{\alpha_t} =: g^\star_{t-1} + \Psi_t, \tag{3.6}$$

$$\widehat{h}_t := h^\star_{t-1} + \sum_{j=1}^{P_{\mathfrak{m}}} p_{t,j} \cdot \varphi_{t,j}^{\alpha_t} =: h^\star_{t-1} + \Phi_t, \tag{3.7}$$

where the parameters $p_{t,j}, q_{t,k} \in (0,1]$ and the exponent $\alpha_t \gg 1$ are to be determined later by sophisticated reasoning. Thus we obtain an entire curve $\widehat{F}_t := (\widehat{g}_t, \widehat{h}_t)$ in $\mathbb{C}^2$.

The goal of **Step** t is to select some appropriate parameters $\alpha_t, p_{t,\bullet}, q_{t,\bullet}$, so that the obtained entire curve $\widehat{F}_t$ satisfies the following Propositions 3.1 and 3.2, whose proofs will be postponed to

Section 3.2. Define the following open subsets

$$U_t := \bigcup_{j=1}^{P_\mathfrak{m}} \mathbb{D}(x_j, 2\delta_t) \times E, \quad V_t := \mathbb{CP}^1 \times \bigcup_{k=1}^{Q_\mathfrak{m}} \mathbb{D}(y_k, 2\delta_t)$$

of $\mathbb{CP}^1 \times E$, and the following open subsets

$$\widehat{U}_t := \bigcup_{j=1}^{P_\mathfrak{m}} \mathbb{D}(\widehat{x}_{t,j}, 1), \quad \widehat{V}_t := \bigcup_{k=1}^{Q_\mathfrak{m}} \mathbb{D}(\widehat{y}_{t,k}, 1)$$

of $\mathbb{C}$. For every possible j, k, ℓ, we put

$$\mathscr{A}_{t,j} := \mathrm{Area}_\omega\Big(F_t\Big(\mathbb{D}(\widehat{x}_{t,j}, 1) \cap \mathbb{D}_{R_t}\Big)\Big), \quad \mathscr{A}_{t,j}^{\mathcal{N}} := \int_0^{R_t} \mathrm{Area}_\omega\big(F_t\big(\mathbb{D}(\widehat{x}_{t,j}, 1) \cap \mathbb{D}_r\big)\big) \frac{dr}{r};$$

$$\mathscr{B}_{t,k} := \mathrm{Area}_\omega\Big(F_t\Big(\mathbb{D}(\widehat{y}_{t,k}, 1) \cap \mathbb{D}_{R_t}\Big)\Big), \quad \mathscr{B}_{t,k}^{\mathcal{N}} := \int_0^{R_t} \mathrm{Area}_\omega\big(F_t\big(\mathbb{D}(\widehat{y}_{t,k}, 1) \cap \mathbb{D}_r\big)\big) \frac{dr}{r}.$$

Proposition 3.1. *In every Step $t \geqslant 1$, there exists certain large integer $M_t \gg 1$ such that, if $\alpha_t \geqslant M_t$ and* $\max_j\{p_{t,j} \cdot \alpha_t\} \geqslant 1$, $\max_k\{q_{t,k} \cdot \alpha_t\} \geqslant 1$, *then the following estimates hold simultaneously.*

(♣). *On $\mathbb{D}_{R_{t-1}+1}$, $\widehat{F}_t$ is very close to $\widehat{F}_{t-1}$, more precisely*

$$\|\widehat{F}_t(z) - \widehat{F}_{t-1}(z)\|_{\mathbb{C}^2} < 2^{-t} \cdot \epsilon_{t-1} \quad \textit{for } z \in \mathbb{D}_{R_{t-1}+1}. \tag{3.8}$$

(♠). *The entire curve $F_t := \Pi \circ \widehat{F}_t$ satisfies the length–area conditions*

$$\frac{\mathrm{Length}_\omega(F_t(\mathbb{D}_{R_t}))}{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}))} < 2^{-t}, \quad \frac{\mathfrak{L}_{F_t}(R_t)}{\mathfrak{A}_{F_t}(R_t)} < 2^{-t}. \tag{3.9}$$

(♡). *One has*

$$\frac{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}) \setminus (U_t \cup V_t))}{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}))} < 2^{-t}, \tag{3.10}$$

$$\frac{1}{\mathfrak{A}_{F_t}(R_t)} \int_0^{R_t} \frac{dr}{r} \mathrm{Area}_\omega(F_t(\mathbb{D}_r) \setminus (U_t \cup V_t)) < 2^{-t}. \tag{3.11}$$

Proposition 3.2. *Let M_t be as in Proposition 3.1. In every Step $t \geqslant 1$, given*

$$\Omega_0 := [A_1 : \cdots : A_{P_\mathfrak{m}} : B_1 : \cdots : B_{Q_\mathfrak{m}}] \in \mathbb{RP}^{P_\mathfrak{m}+Q_\mathfrak{m}-1}$$

with all coordinates positive. There exist $\alpha_t, p_{t,\bullet}, q_{t,\bullet}$ satisfying $\alpha_t \geqslant M_t$ and $\max_j\{p_{t,j} \cdot \alpha_t\} \geqslant 1$, $\max_k\{q_{t,k} \cdot \alpha_t\} \geqslant 1$, *such that, on $\mathbb{RP}^{P_\mathfrak{m}+Q_\mathfrak{m}-1}$ with a distance* dist, *one has*

$$\mathrm{dist}\Big([\mathscr{A}_{t,1} : \cdots : \mathscr{A}_{t,P_\mathfrak{m}} : \mathscr{B}_{t,1} : \cdots : \mathscr{B}_{t,Q_\mathfrak{m}}],\, \Omega_0\Big) < 2^{-t},$$

$$\mathrm{dist}\Big([\mathscr{A}_{t,1}^{\mathcal{N}} : \cdots : \mathscr{A}_{t,P_\mathfrak{m}}^{\mathcal{N}} : \mathscr{B}_{t,1}^{\mathcal{N}} : \cdots : \mathscr{B}_{t,Q_\mathfrak{m}}^{\mathcal{N}}],\, \Omega_0\Big) < 2^{-t}.$$

Applying Proposition 3.2 with $\Omega_0 = [\mathbf{A}_{\mathfrak{m}} : \mathbf{B}_{\mathfrak{m}}]$, we can obtain some admissible parameters $\alpha_t, p_{t,\bullet}, q_{t,\bullet}$.

Finally, we finish the **Step** t by selecting a small positive error bound $\epsilon_t \ll \epsilon_{t-1}/2$, such that, for any mild holomorphic perturbation $\widehat{\mathbf{F}}$ of $\widehat{F}_t$ with $\|\widehat{\mathbf{F}} - \widehat{F}_t\|_{\mathbb{C}^2} < \epsilon_t$ on $\mathbb{D}_{R_t+1}$, after replacing F_t by $\mathbf{F} := \Pi \circ \widehat{\mathbf{F}}$ and enlarging 2^{-t} to 2^{-t+1} on the right-hand sides,

($\diamond$). all the corresponding estimates in Propositions 3.1 and 3.2 maintain.

The existence of such ϵ_t in the consideration ($\diamond$) can be proved by a *reductio ad absurdum* argument, as the $\mathscr{C}^0$-convergence of a sequence of holomorphic functions on a larger open set $\mathbb{D}_{R_t+1}$ can guarantee the $\mathscr{C}^1$-convergence on the smaller compact set $\overline{\mathbb{D}}_{R_t}$ by Cauchy's integral formula.

Let us we briefly explain the motivation. ($\clubsuit$) ensures that $\{\widehat{F}_t\}_{t\geqslant 1}$ converges to an entire curve $\widehat{\mathbf{F}} : \mathbb{C} \to \mathbb{C}^2$. ($\spadesuit$) can guarantee that $\mathbf{F} = \Pi \circ \widehat{\mathbf{F}}$ also satisfies a similar length–area condition, provided that $\{\epsilon_t\}_{t\geqslant 1}$ shrink to zero sufficiently fast. Similarly, ($\heartsuit$) is to force $\mathbf{F}$ to produce Nevanlinna/Ahlfors currents having singular parts supported in precise locations.

Now we sketch our idea of generating singular parts of Nevanlinna/Ahlfors currents. For any $\mathfrak{m} \geqslant 1$, write $\sigma^{-1}(\mathfrak{m})$ in the increasing order as $\{t_{\mathfrak{m},s}\}_{s\geqslant 1}$. By Propositions 3.1 and 3.2, we can check that any Nevanlinna/Ahlfors current $\mathcal{T}$ produced by some subsequence of $\{\mathbf{F}|_{\overline{\mathbb{D}}_{R_{t_{\mathfrak{m},s}}}}\}_{s\geqslant 1}$ has zero mass outside $\bigcup_{j=1}^{P_{\mathfrak{m}}}\{x_j\} \times E$ and $\mathbb{CP}^1 \times \bigcup_{k=1}^{Q_{\mathfrak{m}}}\{y_k\}$. Moreover, we can make sure that $\mathcal{T} = \mathcal{T}_{\mathrm{alg}}$ charges preassigned positive masses $\mathbf{A}_{\mathfrak{m}}$ and $\mathbf{B}_{\mathfrak{m}}$ on each of these irreducible algebraic curves.

3.2 | Proofs of Propositions 3.1 and 3.2

Proof of (3.8). Note that on $\mathbb{D}_{R_{t-1}+1}$, all $\varphi_{t,\bullet}, \psi_{t,\bullet}$ have moduli strictly less than 1, and $p_{t,\bullet}, q_{t,\bullet} \in (0, 1]$ by our construction. Thus, by letting $\alpha_t \geqslant M_t \gg 1$, from (3.6), (3.7), on $\mathbb{D}_{R_{t-1}+1}$ we have

$$|\widehat{g}_t - g^{\star}_{t-1}| < \epsilon_{t-1} \cdot 2^{-t-2}, \qquad |\widehat{h}_t - h^{\star}_{t-1}| < \epsilon_{t-1} \cdot 2^{-t-2}.$$

This combining with (3.3) finishes the proof. $\square$

To prove the condition ($\spadesuit$), we introduce the following notation

$$M := \max_{\overline{\mathbb{D}}_{R_t}} \left\{ |(g^{\star}_{t-1})'|, |(h^{\star}_{t-1})'| \right\}, \qquad m := \max_{\overline{\mathbb{D}}_{R_t} \cap \overline{\mathbb{D}}(\widehat{x}_{t,1},1)} |\varphi_{t,1}| > 1,$$

$$m' := \max_{\overline{\mathbb{D}}_{R_t}} |\varphi'_{t,1}| > 0, \qquad m_0 := \max_{\overline{\mathbb{D}}_{R_t} \setminus \mathbb{D}(\widehat{x}_{t,1},1)} |\varphi_{t,1}| < 1.$$

As every $\varphi_{t,j}$ for $j = 2, \dots, P_{\mathfrak{m}}$ is obtained by composing $\varphi_{t,1}$ with some rotation, we have

$$m = \max_{\overline{\mathbb{D}}_{R_t} \cap \overline{\mathbb{D}}(\widehat{x}_{t,j},1)} |\varphi_{t,j}|, \qquad m' = \max_{\overline{\mathbb{D}}_{R_t}} |\varphi'_{t,j}|, \qquad m_0 = \max_{\overline{\mathbb{D}}_{R_t} \setminus \mathbb{D}(\widehat{x}_{t,j},1)} |\varphi_{t,j}|.$$

Similar equalities also hold for the functions $\psi_{t,\bullet}$.

Convention. From now on, we use $O_t(1)$ to denote some positive constants independent of α_t (but can depend on $t, R_t, \varphi_{t,1}$, etc), which vary according to the context.

Proof of the first inequality of (3.9). Write $\wp^*\omega_{\mathrm{FS}} = \vartheta(z)\, idz \wedge d\bar{z}$ for some smooth function $\vartheta \geqslant 0$ on $\mathbb{C}$. As $\wp$ is doubly periodic with respect to the lattice Γ, so is ϑ. In particular, ϑ is bounded from above. By definition,

$$\mathrm{Length}_\omega\Big(F_t(\partial\mathbb{D}_{R_t})\Big) = \int_{\partial\mathbb{D}_{R_t}} |\widehat{F}_t^*\omega|^{1/2} \quad \text{and} \quad \mathrm{Area}_\omega\Big(F_t(\mathbb{D}_{R_t})\Big) = \int_{\mathbb{D}_{R_t}} \widehat{F}_t^*\omega,$$

where

$$\begin{aligned}\widehat{F}_t^*\omega &= \vartheta\left(\widehat{g}_t(z)\right) i\, d\widehat{g}_t(z) \wedge d\overline{\widehat{g}_t(z)} + i/2\, d\widehat{h}_t(z) \wedge d\overline{\widehat{h}_t(z)} \\ &= \vartheta\left(\widehat{g}_t(z)\right) |\widehat{g}_t{}'(z)|^2\, i\, dz \wedge d\bar{z} + |\widehat{h}_t'(z)|^2\, i/2\, dz \wedge d\bar{z}.\end{aligned} \tag{3.12}$$

Taking derivatives on (3.6), (3.7), we get

$$\widehat{g}_t{}' = (g_{t-1}^\star)' + \Psi_t' = (g_{t-1}^\star)' + \sum_{k=1}^{Q_{\mathrm{m}}} q_{t,k} \cdot \alpha_t\, \psi_{t,k}^{\alpha_t-1} \psi_{t,k}',$$

$$\widehat{h}_t' = (h_{t-1}^\star)' + \Phi_t' = (h_{t-1}^\star)' + \sum_{j=1}^{P_{\mathrm{m}}} p_{t,j} \cdot \alpha_t\, \varphi_{t,j}^{\alpha_t-1} \varphi_{t,j}'.$$

Note that $\varphi_{t,\bullet}, \psi_{t,\bullet}$ satisfy the estimates (3.1), (3.2), while $p_{t,\bullet}, q_{t,\bullet} \leqslant 1$. By much the same argument as (2.8), on $\overline{\mathbb{D}}_{R_t}$, for $\alpha_t \gg 1$, we have

$$|\Phi_t'| \lesssim \alpha_t m^{\alpha_t-1} m', \qquad |\Psi_t'| \lesssim \alpha_t m^{\alpha_t-1} m'. \tag{3.13}$$

Together with the boundedness of $\vartheta, (g_{t-1}^\star)', (h_{t-1}^\star)'$ on $\overline{\mathbb{D}}_{R_t}$, we obtain that

$$\mathrm{Length}_\omega\Big(F_t(\partial\mathbb{D}_{R_t})\Big) \lesssim R_t \cdot \alpha_t m^{\alpha_t-1} m'. \tag{3.14}$$

On the other hand, by Lemma 3.3, $\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t})) \gtrsim (m^{3/2})^{\alpha_t-1}$. Thus,

$$\frac{\mathrm{Length}_\omega(F_t(\partial\mathbb{D}_{R_t}))}{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}))} \leqslant O_t(1) \cdot \frac{\alpha_t m^{\alpha_t-1} m'}{(m^{3/2})^{\alpha_t-1}},$$

which tends to 0 as $\alpha_t \to \infty$. This concludes the proof. □

Lemma 3.3. *For $\alpha_t \gg 1$, for every possible j, k, one has*

$$\int_{\mathbb{D}(\widehat{x}_{t,j},1)\cap\mathbb{D}_{R_t}} (\pi\circ\widehat{h}_t)^*\omega_E \gtrsim p_{t,j}^2 \cdot \alpha_t^2 (m^{3/2})^{\alpha_t-1},$$

$$\int_{\mathbb{D}(\widehat{y}_{t,k},1)\cap\mathbb{D}_{R_t}} (\wp\circ\widehat{g}_t)^*\omega_{\mathrm{FS}} \gtrsim q_{t,k}^2 \cdot \alpha_t^2 (m^{3/2})^{\alpha_t-1}.$$

Proof. We only prove the second inequality, as the first one is simpler. Recall

Bloch's Theorem (see [3, p. 295]). Let g be a holomorphic function on a region containing $\mathbb{D}(z,r)$. Then $g(\mathbb{D}(z,r))$ contains a disc of radius $r|g'(z)|/72$.

For each $k = 1, \dots, Q_{\mathfrak{m}}$, we can select some point $z_k \in \mathbb{D}_{R_t} \cap \mathbb{D}(\hat{y}_{t,k}, 1)$ with $\psi'_{t,k}(z_k) \neq 0$ and $|\psi_{t,k}(z_k)| > m^{3/4}$. As $|\psi_{t,k'}| < 1$ for other $k' \neq k$ on $\mathbb{D}(\hat{y}_{t,k}, 1)$, we have

$$|\hat{g}_t{}'(z_k)| \geqslant |q_{t,k} \cdot \alpha_t \, \psi'_{t,k}(z_k) \, \psi_{t,k}^{\alpha_t - 1}(z_k)| - O_t(1) \gtrsim q_{t,k} \cdot \alpha_t (m^{3/4})^{\alpha_t - 1}.$$

Let $r_k := (R_t - |z_k|)/2$. By Bloch's theorem, $\hat{g}_t\big(\mathbb{D}(z_k, r_k)\big)$ contains a disc of radius $\gtrsim q_{t,k} \cdot \alpha_t (m^{3/4})^{\alpha_t - 1}$. In particular, this disc strictly contains at least $\gtrsim q_{t,k}^2 \cdot \alpha_t^2 (m^{3/2})^{\alpha_t - 1}$ copies of fundamental domains of the lattice Γ, while each copy contributes some constant positive area with respect to $\wp^* \omega_{\mathrm{FS}}$. Thus, we have

$$\int_{\mathbb{D}(z_k, r_k)} (\wp \circ \hat{g}_t)^* \, \omega_{\mathrm{FS}} \geqslant \int_{\hat{g}_t(\mathbb{D}(z_k, r_k))} \wp^* \, \omega_{\mathrm{FS}} \gtrsim q_{t,k}^2 \cdot \alpha_t^2 (m^{3/2})^{\alpha_t - 1}. \tag{3.15}$$

This ends the proof. □

Proof of the second inequality of (3.9). Recall the definition

$$\mathfrak{L}_{F_t}(R_t) := \int_0^{R_t} \mathrm{Length}_\omega(F_t(\partial \mathbb{D}_r)) \, \frac{dr}{r} \quad \text{and} \quad \mathfrak{A}_{F_t}(R_t) := \int_0^{R_t} \mathrm{Area}_\omega(F_t(\mathbb{D}_r)) \, \frac{dr}{r}.$$

By the same argument as (3.14), for every $r \leqslant R_t$, we have

$$\mathrm{Length}_\omega(F_t(\partial \mathbb{D}_r)) \lesssim r \cdot \alpha_t m^{\alpha_t - 1} m'.$$

Hence

$$\mathfrak{L}_{F_t}(R_t) \lesssim \int_0^{R_t} r \cdot \alpha_t m^{\alpha_t - 1} m' \, \frac{dr}{r} \lesssim R_t \cdot \alpha_t m^{\alpha_t - 1} m'.$$

Now we bound $\mathfrak{A}_{F_t}(R_t)$ from below. Take a k_0 such that $q_{t,k_0} \cdot \alpha_{k_0} \geqslant 1$. Let z_{k_0}, r_{k_0} be chosen as in the proof of Lemma 3.3. Set $R_t^- := |z_{k_0}| + r_{k_0} < R_t$. By (3.15), we have $\mathrm{Area}_\omega\big(F_t(\mathbb{D}_r)\big) \gtrsim (m^{3/2})^{\alpha_t - 1}$ for all $r \in [R_t^-, R_t]$. Hence,

$$\mathfrak{A}_{F_t}(R_t) \geqslant \int_{R_t^-}^{R_t} \mathrm{Area}_\omega(F_t(\mathbb{D}_r)) \, \frac{dr}{r} \gtrsim (m^{3/2})^{\alpha_t - 1}. \tag{3.16}$$

Thus, we conclude that

$$\frac{\mathfrak{L}_{F_t}(R_t)}{\mathfrak{A}_{F_t}(R_t)} \leqslant O_t(1) \cdot \frac{\alpha_t m^{\alpha_t - 1} m'}{(m^{3/2})^{\alpha_t - 1}},$$

which tends to 0 as $\alpha_t \to \infty$. This concludes the proof. □

Proofs of (3.10) and (3.11). If $z \in \mathbb{D}(\hat{x}_{t,j}, 1)$ for some j, then $|\psi_{t,k}(z)| \leqslant m_0 < 1$ for every k, as $\overline{\mathbb{D}}(\hat{y}_{t,k}, 1)$ and $\overline{\mathbb{D}}(\hat{x}_{t,j}, 1)$ are disjoint by our construction. This gives

$$|\hat{g}_t(z) - g_{t-1}^{\star}(z)| \leqslant \sum_{k=1}^{Q_{\mathfrak{m}}} q_{t,k} |\psi_{t,k}(z)|^{\alpha_t} \ll 1 \quad \text{as} \quad \alpha_t \gg 1.$$

This combining with (3.4) yields $\mathrm{dist}_{\omega_{\mathrm{FS}}}\left(x_j, \wp\circ\widehat{g}_t(z)\right) < 2\delta_t$. Similarly, for $z \in \mathbb{D}(\widehat{y}_{t,k}, 1)$, we have $\mathrm{dist}_{\omega_E}\left(y_k, \pi\circ\widehat{h}_t(z)\right) < 2\delta_t$ from (3.5). Therefore,

$$F_t\left(\mathbb{D}(\widehat{x}_{t,j}, 1)\right) \subset \mathbb{D}(x_j, 2\delta_t) \times E, \quad F_t\left(\mathbb{D}(\widehat{y}_{t,k}, 1)\right) \subset \mathbb{CP}^1 \times \mathbb{D}(y_k, 2\delta_t). \tag{3.17}$$

It follows that

$$\mathrm{Area}_\omega\Big(F_t(\mathbb{D}_{R_t}) \setminus (U_t \cup V_t)\Big) \leqslant \mathrm{Area}_\omega\Big(F_t\Big(\mathbb{D}_{R_t} \setminus (\widehat{U}_t \cup \widehat{V}_t)\Big)\Big).$$

Observe from (3.13) that, for $z \notin \widehat{U}_t \cup \widehat{V}_t$, both $|\widehat{g}_t'|$ and $|\widehat{h}_t'|$ are $O_t(1)$. Thus, (3.12) implies that $\mathrm{Area}_\omega\left(F_t\left(\mathbb{D}_{R_t} \setminus (\widehat{U}_t \cup \widehat{V}_t)\right)\right) \leqslant O_t(1)$. Plugging in Lemma 3.3, we conclude that

$$\frac{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}) \setminus (U_t \cup V_t))}{\mathrm{Area}_\omega(F_t(\mathbb{D}_{R_t}))} \lesssim \frac{O_t(1)}{(m^{3/2})^{\alpha_t - 1}}.$$

Whence (3.10) follows.

Repeating the same argument, we can likewise obtain that

$$\mathrm{Area}_\omega(F_t(\mathbb{D}_r) \setminus (U_t \cup V_t)) \leqslant r^2 \cdot O_t(1).$$

Hence,

$$\int_0^{R_t} \mathrm{Area}_\omega(F_t(\mathbb{D}_r) \setminus (U_t \cup V_t))\, \frac{dr}{r} \leqslant \int_0^{R_t} r \cdot O_t(1)\, dr \leqslant O_t(1).$$

By (3.16), we finish the proof of (3.11). □

It remains to prove Proposition 3.2. To achieve this, we state a lemma first. Define

$$\kappa_t := \int_{\mathbb{D}(\widehat{x}_{t,1}, 1) \cap \mathbb{D}_{R_t}} \left(\varphi_{t,1}^{\alpha_t}\right)^*(\pi^*\omega_E) \quad \text{and} \quad \kappa_t^{\mathcal{N}} := \int_0^{R_t} \int_{\mathbb{D}(\widehat{x}_{t,1}, 1) \cap \mathbb{D}_r} \left(\varphi_{t,1}^{\alpha_t}\right)^*(\pi^*\omega_E)\, \frac{dr}{r}.$$

Lemma 3.4. *There exists a constant $c > 0$ independent of $t \geqslant 1$, such that for every j, k,*

$$\mathscr{A}_{t,j} = p_{t,j}^2 \cdot \kappa_t + O_t(1) m^{\alpha_t - 1}, \quad \mathscr{B}_{t,k} = c\, q_{t,k}^2 \cdot \kappa_t + O_t(1) m^{\alpha_t - 1},$$

$$\mathscr{A}_{t,j}^{\mathcal{N}} = p_{t,j}^2 \cdot \kappa_t^{\mathcal{N}} + O_t(1) m^{\alpha_t - 1}, \quad \mathscr{B}_{t,k}^{\mathcal{N}} = c\, q_{t,k}^2 \cdot \kappa_t^{\mathcal{N}} + O_t(1) m^{\alpha_t - 1}.$$

Proof. We prove the equalities for $\mathscr{A}_{t,j}$ and $\mathscr{B}_{t,k}$. The others are similar. Using (3.1) and (3.2), and considering the order of m, we obtain

$$\mathscr{A}_{t,j} = \int_{\mathbb{D}(\widehat{x}_{t,j}, 1) \cap \mathbb{D}_{R_t}} \left(p_{t,j} \varphi_{t,j}^{\alpha_t}\right)^*(\pi^*\omega_E) + O_t(1) m^{\alpha_t - 1}$$

$$\mathscr{B}_{t,k} = \int_{\mathbb{D}(\widehat{y}_{t,k}, 1) \cap \mathbb{D}_{R_t}} \left(q_{t,k} \psi_{t,k}^{\alpha_t}\right)^*(\wp^*\omega_{\mathrm{FS}}) + O_t(1) m^{\alpha_t - 1}.$$

As $\widetilde{\wp}^*\omega_{\mathrm{FS}}$ and ω_E are real closed 2-forms on E, by the fact $\dim_{\mathbb{R}} H^2(E, \mathbb{R}) = 1$ of the top degree de Rham cohomology group of E, there exists a constant $c > 0$ and a smooth 1-form θ such that

$\widetilde{\wp}^*\omega_{\mathrm{FS}} = c\,\omega_E + \mathrm{d}\theta$. Here c is determined by the area relation $\int_E \widetilde{\wp}^*\omega_{\mathrm{FS}} = c \cdot \int_E \omega_E$. As $\widetilde{\wp}$ is a $2:1$ ramified covering, $\int_E \widetilde{\wp}^*\omega_{\mathrm{FS}} = 2\int_{\mathbb{CP}^1}\omega_{\mathrm{FS}} = 2$. Thus, $c = 2/\varrho$. Using that $\wp = \widetilde{\wp}\circ\pi$, we have

$$\wp^*\omega_{\mathrm{FS}} = \pi^*(\widetilde{\wp}^*\omega_{\mathrm{FS}}) = \pi^*(c\,\omega_E + \mathrm{d}\theta). \tag{3.18}$$

Hence,

$$\mathscr{B}_{t,k} = c\int_{\mathbb{D}(\hat{y}_{t,k},1)\cap\mathbb{D}_{R_t}} \left(q_{t,k}\psi_{t,k}^{\alpha_t}\right)^*(\pi^*\omega_E) + \int_{\mathbb{D}(\hat{y}_{t,k},1)\cap\mathbb{D}_{R_t}} \left(q_{t,k}\psi_{t,k}^{\alpha_t}\right)^*(\pi^*\mathrm{d}\theta) + O_t(1)m^{\alpha_t-1}. \tag{3.19}$$

Denote the middle term by $\eta_{t,k}$. Noting that $\varphi_{t,j}, \psi_{t,k}$ are obtained by composing $\varphi_{t,1}$ with some rotations, and that $\pi^*\omega_E$ is a constant (1,1)-form, we see that

$$\mathscr{A}_{t,j} = p_{t,j}^2 \cdot \kappa_t + O_t(1)m^{\alpha_t-1} \quad \text{and} \quad \mathscr{B}_{t,j} = q_{t,k}^2 \cdot \kappa_t + \eta_{t,k} + O_t(1)m^{\alpha_t-1}.$$

The proof will be done if we can show that $\eta_{t,k} = O_t(1)m^{\alpha_t-1}$. This just follows by using Stoke's formula and the same proof as (3.14). □

Proof of Proposition 3.2. Borrow the constant c in Lemma 3.4. Take a sufficiently small positive number c' such that all the parameters

$$p_{t,j} = c' \cdot \sqrt{A_j}, \quad q_{t,k} = c' \cdot \sqrt{B_k/c}$$

are in the interval $(0,1)$. By Lemma 3.4, the proposition follows from $\kappa_t, \kappa_t^{\mathcal{N}} \gtrsim m^{\alpha_t}$, which is a corollary of (3.15) and (3.16). □

3.3 | Proof of Theorem B

Read $\mathbf{A}_{\mathfrak{m}} = (A_1^{[\mathfrak{m}]}, \ldots, A_{P_{\mathfrak{m}}}^{[\mathfrak{m}]})$ and $\mathbf{B}_{\mathfrak{m}} = (B_1^{[\mathfrak{m}]}, \ldots, B_{Q_{\mathfrak{m}}}^{[\mathfrak{m}]})$.

Proposition 3.5. *For any $\mathfrak{m} \geqslant 1$, there exists an increasing sequence $\{t_s\}_{s\geqslant 1}$ in $\mathbb{Z}_+$ such that the sequence of holomorphic discs $\{\mathbf{F}\,|_{\overline{\mathbb{D}}_{R_{t_s}}}\}_{s\geqslant 1}$ produces Nevanlinna/Ahlfors current*

$$\mathcal{T} = \sum_{j=1}^{P_{\mathfrak{m}}} \frac{A_j^{[\mathfrak{m}]}}{\varrho} \cdot [\mathsf{E}_j] + \sum_{k=1}^{Q_{\mathfrak{m}}} B_k^{[\mathfrak{m}]} \cdot [\mathsf{R}_k],$$

where $\mathsf{E}_j := \pi_1^{-1}(x_j)$ are elliptic curves, and where $\mathsf{R}_k := \pi_2^{-1}(y_k)$ are rational curves.

Proof. As $\widehat{\mathbf{F}} = \lim_{j\to\infty}\widehat{F}_j$, for every $t \geqslant 1$, on $\mathbb{D}_{R_t+1}$ we have

$$\|\widehat{\mathbf{F}} - F_t\| \leqslant \sum_{j\geqslant t}\|F_{j+1} - F_j\| \leqslant \sum_{j\geqslant t} 2^{-j-1}\cdot\epsilon_j < \sum_{j\geqslant t} 2^{-j-1}\cdot\epsilon_t < \epsilon_t.$$

Hence, by the stability condition ($\diamond$), the entire curve $\mathbf{F} = \Pi\circ\widehat{\mathbf{F}}$ satisfies all the estimates in Propositions 3.1 and 3.2 after replacing 2^{-t} by 2^{-t+1}.

Write elements of $\sigma^{-1}(\mathfrak{m}) \subset \mathbb{Z}_+$ in the increasing order as $\{t_s\}_{s\geqslant 1}$. Due to the length–area estimate (3.9), after passing to certain subsequence of $\{\mathbf{F}\,|_{\overline{\mathbb{D}}_{R_{t_s}}}\}_{s\geqslant 1}$, we can obtain some Nevanlinna/Ahlfors current $\mathcal{T}$. These two currents happen to coincide. We only prove the Ahlfors part, while the argument works for the Nevanlinna part as well. By definition, the mass of $\mathcal{T}$ on $\mathsf{E}_{j_0} = \{x_{j_0}\} \times E$ is equal to

$$\int_{\{x_{j_0}\}\times E} \mathcal{T} \wedge \omega = \lim_{s\to\infty} \int_{\mathbb{D}(x_{j_0},2\delta_{t_s})\times E} \mathcal{T} \wedge \omega = \lim_{s\to\infty} \frac{\operatorname{Area}_\omega(\mathbf{F}[\mathbf{F}^{-1}(\mathbb{D}(x_{j_0},2\delta_{t_s})\times E)])}{\operatorname{Area}_\omega(\mathbf{F}(\mathbb{D}_{R_{t_s}}))}$$

Recall (3.17) that $\mathbf{F}\big(\mathbb{D}(\hat{x}_{t_s,j_0},1)\big) \subset \mathbb{D}(x_{j_0},2\delta_{t_s}) \times E$. It follows that

$$\int_{\{x_{j_0}\}\times E} \mathcal{T} \wedge \omega \geqslant \lim_{s\to\infty} \frac{\mathscr{A}_{t_s,j_0}}{\operatorname{Area}_\omega(\mathbf{F}(\mathbb{D}_{R_{t_s}}))} = \frac{A_{j_0}^{[\mathfrak{m}]}}{\sum_{j=1}^{P_\mathfrak{m}} A_j^{[\mathfrak{m}]} + \sum_{k=1}^{Q_\mathfrak{m}} B_k^{[\mathfrak{m}]}}$$

by Proposition 3.2 and (3.10). Similarly, the mass of $\mathcal{T}$ on $\mathsf{R}_{k_0} = \mathbb{CP}^1 \times \{y_{k_0}\}$ is

$$\int_{\mathbb{CP}^1\times\{y_{k_0}\}} \mathcal{T} \wedge \omega \geqslant \frac{B_{k_0}^{[\mathfrak{m}]}}{\sum_{j=1}^{P_\mathfrak{m}} A_j^{[\mathfrak{m}]} + \sum_{k=1}^{Q_\mathfrak{m}} B_k^{[\mathfrak{m}]}}.$$

As the total mass of $\mathcal{T}$ is 1, while $\mathcal{T}$ cannot charge positive mass on points $\{x_{j_0}\} \times \{y_{k_0}\}$, the above inequalities must be exactly equalities. Hence, we conclude the proof. □

Lemma 3.6. *Let $f : \mathbb{C} \to X$ be an entire curve which produces Nevanlinna/Ahlfors currents $\{\mathcal{T}_s\}_{s=1}^\infty$ by $\{f\,|_{\overline{\mathbb{D}}_r}\}_{r>0}$. If $\{\mathcal{T}_s\}_{s=1}^\infty$ converges weakly to a current $\mathcal{T}$, then $\mathcal{T}$ is also a Nevanlinna/Ahlfors current generated by $\{f\,|_{\overline{\mathbb{D}}_r}\}_{r>0}$.*

Proof. We only prove the case of Ahlfors currents, as the other case is similar. For each $s \geqslant 1$, let $\mathcal{T}_s$ be generated by the radii $\{r_{s,\ell}\}_{\ell=1}^\infty$. By throwing some smaller radii, we can assume that for all $\ell \geqslant 1$, the length–area ratios are small

$$\frac{\operatorname{Length}_\omega(f(\partial\mathbb{D}_{r_{s,\ell}}))}{\operatorname{Area}_\omega(f(\mathbb{D}_{r_{s,\ell}}))} < 2^{-s}. \tag{3.20}$$

In the Fréchet space $\mathcal{A}^{1,1}(X)$ of smooth $(1,1)$-forms on X, we can take a countable dense subset $\{\theta_j\}_{j=1}^\infty$. For each $j \geqslant 1$ and $\epsilon > 0$, we can choose a large integer $M_{j,\epsilon}$ such that $|\langle \mathcal{T}_s, \theta_j\rangle - \langle \mathcal{T}, \theta_j\rangle| < \epsilon$ for all $s \geqslant M_{j,\epsilon}$. Moreover, by definition of $\mathcal{T}_s$, we can find $N_{j,s,\epsilon} \gg 1$ such that for all $\ell \geqslant N_{j,s,\epsilon}$, there hold

$$\left| \frac{1}{\operatorname{Area}_\omega(f(\mathbb{D}_{r_{s,\ell}}))} \langle f_*[\mathbb{D}_{r_{s,\ell}}], \theta_j\rangle - \langle \mathcal{T}_s, \theta_j\rangle \right| < \epsilon.$$

Hence, for all $j \geqslant 1$, $\epsilon > 0$, $s \geqslant M_{j,\epsilon}$, $\ell \geqslant N_{j,s,\epsilon}$, we have

$$\left| \frac{1}{\operatorname{Area}_\omega(f(\mathbb{D}_{r_{s,\ell}}))} \langle f_*[\mathbb{D}_{r_{s,\ell}}], \theta_j\rangle - \langle \mathcal{T}, \theta_j\rangle \right| < 2\epsilon. \tag{3.21}$$

By the diagonal argument of the Arzelà–Ascoli Theorem, from $\{r_{s,\ell}\}_{s,\ell\geqslant 1}$ we can extract a sequence of increasing radii $\{r_k\}_{k=1}^{\infty}$ tending to infinity, such that the holomorphic discs $\{f\,|_{\overline{\mathbb{D}}_{r_k}}\}_{k\geqslant 1}$ have length–area ratios tending to zero as guaranteed by (3.20), and that they generate $\mathcal{T}$ as an Ahlfors current by (3.21). □

We have the following two observations.

Observation 1. As $\mathbb{Q}_+ \subset \mathbb{R}_{\geqslant 0}$ is dense, by Proposition 3.5 and Lemma 3.6, for all real numbers $a_j, b_k \geqslant 0$ with $\sum_{j=1}^{\infty} a_j + \sum_{k=1}^{\infty} b_k = 1$, $\{\mathbf{F}\,|_{\overline{\mathbb{D}}_r}\}_{r>0}$ can generate Nevanlinna/Ahlfors currents

$$\mathcal{T} = \sum_{j=1}^{\infty} \frac{a_j}{\varrho} \cdot \left[\pi_1^{-1}(x_j)\right] + \sum_{k=1}^{\infty} b_k \cdot \left[\pi_2^{-1}(y_k)\right].$$

Observation 2. As both $\{x_j\}_{j\geqslant 1} \subset \mathbb{CP}^1$ and $\{y_k\}_{k\geqslant 1} \subset E$ are dense, by **Observation 1** above and Lemma 3.6, for arbitrary points $\{x'_j\}_{j=1}^{\infty} \subset \mathbb{CP}^1$ and $\{y'_k\}_{k=1}^{\infty} \subset E$, for all real numbers $a_j, b_k \geqslant 0$ with $\sum_{j=1}^{\infty} a_j + \sum_{k=1}^{\infty} b_k = 1$, $\{\mathbf{F}\,|_{\overline{\mathbb{D}}_r}\}_{r>0}$ can generate Nevanlinna/Ahlfors currents

$$\mathcal{T} = \sum_{j=1}^{\infty} \frac{a_j}{\varrho} \cdot \left[\pi_1^{-1}(x'_j)\right] + \sum_{k=1}^{\infty} b_k \cdot \left[\pi_2^{-1}(y'_k)\right].$$

Proof of Theorem B. **Observation 2** above shows that **F** can produce all the shapes of Nevanlinna/Ahlfors currents with trivial diffuse part. It remains to show examples of the shape $\big(|J_{\mathsf{E}}| \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}, |J_{\mathsf{R}}| \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}, \mathcal{T}_{\mathrm{diff}}$ is nontrivial$\big)$.

For every $s \geqslant 1$, select some "equidistributed" points $\{w_{s,\ell}\}_{\ell=1}^{s} \subset E$ such that the sequence of averaged Dirac measures $\mu_s := \frac{1}{s}\sum_{\ell=1}^{s} \delta_{w_{s,\ell}}$ converges weakly to ω_E/ϱ as s tends to infinity. Choose arbitrary distinct points $\{x'_j\}_{j=1}^{|J_{\mathsf{E}}|} \subset \mathbb{CP}^1$, $\{y'_k\}_{k=1}^{|J_{\mathsf{R}}|} \subset E$. Take strictly positive real coefficients $\{a_j\}_{j=1}^{|J_{\mathsf{E}}|}, \{b_k\}_{k=1}^{|J_{\mathsf{R}}|}$ with total sum $\sum_{j=1}^{|J_{\mathsf{E}}|} a_j + \sum_{k=1}^{|J_{\mathsf{R}}|} b_k < 1$. By **Observation 2**, $\{\mathbf{F}\,|_{\overline{\mathbb{D}}_r}\}_{r>0}$ can generate Nevanlinna/Ahlfors currents

$$\mathcal{T}_s := \left(\sum_{j=1}^{\min\{s,|J_{\mathsf{E}}|\}} \frac{a_j}{\varrho} \cdot \left[\pi_1^{-1}(x'_j)\right] + \sum_{k=1}^{\min\{s,|J_{\mathsf{R}}|\}} b_k \cdot \left[\pi_2^{-1}(y'_k)\right] \right) + \left(\frac{c_s}{s} \cdot \sum_{\ell=1}^{s} \left[\pi_2^{-1}(w_{s,\ell})\right] \right), \tag{3.22}$$

where $c_s := 1 - \sum_{j=1}^{\min\{s,|J_{\mathsf{E}}|\}} a_j - \sum_{k=1}^{\min\{s,|J_{\mathsf{R}}|\}} b_k > 0$. As s tends to infinity, the first bracket $(\cdots)$ of (3.22) converges weakly to a singular current

$$\mathcal{T}_{\mathrm{alg}} := \sum_{j=1}^{|J_{\mathsf{E}}|} \frac{a_j}{\varrho} \cdot \left[\pi_1^{-1}(x'_j)\right] + \sum_{k=1}^{|J_{\mathsf{R}}|} b_k \cdot \left[\pi_2^{-1}(y'_k)\right],$$

while the second bracket $(\cdots)$ of (3.22) converges weakly to a nontrivial diffuse current $\mathcal{T}_{\mathrm{diff}}$. Indeed, it is easy to check that, for any smooth $(1,1)$-form η on X,

$$\langle \mathcal{T}_{\mathrm{diff}}, \eta \rangle = \left(1 - \sum_{j=1}^{|J_{\mathsf{E}}|} a_j - \sum_{k=1}^{|J_{\mathsf{R}}|} b_k \right) \cdot \frac{1}{\varrho} \int_X \eta \wedge \pi_2^* \omega_E.$$

Thus, by Lemma 3.6, we obtain a Nevanlinna/Ahlfors current $\mathcal{T} := \mathcal{T}_{\mathrm{alg}} + \mathcal{T}_{\mathrm{diff}}$ with the desired shape. □

ACKNOWLEDGMENTS

This work is inspired by insightful questions of Sibony, Yau, and Zhou. We thank Dinh Tuan Huynh, Bin Guo (AMSS), Lei Hou, Yi C. Huang for helpful suggestions which improved the exposition, and the referees for helpful comments. Song-Yan Xie is partially supported by National Key R&D Program of China Grant Numbers: 2021YFA1003100, 2023YFA1010500 and National Natural Science Foundation of China Grant Numbers: 12288201, 12471081.

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The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. R. Brody, *Compact manifolds and hyperbolicity*, Trans. Amer. Math. Soc. **235** (1978), 213–219.
2. M. Brunella, *Courbes entières et feuilletages holomorphes*, L'Enseignement Mathématique **45** (1999), 195–216.
3. J. B. Conway, *Functions of one complex variable*, 2nd ed., Graduate Texts in Mathematics, vol. 11, Springer, New York–Berlin, 1978.
4. T.-C. Dinh and N. Sibony, *Unique ergodicity for foliations in* $\mathbb{P}^2$ *with an invariant curve*, Invent. Math. **211** (2018), no. 1, 1–38.
5. T.-C. Dinh and D.-V. Vu, *Algebraic flows on commutative complex Lie groups*, Comment. Math. Helv. **95** (2020), no. 3, 421–460.
6. J. Duval, *Singularités des courants d'Ahlfors*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 3, 527–533.
7. J. Duval, *Sur le lemme de Brody*, Invent. Math. **173** (2008), no. 2, 305–314.
8. J. Duval, *Sur la théorie d'Ahlfors des surfaces*, Enseign. Math. **60** (2014), no. 3–4, 417–420.
9. J. Duval, *Around Brody Lemma*, Hyperbolicity properties of algebraic varieties, Panor. Synthèses, vol. 56, Soc. Math. France, Paris, 2021, pp. 1–12.
10. J. Duval and D. T. Huynh, *A geometric second main theorem*, Math. Ann. **370** (2018), no. 3–4, 1799–1804.
11. F. Forstnerič, *Stein manifolds and holomorphic mappings*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 56, Springer, Cham, 2017.
12. M. Green and P. Griffiths, *Two applications of algebraic geometry to entire holomorphic mappings*, The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, CA, 1979), Springer, New York–Berlin, 1980, pp. 41–74.
13. P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994, Reprint of the 1978 original.
14. M. Gromov, *Spaces and questions, Geom. Funct. Anal., Special Volume, Part I*, 2000, pp. 118–161.
15. D. T. Huynh, *Sur le Second Théorème Principal*, Ph.D. thesis, Orsay, 2016.
16. D. T. Huynh and D.-V. Vu, *On the set of divisors with zero geometric defect*, J. Reine Angew. Math. **771** (2021), 193–213.
17. D. T. Huynh and S.-Y. Xie, *On Ahlfors currents*, J. Math. Pures Appl. (9) **156** (2021), 307–327.
18. S. Lang, *Hyperbolic and diophantine analysis*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), no. 2, 159–205.
19. S. Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer, New York, 2002.
20. M. McQuillan, *Diophantine approximations and foliations*, Inst. Hautes Études Sci. Publ. Math. **87** (1998), 121–174.
21. M. McQuillan, *Integrating* $\partial\bar{\partial}$, Proceedings of the International Congress of Mathematicians, vol. I (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 547–554.
22. R. Nevanlinna, *Analytic functions*, Die Grundlehren der mathematischen Wissenschaften, vol. 162, Springer, New York–Berlin, 1970.
23. Y.-T. Siu, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. **27** (1974), 53–156.

24. C. Voisin, *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2002, Translated from the French original by Leila Schneps.
25. P. Vojta, *Diophantine approximation and Nevanlinna theory*, Arithmetic geometry, Lecture Notes in Mathematics, vol. 2009, Springer, Berlin, 2011, pp. 111–224.
26. J. Xie and X. Yuan, *The geometric Bombieri–Lang conjecture for ramified covers of abelian varieties*, arXiv:2308.08117, 2023.
27. S.-Y. Xie, *Entire curves producing distinct Nevanlinna currents*, Int. Math. Res. Not. IMRN **10** (2024), 8350–8362.