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On the ampleness of the cotangent bundles of complete intersections

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Abstract For the intersection family $\mathscr{X}$ of general Fermat-type hypersurfaces in $\mathbb{P}^N_{\mathbb{K}}$ defined over an algebraically closed field $\mathbb{K}$, we extend Brotbek's symmetric differential forms by a geometric approach, and we further exhibit unveiled families of lower degree symmetric differential forms on all possible intersections of $\mathscr{X}$ with coordinate hyperplanes. Thereafter, we develop what we call the '*moving coefficients method*' to prove a conjecture made by Olivier Debarre: *for a generic choice of $c \geqslant N/2$ hypersurfaces $H_1, \dots, H_c \subset \mathbb{P}^N_{\mathbb{C}}$ of degrees $d_1, \dots, d_c$ sufficiently large, the intersection $X := H_1 \cap \cdots \cap H_c$ has ample cotangent bundle Ω_X*, and concerning effectiveness, the lower bound $d_1, \dots, d_c \geqslant \prod_{l=\lceil N/2 \rceil}^{N} l^4 \left(l+1\right)^{2l}$ works. Lastly, thanks to known results about the Fujita Conjecture, we establish the very-ampleness of $\mathsf{Sym}^{\kappa}\,\Omega_X$ for all $\kappa \geqslant \kappa_0$, with a uniform lower bound $\kappa_0 = 64\left(\sum_{i=1}^{c} d_i\right)^2$.

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1 Introduction

The goal of this article is to answer the Debarre Ampleness Conjecture.

Theorem 1.1 *The cotangent bundle Ω_X of the complete intersection $X := H_1 \cap \cdots \cap H_c \subset \mathbb{P}^N_{\mathbb{C}}$ of $c \geqslant N/2$ generic hypersurfaces $H_1, \ldots, H_c$ with degrees $\geqslant \mathsf{d}(N)$ is ample, where:*

$$\mathsf{d}(N) := \prod_{l=\lceil N/2 \rceil}^{N} l^4 \left(l+1\right)^{2l} \leqslant N^{\frac{3}{4} N^2 + \mathrm{o}(N^2)}.$$

Precisely, ampleness means that, for all large $k \geqslant \mathrm{k}_0 \gg 1$, for all pairs of distinct points $x, y \in X$, the two evaluation maps:

$$\mathsf{H}^0\left(X,\, \mathsf{Sym}^k\, \Omega_X\right) \twoheadrightarrow \mathsf{Jet}_1\, \mathsf{Sym}^k\, \Omega_X\big|_x,$$
$$\mathsf{H}^0\left(X, \mathsf{Sym}^k\, \Omega_X\right) \twoheadrightarrow \mathsf{Sym}^k\, \Omega_X\big|_x \oplus \mathsf{Sym}^k\, \Omega_X\big|_y$$

are both surjective, where $\mathsf{Jet}_1\, E\big|_x := \mathscr{O}_x(E) \Big/ (\mathfrak{m}_x)^2\, \mathscr{O}_x(E)$ for every vector bundle E over X.

The hypothesis $c \geqslant N/2$ appears *optimal*, for otherwise $\mathsf{H}^0\big(X, \mathsf{Sym}^k\, \Omega_X\big) = 0$ for all $k \geqslant 1$, according to Brückmann-Rackwitz [3] and Schneider [15].

As highlighted in [4], projective algebraic varieties X with ample cotangent bundles have several interesting properties, for instance, the canonical line bundles of all subvarieties of X are ample; there are finitely many nonconstant rational maps from any fixed projective variety to X ([14]); if X is defined over $\mathbb{C}$, then every holomorphic map $\mathbb{C} \to X$ must be constant ([9]); if X is defined over a number field K, the K-rational points of X are conjectured by Lang to be finite ([10,13]).

In [1], Brotbek reached a proof of the Debarre Ampleness Conjecture for complex surfaces: $\dim_{\mathbb{C}} X = N - c = 2$. Recently, Brotbek [2] constructed explicit global symmetric differential forms in coordinates by a cohomological approach, and thereby proved the Debarre Ampleness Conjecture in the case $4c \geqslant 3\,N - 2$ for equal degrees $d_1 = \cdots = d_c \geqslant 2N + 3$.

In this article, we work over an algebraically closed field $\mathbb{K}$ with any characteristic. Our main theorem is the following, which coincides with Theorem 1.1 for $r = 0$ and $\mathbb{K} = \mathbb{C}$.

Theorem 1.2 (Ampleness) *For all positive integers $N \geqslant 1$, for any nonnegative integers $c,\ r \geqslant 0$ with $2c + r \geqslant N$, for all large integers $d_1, \ldots, d_c, d_{c+1}, \ldots, d_{c+r} \geqslant \mathsf{d}(N)$, for generic choices of $c + r$ hypersurfaces $H_1, \ldots, H_{c+r} \subset \mathbb{P}^N_{\mathbb{K}}$ with the corresponding degrees $d_1, \ldots, d_{c+r}$,* *the cotangent bundle* Ω_V *of the intersection of the first* c *hypersurfaces* $V := H_1 \cap \cdots \cap H_c$ *restricted to the intersection of all the* $c+r$ *hypersurfaces* $X := H_1 \cap \cdots \cap H_c \cap H_{c+1} \cap \cdots \cap H_{c+r}$ *is ample.*

Lastly, taking account Siu's result [16] about the Fujita Conjecture, we will prove in Sect. 13 the following

Theorem 1.3 (Effective Very Ampleness) *Under the same assumption and notation as in the Ampleness Theorem 1.2, if in addition the ambient field* $\mathbb{K}$ *has characteristic zero, then for generic choices of* $H_1, \ldots, H_{c+r}$*, the restricted cotangent bundle* $\mathsf{Sym}^{\kappa}\,\Omega_V\big|_X$ *is very ample on* X*, for every* $\kappa \geqslant \kappa_0$*, with the uniform lower bound* $\kappa_0 = 16\left(\sum_{i=1}^{c} d_i + \sum_{i=1}^{c+r} d_i\right)^2$.

Concerning organization, we provide basic preliminaries in Sects. 2, 3, 4. In Sect. 5, we present the proof blueprint of Theorem 1.2, which extends the previous method of [2] by adding new ingredients in four aspects as follows.

As a matter of fact, our initial idea is to reconstruct Brotbek's symmetric differential forms [2] by means of a geometric approach, which turns out to be fruitful in that, not only we obtain symmetric forms on intersections of general Fermat-type hypersurfaces, but also in that we discover unveiled symmetric forms on all possible further intersections with coordinate hyperplanes—see Sect. 6.

Fundamentally, our central idea is to design more 'flexible' hypersurfaces than the pure Fermat-type ones of [2], so that we can construct many more negatively twisted symmetric differential forms. To this aim, we develop the *moving coefficients method* (MCM) in Sect. 7. Then, we devote Sect. 8 to control the base locus of the obtained symmetric forms over the part where all coordinates are nonvanishing. Lastly, the ultimate difficulty lies in the *Core Lemma*, whose proof constitutes the technical heart of this article, and which will be accomplished in Sect. 11.

Playing an essential role, another key idea is to make use of symmetric differential forms constructed over the part where some coordinates vanish, a new feature which removes the last obstacle towards ampleness, and that therefore determines the shape of our desired hypersurfaces—see Sect. 9.

Surprisingly, one neat idea, which we term *product coup*, enables us to construct ample examples of intersections of hypersurfaces with any large degrees. This settles the Debarre Ampleness Conjecture and shapes the formulation of our main theorem—see Sect. 5.4.

Finally, we achieve the effective degree estimates of Theorem 1.2 in Sect. 12.

2 Restatements of the Ampleness Theorem 1.2

Recalling the projective parameter space of $c+r$ hypersurfaces with degrees $d_1, \dots, d_{c+r} \geqslant 1$ is:

$$\mathbb{P}\Big(\oplus_{i=1}^{c+r} \, \mathsf{H}^0\big(\mathbb{P}^N_{\mathbb{K}}, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(d_i)\big)\Big) = \mathbb{P}\left\{ \oplus_{i=1}^{c+r} \sum_{|\alpha|=d_i} A^i_\alpha \, z^\alpha \, : \, A^i_\alpha \in \mathbb{K} \right\},$$

for shortness we write it as:

$$\mathbb{P}^{\diamondsuit}_{\mathbb{K}} = \operatorname{Proj} \mathbb{K}\big[\{A^i_\alpha\}_{\substack{1\leqslant i\leqslant c+r \\ |\alpha|=d_i}}\big]. \tag{1}$$

First, we introduce the two subschemes:

$$\mathscr{X} \subset \mathscr{V} \subset \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}},$$

where $\mathscr{X}$ is the family of intersections of all the $c+r$ hypersurfaces:

$$\begin{aligned} \mathscr{X} := \mathsf{V}\Big(\sum_{|\alpha|=d_1} A^1_\alpha \, z^\alpha, \dots, \sum_{|\alpha|=d_c} A^c_\alpha \, z^\alpha, \\ \sum_{|\alpha|=d_{c+1}} A^{c+1}_\alpha \, z^\alpha, \dots, \sum_{|\alpha|=d_{c+r}} A^{c+r}_\alpha \, z^\alpha \Big), \end{aligned} \tag{2}$$

and where $\mathscr{V}$ is the family of intersections of the first c hypersurfaces:

$$\mathscr{V} := \mathsf{V}\Big(\sum_{|\alpha|=d_1} A^1_\alpha \, z^\alpha, \dots, \sum_{|\alpha|=d_c} A^c_\alpha \, z^\alpha \Big). \tag{3}$$

Now, let $\mathrm{pr}_1, \mathrm{pr}_2$ be the two canonical projections of $\mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}$ onto its two components. By restricting $\mathrm{pr}_1\big|_{\mathscr{V}} : \mathscr{V} \to \mathbb{P}^{\diamondsuit}_{\mathbb{K}}$, we receive the sheaf $\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}}$ of relative differentials of degree 1 of $\mathscr{V}$ over $\mathbb{P}^{\diamondsuit}_{\mathbb{K}}$, which should be viewed as the family of cotangent bundles of the intersections. Then, we consider the projectivization $\mathbf{P}\big(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}}\big)$, equipped with the *Serre line bundle* $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}})}(1)$ on it. Recalling the cotangent bundle $\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}$ of $\mathbb{P}^N_{\mathbb{K}}$ and its projectivization $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$, now looking at:

$$\begin{tikzcd}
\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}}) \arrow[r, hook, "\widetilde{i}"] \arrow[d] & \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \arrow[r, "\widetilde{\pi}_2"] \arrow[d] & \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \\
\mathscr{V} \arrow[r, hook, "i"] & \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}, &
\end{tikzcd} \tag{4}$$

we see that $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}})}(1)$ is the pullback of $\widetilde{\pi}_2^* \, \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ by $\widetilde{i}$, where $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ is the Serre line bundle of $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$.

Next, let $\widetilde{\pi}\colon \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \to \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}$ be the canonical projection, and let π_1, π_2 be the compositions of $\widetilde{\pi}$ with $\mathrm{pr}_1, \mathrm{pr}_2$:

$$\begin{tikzcd}
 & \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \arrow[d, "\widetilde{\pi}"] \arrow[ddl, "\pi_1 := \mathrm{pr}_1 \circ \widetilde{\pi}"'] \arrow[ddr, "\pi_2 := \mathrm{pr}_2 \circ \widetilde{\pi}"] & \\
 & \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}} \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\
\mathbb{P}^{\diamondsuit}_{\mathbb{K}} & & \mathbb{P}^N_{\mathbb{K}}.
\end{tikzcd} \tag{5}$$

Let:

$$\mathbf{P} \;:=\; \widetilde{\pi}^{-1}(\mathscr{X}) \cap \mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}}) \;\subset\; \mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}}) \;\subset\; \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \tag{6}$$

be the 'pullback' of $\mathscr{X} \subset \mathscr{V} \subset \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times_{\mathbb{K}} \mathbb{P}^N_{\mathbb{K}}$ under the map $\widetilde{\pi}$, and let:

$$\mathscr{O}_{\mathbf{P}}(1) \;:=\; \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}})}(1)\big|_{\mathbf{P}} \;=\; \widetilde{\pi}_2^* \, \mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)\big|_{\mathbf{P}} \tag{7}$$

be the restricted Serre line bundle. Then, Theorem 1.2 can be reformulated as below, with the assumption that the hypersurface degrees are sufficiently large $d_1, \ldots, d_{c+r} \gg 1$.

Theorem 1.2 (Version A) *For a generic point $t \in \mathbb{P}^{\diamondsuit}_{\mathbb{K}}$, over the fibre $\mathbf{P}_t := \pi_1^{-1}(t) \cap \mathbf{P}$, the restricted Serre line bundle $\mathscr{O}_{\mathbf{P}_t}(1) := \mathscr{O}_{\mathbf{P}}(1)\big|_{\mathbf{P}_t}$ is ample.*

We will abbreviate every closed point $t = \big[\{A^i_\alpha\}_{\substack{1 \leqslant i \leqslant c+r \\ |\alpha| = d}}\big] \in \mathbb{P}^{\diamondsuit}_{\mathbb{K}}$ as $t = [F_1 \colon \cdots \colon F_{c+r}]$, where $F_i := \sum_{|\alpha|=d_i} A^i_\alpha \, z^\alpha$. Then $\mathbf{P}_t = \{t\} \times_{\mathbb{K}} {}_{F_{c+1}, \ldots, F_{c+r}}\mathbf{P}_{F_1, \ldots, F_c}$ for a uniquely defined subscheme:

$$_{F_{c+1},\dots,F_{c+r}}\mathbf{P}_{F_1,\dots,F_c} \subset \mathbf{P}\big(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}\big). \tag{8}$$

Theorem 1.2 (Version B) *For a generic closed point* $[F_1 : \cdots : F_{c+r}] \in \mathbb{P}^{\diamondsuit}_{\mathbb{K}}$, *the Serre line bundle* $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ *is ample on* $_{F_{c+1},\dots,F_{c+r}}\mathbf{P}_{F_1,\dots,F_c}$.

To understand better the above statements, we now show the geometric picture.

3 The background geometry

For each scheme mentioned above, we consider its underlying topological space ($\mathbb{K}$-variety) that consists of all the closed points.

3.1 The geometry of $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ and $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$

Recalling the tangent space of $\mathbb{P}^N_{\mathbb{K}}$ at every point $[z]$ is $\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}\big|_{[z]} = \mathbb{K}^{N+1}\big/\mathbb{K}\cdot z$, thus the total tangent space of $\mathbb{P}^N_{\mathbb{K}}$ is $\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}} := \mathrm{T}_{\mathrm{hor}}\mathbb{K}^{N+1}\big/\sim$, where:

$$\mathrm{T}_{\mathrm{hor}}\mathbb{K}^{N+1} := \Big\{(z, [\xi]) : \; z \in \mathbb{K}^{N+1} \setminus \{0\} \text{ and } [\xi] \in \mathbb{K}^{N+1}\big/\mathbb{K}\cdot z\Big\}, \tag{9}$$

and where the quotient relation is $(z, [\xi]) \sim (\lambda z, [\lambda\xi])$, for all $\lambda \in \mathbb{K}^{\times}$.

Note that the $\mathbb{K}$-variety associated to $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ is just the projectivized tangent space $\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$. Also, the Serre line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ on $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ corresponds to the Serre line bundle $\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(1)$ on $\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$, whose dual is the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(-1)$ defined by:

$$\begin{aligned}\mathscr{O}_{\mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})}(-1)\big|_{([z,\xi])} &:= \mathbb{K}\cdot[\xi] \;\subset\; \mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}\big|_{[z]} \\ &= \mathbb{K}^{N+1}\Big/\mathbb{K}\cdot z \qquad (\forall\,([z,\xi]) \in \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})).\end{aligned}$$

3.2 The geometry of $\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}})$, $\mathbf{P}$ and $\mathbf{P}_t$

Recalling (3), the $\mathbb{K}$-variety of $\mathscr{V}$ is:

$$\mathbb{V} := \Big\{\big([F_1, \dots, F_{c+r}], [z]\big) \in \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times \mathbb{P}^N_{\mathbb{K}} : \; F_i(z) = 0, \; \forall\, i = 1 \dots c\Big\}.$$

Moreover, recalling (4), the $\mathbb{K}$-variety of $\mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}}) \subset \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times_{\mathbb{K}} \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$ is:

$$\begin{aligned}\mathbb{P}(\mathrm{T}_{\mathbb{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}}) := \Big\{\big([F_1, \dots, F_{c+r}], ([z], [\xi])\big) \in \mathbb{P}^{\diamondsuit}_{\mathbb{K}} \times \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}}) :\ & F_i(z) = 0, \\ & dF_i\big|_z(\xi) = 0,\ \forall\, i = 1 \dots c\Big\}.\end{aligned}$$

Similarly, the $\mathbb{K}$-variety $\mathbb{P} \subset \mathbb{P}(\mathrm{T}_{\mathbb{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}})$ associated to $\mathbf{P} \subset \mathbf{P}(\Omega^1_{\mathscr{V}/\mathbb{P}^{\diamondsuit}_{\mathbb{K}}})$ is:

$$\begin{aligned}\mathbb{P} := \Big\{\big([F_1, \dots, F_{c+r}], ([z], [\xi])\big) :\ & F_i(z) = 0, \\ & dF_j\big|_z(\xi) = 0,\ \forall\, i = 1 \dots c + r, \forall\, j = 1 \dots c\Big\},\end{aligned}$$

and the $\mathbb{K}$-variety ${}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c} \subset \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$ associated to (8) is:

$$\begin{aligned}{}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c} := \Big\{([z], [\xi]) :\ & F_i(z) = 0,\ dF_j\big|_z(\xi) = 0, \\ & \forall\, i = 1 \dots c + r,\ \forall\, j = 1 \dots c\Big\}.\end{aligned} \tag{10}$$

4 Some hints on the Ampleness Theorem 1.2

4.1 Ampleness is Zariski open

The foundation of our approach is the following theorem due to Grothendieck (cf. [6, III.4.7.1], [11, p. 29, Theorem 1.2.17]).

Theorem 4.1 (Amplitude in families) *Let $f : X \to T$ be a proper morphism of schemes, and let $\mathscr{L}$ be a line bundle on X. For every point $t \in T$, denote by $X_t := f^{-1}(t)$ and $\mathscr{L}_t := \mathscr{L}\big|_{X_t}$. If $\mathscr{L}_0$ is ample on X_0 for some point $0 \in T$, then there exists a Zariski open set $U \subset T$ containing 0 such that $\mathscr{L}_t$ is ample on X_t, for all $t \in U$.*

Note that in (5), $\pi_1 = \mathrm{pr}_1 \circ \pi$ is proper, and so is its restriction to $\mathbf{P}$. Therefore, by virtue of the above theorem, one only needs to find one point $t \in \mathbb{P}^{\diamondsuit}_{\mathbb{K}}$ such that $\mathscr{O}_{\mathbf{P}_t}(1)$ is ample on $\mathbf{P}_t$.

4.2 Largely twisted Serre line bundle is (very) ample

Let $\pi_0\colon\ \mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}}) \to \mathbb{P}^N_{\mathbb{K}}$ be the canonical projection. By a classical result [8, p. 161, Proposition 7.10], for all large integers $\ell \gg 1$, the twisted line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes \pi_0^*\, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell)$ is ample on $\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})$.

In fact, one can check by hand that, for all $\ell \geqslant 3$, the following global sections:

$$z_k\, z_j^{\ell-1}\, d\left(\frac{z_i}{z_j}\right) \qquad (i, j, k = 0 \cdots N, i \neq j) \tag{11}$$

even guarantee the very-ampleness.

Consequently, when $\ell \geqslant 3$, for every point $t \in \mathbb{P}_{\mathbb{K}}^{\diamond}$, the positively twisted Serre line bundle $\mathscr{O}_{\mathbf{P_t}}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}_{\mathbb{K}}^N}(\ell)$ is (very) ample on $\mathbf{P}_t = \pi_1^{-1}(t) \cap \mathbf{P}$.

4.3 Nefness of negatively twisted cotangent sheaf suffices

Thanks to the above ampleness, now we may relax the ampleness goal mentioned below Theorem 4.1.

Proposition 4.2 *For every point $t \in \mathbb{P}_{\mathbb{K}}^{\diamond}$, the following properties are equivalent.*

(i) *$\mathscr{O}_{\mathbf{P}_t}(1)$ is ample on $\mathbf{P}_t$.*
(ii) *There exist two positive integers $a, b \geqslant 1$ such that $\mathscr{O}_{\mathbf{P_t}}(a) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}_{\mathbb{K}}^N}(-b)$ is ample on $\mathbf{P}_t$.*
(iii) *There exist two positive integers $a, b \geqslant 1$ such that $\mathscr{O}_{\mathbf{P_t}}(a) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}_{\mathbb{K}}^N}(-b)$ is nef on $\mathbf{P}_t$.*

Proof (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is clear. To show (iii) $\Longrightarrow$ (i), noting that "ample $\otimes$ nef $=$ ample" (cf. [11, p. 53, Corollary 1.4.10]), one can use any ample positively twisted Serre line bundle to compensate the negative twisted degree of the nef line bundle. $\square$

Repeating the same reasoning, we may obtain:

Proposition 4.3 *For every point $t \in \mathbb{P}_{\mathbb{K}}^{\diamond}$, if $\mathscr{O}_{\mathbf{P_t}}(\ell_1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}_{\mathbb{K}}^N}(-\ell_2)$ is nef on $\mathbf{P}_t$ for some positive integers $\ell_1, \ell_2 \geqslant 1$, then for any positive integers $\ell_1', \ell_2' \geqslant 1$ with $\ell_2'/\ell_1' < \ell_2/\ell_1$, the twisted line bundle $\mathscr{O}_{\mathbf{P_t}}(\ell_1') \otimes \pi_2^* \mathscr{O}_{\mathbb{P}_{\mathbb{K}}^N}(-\ell_2')$ is ample on $\mathbf{P}_t$.* $\square$

By definition, the nefness of $\mathscr{S}_t^a(-b) := \mathscr{O}_{\mathbf{P_t}}(a) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}_{\mathbb{K}}^N}(-b)$ means that, for every irreducible curve $C \subset \mathbf{P}_t$, the intersection number $C \cdot \mathscr{S}_t^a(-b)$ is $\geqslant 0$. Recalling now a classical result [8, p. 295, Lemma 1.2], one only needs to show $\mathsf{H}^0\big(C, \mathscr{S}_t^a(-b)\big) \neq \{0\}$. To this end, one would desire sufficiently many global nonzero sections $s_1, \ldots, s_m \in \mathsf{H}^0\big(\mathbf{P}_t, \mathscr{S}_t^a(-b)\big)$ such that their base locus is empty or discrete, whence one of $s_1\big|_C, \ldots, s_m\big|_C$ suffices to conclude.

More flexibly, without fixing a and b, we have

Corollary 4.4 *Suppose that there exist some $m \geqslant 1$ nonzero sections:*

$$s_i \in \mathsf{H}^0\big(\mathbf{P}_t, \mathscr{S}_t^{a_i}(-b_i)\big) \qquad (i = 1 \cdots m; a_i, b_i \geqslant 1)$$

such that their base locus is discrete or empty, then for all positive integers a, b with:

$$b/a \leqslant \min\left\{b_1/a_1, \ldots, b_m/a_m\right\},$$

the negatively twisted Serre line bundle $\mathscr{S}_t^a(-b)$ is nef. □

5 A proof blueprint of the Ampleness Theorem 1.2

5.1 Main Nefness Theorem

Recalling Theorem 4.2, the Ampleness Theorem 1.2 is a consequence of the theorem below, whose effective bound $\mathtt{d}_0(\heartsuit)$ for $\heartsuit = 1$ will be given in Theorem 12.2.

Theorem 5.1 *Given any positive integer $\heartsuit \geqslant 1$, there exists a lower degree bound $\mathtt{d}_0(\heartsuit) \gg 1$ such that, for all degrees $d_1, \ldots, d_{c+r} \geqslant \mathtt{d}_0(\heartsuit)$, for a very generic*[1] *$t \in \mathbb{P}_{\mathbb{K}}^{\diamondsuit}$, the negatively twisted Serre line bundle $\mathscr{O}_{\mathbf{P}_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}_{\mathbb{K}}^N}(-\heartsuit)$ is nef on $\mathbf{P}_t$.*

It suffices to find one such $t \in \mathbb{P}_{\mathbb{K}}^{\diamondsuit}$ to guarantee'very generic' (cf. [11, p. 56, Proposition 1.4.14]).

We will prove Theorem 5.1 in two steps. At first, in Sect. 5.3, we sketch the proof in the central cases when all $c + r$ hypersurfaces are approximately of the same large degrees. Then, in Sect. 5.4, we play a *product coup* to embrace all large degrees.

5.2 A refined nefness criterion

In practice, it is delicate to gather enough global sections with empty/discrete base locus to guarantee the desired nefness. To overcome this difficulty, we will also use nonzero sections of the same bundle restricted to proper subvarieties, and such an idea will be a key feature of our approach compared with that of [2].

Definition 5.2 Let X be a variety, and let $Y \subset X$ be a subvariety. A line bundle $\mathscr{L}$ on X is said to be *nef outside* Y if, for every irreducible curve $C \subset X$ with $C \not\subset Y$, the intersection number $C \cdot \mathscr{L}$ is $\geqslant 0$.

A standard reasoning on irreducibility and Noetherian topology gives ([17, p. 16, Theorem 4.6])

[1] $t \in \mathbb{P}_{\mathbb{K}}^{\diamondsuit} \setminus \cup_{i=1}^{\infty} Z_i$ for some countable proper subvarieties $Z_i \subsetneq \mathbb{P}_{\mathbb{K}}^{\diamondsuit}$.

Lemma 5.3 *Let X be a Noetherian variety, and let $\mathscr{L}$ be a line bundle on X. Assume that there exists a set $\mathscr{P}$ of closed subvarieties of X satisfying:*

(i) $\emptyset \in \mathscr{P}$ *and* $X \in \mathscr{P}$*;*

(ii) *for every element $Y \in \mathscr{P}$ with $Y \neq \emptyset$, there exist finitely many elements $Z_1, \ldots, Z_m \in \mathscr{P}$ with $Z_1, \ldots, Z_m \subsetneqq Y$ such that the restricted line bundle $\mathscr{L}\big|_Y$ is nef outside the union $Z_1 \cup \cdots \cup Z_m$.*

Then $\mathscr{L}$ is nef on X. □

5.3 The central cases of relatively the same large degrees

Theorem 5.4 *For any fixed $c+r$ positive integers $\epsilon_1, \ldots, \epsilon_{c+r} \geqslant 1$, for every sufficiently large integer $d \gg 1$, Theorem 5.1 holds with $d_i = d + \epsilon_i$, $i = 1 \ldots c+r$.*

When $c+r \geqslant N$, generically X is discrete or empty, so there is nothing to prove. Assuming $c+r \leqslant N-1$, we now outline the proof.
Step 1 In the entire family of $c+r$ hypersurfaces with degrees $d+\epsilon_1, \ldots, d+\epsilon_{c+r}$, whose projective parameter space is $\mathbb{P}_{\mathbb{K}}^{\diamondsuit}$, we will work with some specific subfamily which suits our coming *moving coefficients method*, and whose projective parameter space will be denoted by $\mathbb{P}_{\mathbb{K}}^{\blacklozenge} \subset \mathbb{P}_{\mathbb{K}}^{\diamondsuit}$. The presentation of $\mathbb{P}_{\mathbb{K}}^{\blacklozenge}$ will be made partially in Sect. 7, and then completely in Sect. 9.2.

Recalling (2) and (5), we then consider the subfamily of intersections $\mathscr{Y} \subset \mathscr{X}$:

$$\mathrm{pr}_1^{-1}\,(\mathbb{P}_{\mathbb{K}}^{\blacklozenge}) \cap \mathscr{X} \;=:\; \mathscr{Y} \;\subset\; \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \times_{\mathbb{K}} \mathbb{P}_{\mathbb{K}}^{N} \;=\; \mathrm{pr}_1^{-1}\,(\mathbb{P}_{\mathbb{K}}^{\blacklozenge}).$$

Recalling (5), (6), we introduce the subscheme $\mathbf{P}' := \widetilde{\pi}^{-1}(\mathscr{Y}) \cap \mathbf{P} \subset \mathbf{P}$. By restriction, (5) yields the commutative diagram:

$$\begin{tikzcd}
 & \mathbf{P}' \arrow[d, "\widetilde{\pi}"] \arrow[ddl, "\pi_1 = \mathrm{pr}_1 \circ \widetilde{\pi}"'] \arrow[ddr, "\pi_2 = \mathrm{pr}_2 \circ \widetilde{\pi}"] & \\
 & \mathscr{Y} \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\
\mathbb{P}_{\mathbb{K}}^{\blacklozenge} & & \mathbb{P}_{\mathbb{K}}^{N}.
\end{tikzcd} \tag{12}$$

Introducing the restricted Serre line bundle $\mathscr{O}_{\mathbf{P}'}(1) := \mathscr{O}_{\mathbf{P}}(1)\big|_{\mathbf{P}'}$ over $\mathbf{P}'$, in order to establish Theorem 5.4, it suffices to prove the

Theorem 5.5 *For a generic closed point $t \in \mathbb{P}_{\mathbb{K}}^{\blacklozenge}$, the bundle $\mathscr{O}_{\mathbf{P}'_{\mathbf{t}}}(1) \otimes \pi_2^{*}\mathscr{O}_{\mathbb{P}_{\mathbb{K}}^{N}}(-\heartsuit)$ is nef on $\mathbf{P}'_t := \mathbf{P}_t$.*

Step 2 The central objects now are the *universal negatively twisted Serre line bundles*:

$$\mathscr{O}_{\mathbf{P}'}(a, b, -c) \;:=\; \mathscr{O}_{\mathbf{P}'}(a) \otimes \pi_1^* \,\mathscr{O}_{\mathbb{P}_{\mathbb{K}}^{\blacklozenge}}(b) \otimes \pi_2^* \,\mathscr{O}_{\mathbb{P}_{\mathbb{K}}^{N}}(-c),$$

where a, c are positive integers such that $c/a \geqslant \heartsuit$, and where b are any integers.

Using the moving coefficients method, we will firstly construct a series of global universal negatively twisted symmetric differential n-forms:

$$S_\ell \;\in\; \mathsf{H}^0\big(\mathbf{P}',\ \mathscr{O}_{\mathbf{P}'}(n, N, -\heartsuit_\ell)\big) \qquad (\ell = 1 \cdots ❉), \tag{13}$$

where $n := N - (c + r) \geqslant 1$ and all $\heartsuit_\ell / n \geqslant \heartsuit$, and where *we always use* the symbol '❉' to denote auxiliary positive integers, which vary according to the context.

Secondly, for every integer $1 \leqslant \eta \leqslant n - 1$, for every $0 \leqslant v_1 < \cdots < v_\eta \leqslant N$, considering:

$${}_{v_1,\ldots,v_\eta}\mathbf{P}' \;:=\; \mathbf{P}' \cap \pi_2^{-1}\{z_{v_1} = \cdots = z_{v_\eta} = 0\},$$

we will construct a series of universal negatively twisted symmetric differential $(n - \eta)$-forms:

$${}_{v_1,\ldots,v_\eta}S_\ell \;\in\; \mathsf{H}^0\big({}_{v_1,\ldots,v_\eta}\mathbf{P}',\ \mathscr{O}_{\mathbf{P}'}(n - \eta, N - \eta, -\,{}_{v_1,\ldots,v_\eta}\heartsuit_\ell)\big) \qquad (\ell = 1 \cdots ❉), \tag{14}$$

with all ${}_{v_1,\ldots,v_\eta}\heartsuit_\ell / (n - \eta) \geqslant \heartsuit$.

This step will be accomplished partially in Sect. 7, and then completely in Sect. 9, and thus determines the *Algorithm* of MCM in Sect. 12.1.

Step 3 From now on, we only consider the closed points of every scheme.

Firstly, we will control the base locus of all the global sections obtained in (13):

$$\mathsf{BS} := \text{Base Locus of } \{S_\ell\}_{1 \leqslant \ell \leqslant ❉} \;\subset\; \mathbf{P}' \tag{15}$$

by proving that, on the 'coordinates nonvanishing part' $\mathring{\mathbf{P}}' := \mathbf{P}' \cap \pi_2^{-1}\{z_0 \cdots z_N \neq 0\}$, there holds:

$$\dim \mathsf{BS} \cap \mathring{\mathbf{P}}' < \dim \mathbb{P}_{\mathbb{K}}^{\blacklozenge}. \tag{16}$$

Secondly, for $\eta = 1 \ldots n - 1$, we will control the base locus of all the sections obtained in (14):

$${}_{v_1,\ldots,v_\eta}\mathsf{BS} := \text{Base Locus of } \{{}_{v_1,\ldots,v_\eta}S_\ell\}_{1 \leqslant \ell \leqslant ❉} \;\subset\; {}_{v_1,\ldots,v_\eta}\mathbf{P}'.$$

by proving that, on ${}_{v_1,\ldots,v_\eta}\mathring{\mathbf{P}}' := {}_{v_1,\ldots,v_\eta}\mathbf{P}' \cap \pi_2^{-1}\{z_{r_0}\cdots z_{r_{N-\eta}} \neq 0\}$, where:

$$\{r_0, \ldots, r_{N-\eta}\} := \{0, \ldots, N\} \setminus \{v_1, \ldots, v_\eta\}, \tag{17}$$

there holds:

$$\dim\, {}_{v_1,\ldots,v_\eta}\mathsf{BS} \cap {}_{v_1,\ldots,v_\eta}\mathring{\mathbf{P}}' < \dim \mathbb{P}_{\mathbb{K}}^{\blacklozenge}. \tag{18}$$

This step will be performed mainly by the *Core Lemma* in Sect. 8.

Step 4 For the regular map $\pi_1\colon \mathbf{P}' \to \mathbb{P}_{\mathbb{K}}^{\blacklozenge}$, noting the dimension estimates (16), (18), applying now a classical theorem [7, p. 132, Theorem 11.12], we know that there exists a proper subvariety $\Sigma \subsetneqq \mathbb{P}_{\mathbb{K}}^{\blacklozenge}$ such that, for every closed point $t \in \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \setminus \Sigma$:

(i) the base locus of the restricted symmetric differential n-forms:

$$\mathsf{BS}_t := \text{Base Locus of } \left\{S_\ell(t) := S_\ell\big|_{\mathbf{P}'_t}\right\}_{1\leqslant \ell \leqslant \circledast} \subset \mathbf{P}'_t$$

is discrete or empty over the coordinates nonvanishing part:

$$\dim\, \mathsf{BS}_t \cap \mathring{\mathbf{P}}'_t \leqslant 0, \tag{19}$$

where $\mathring{\mathbf{P}}'_t := \mathring{\mathbf{P}}' \cap \pi_1^{-1}(t)$;

(ii) for $\eta = 1 \ldots n-1$, the base locus of the restricted symmetric differential $(n-\eta)$-forms:

$$\begin{aligned} {}_{v_1,\ldots,v_\eta}\mathsf{BS}_t := \text{Base Locus of } &\left\{{}_{v_1,\ldots,v_\eta}S_\ell(t) := {}_{v_1,\ldots,v_\eta}S_\ell\big|_{{}_{v_1,\ldots,v_\eta}\mathbf{P}'_t}\right\}_{1\leqslant \ell \leqslant \circledast} \\ &\subset {}_{v_1,\ldots,v_\eta}\mathbf{P}'_t \end{aligned}$$

is discrete or empty over the coordinates nonvanishing part:

$$\dim\, {}_{v_1,\ldots,v_\eta}\mathsf{BS}_t \cap {}_{v_1,\ldots,v_\eta}\mathring{\mathbf{P}}'_t \leqslant 0, \tag{20}$$

where ${}_{v_1,\ldots,v_\eta}\mathring{\mathbf{P}}'_t := {}_{v_1,\ldots,v_\eta}\mathring{\mathbf{P}}' \cap \pi_1^{-1}(t)$.

Lastly, for $\eta = n$, there exists a proper subvariety $\Sigma' \subsetneqq \mathbb{P}_{\mathbb{K}}^{\blacklozenge}$ such that, for every closed point $t \in \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \setminus \Sigma'$, the fibre $\mathscr{Y}_t := \mathscr{Y} \cap \mathrm{pr}_1^{-1}(t)$—corresponding to the intersection of the $c+r$ hypersurfaces defined by t in $\mathbb{P}_{\mathbb{K}}^N$—is of dimension $n = N - (c+r)$, and moreover all intersections:

$$\mathscr{Y}_t \cap \mathrm{pr}_2^{-1}\left({}_{v_1,\ldots,v_n}\mathbb{P}^N\right) = \{t\} \times {}_{v_1,\ldots,v_n}\mathscr{Y}_t \qquad (0 \leqslant v_1 < \cdots < v_n \leqslant N) \tag{21}$$

are just finitely many points.

Now, we shall conclude Theorem 5.5 for every closed point $t \in \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \setminus (\Sigma \cup \Sigma')$.

Proof of Theorem 5.5 For the line bundle $\mathscr{L} = \mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}_{\mathbb{K}}^N}(-\heartsuit)$ over the variety $\mathbf{P}'_t$, we claim that the set of subvarieties:

$$\mathscr{V} := \left\{ \emptyset,\ \mathbf{P}'_t,\ {}_{v_1,\ldots,v_\eta}\mathbf{P}'_t \right\}_{\substack{1 \leqslant \eta \leqslant n \\ 0 \leqslant v_1 < \cdots < v_\eta \leqslant N}}$$

satisfies the conditions of Lemma 5.3.

Indeed, firstly, recalling (19), the sections $\{S_\ell(t)\}_{\ell=1\ldots\maltese}$ have empty/discrete base locus over the coordinates nonvanishing part, i.e. outside $\cup_{j=0}^N {}_j\mathbf{P}'_t$. Hence, using an adaptation of Corollary 4.4, remembering $\heartsuit/1 \leqslant \min\{\heartsuit_\ell/n\}_{1 \leqslant \ell \leqslant \maltese}$, the line bundle $\mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N}(-\heartsuit)$ is nef outside $\cup_{j=0}^N {}_j\mathbf{P}'_t$.

Secondly, for every integer $\eta = 1 \ldots n-1$, recalling the dimension estimate (20), again by Corollary 4.4, remembering $\heartsuit/1 \leqslant \min\{{}_{v_1,\ldots,v_\eta}\heartsuit_\ell/(n-\eta)\}_{1 \leqslant \ell \leqslant \maltese}$, the line bundle $\mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N}(-\heartsuit)$ is nef on ${}_{v_1,\ldots,v_\eta}\mathbf{P}'_t$ outside $\cup_{j=0}^{N-\eta} {}_{v_1,\ldots,v_\eta,r_j}\mathbf{P}'_t$ (see (17)).

Lastly, for $\eta = n$, noting that under the projection $\pi : \mathbf{P}'_t \to \mathscr{Y}_t$, thanks to (21), every subvariety ${}_{v_1,\ldots,v_n}\mathbf{P}'_t$ contracts to discrete points ${}_{v_1,\ldots,v_n}\mathscr{Y}_t$, we see that on ${}_{v_1,\ldots,v_n}\mathbf{P}'_t$, the line bundle $\mathscr{O}_{\mathbf{P}'_t}(1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}_{\mathbb{K}}^N}(-\heartsuit) \cong \mathscr{O}_{\mathbf{P}'_t}(1)$ is not only nef, but also ample!

Summarizing the above three parts, by Theorem 5.3, we conclude the proof. $\square$

5.4 Product coup

We will use in an essential way Theorem 5.4 with all ϵ_i equal to either 1 or 2. To begin with, we need an elementary

Observation 5.6 *For all positive integers $d \geqslant 1$, every integer $d_0 \geqslant d^2 + d$ is a sum of nonnegative multiples of $d+1$ and $d+2$.* $\square$

Proof of Theorem 5.1 Take one sufficiently large integer d such that Theorem 5.4 holds for any integers $\epsilon_i \in \{1, 2\}$, $i = 1 \ldots c+r$. Now, the above observation says that all large degrees $d_1, \ldots, d_{c+r} \geqslant d^2 + d$ can be written as $d_i = p_i\,(d+1) + q_i\,(d+2)$, with some nonnegative integers $p_i, q_i \geqslant 0$, $i = 1 \ldots c+r$. Let $F_i := f_1^i \cdots f_{p_i}^i f_{p_i+1}^i \cdots f_{p_i+q_i}^i$ be a product of some p_i generic homogeneous polynomials $f_1^i, \ldots, f_{p_i}^i$ (to be chosen later) each of degree $d+1$ and of some q_i generic homogeneous polynomials $f_{p_i+1}^i, \ldots, f_{p_i+q_i}^i$ (to be chosen later) each of degree $d+2$, so that F_i has degree d_i.

Recalling (10), a point $([z],[\xi]) \in \mathbb{P}(\mathrm{T}_{\mathbb{P}^N_{\mathbb{K}}})$ lies in ${}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c}$ if and only if:

$$F_i(z) = 0,\ dF_j\big|_z(\xi) = 0 \qquad (\forall\, i = 1 \cdots c{+}r, \forall\, j = 1 \cdots c).$$

Note that, for every $j = 1 \dots c$, the pair of equations:

$$F_j(z) = 0,\ dF_j\big|_z(\xi) = 0 \tag{22}$$

is equivalent to either:

$$\exists\, 1 \leqslant v_j \leqslant p_j + q_j \quad s.t. \quad f^j_{v_j}(z) = 0,\ df^j_{v_j}\big|_z(\xi) = 0, \tag{23}$$

or to:

$$\exists\, 1 \leqslant w^1_j < w^2_j \leqslant p_j + q_j \quad s.t. \quad f^j_{w^1_j}(z) = 0,\ f^j_{w^2_j}(z) = 0. \tag{24}$$

Therefore, $([z],[\xi]) \in {}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c}$ is equivalent to say that there exists a subset $\{i_1,\dots,i_k\} \subset \{1,\dots,c\}$ of cardinality k ($k = 0$ for $\emptyset$) such that, firstly, for every index $j \in \{i_1 \dots i_k\}$, (z,ξ) is a solution of (22) of type (23), secondly, for every index $j \in \{1,\dots,c\} \setminus \{i_1 \dots i_k\}$, (z,ξ) is a solution of (22) of type (24), and lastly, for every $j = c+1 \dots c+r$, one of $f^j_1,\dots,f^j_{p_j+q_j}$ vanishes at z. Thus, we see that the variety ${}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c}$ actually decomposes into a union of subvarieties:

$$\begin{aligned}{}_{F_{c+1},\dots,F_{c+r}}\mathbb{P}_{F_1,\dots,F_c} = & \cup_{k=0\cdots c} \cup_{1 \leqslant i_1 < \cdots < i_k \leqslant c} \cup_{\substack{1 \leqslant v_{i_j} \leqslant p_{v_j} + q_{v_j} \\ j = 1 \dots k}} \\ & \cup_{\substack{\{r_1,\dots,r_{c-k}\} = \{1,\dots,c\} \setminus \{i_1,\dots,i_k\} \\ 1 \leqslant w^1_{r_l} < w^2_{r_l} \leqslant p_{r_l} + q_{r_l} \\ l = 1 \dots c-k}} \cup_{\substack{1 \leqslant u_j \leqslant p_j + q_j \\ j = c+1 \dots c+r}} \\ & {}_{f^{r_1}_{w^1_{r_1}}, f^{r_1}_{w^2_{r_1}}, \dots, f^{r_{c-k}}_{w^1_{r_{c-k}}}, f^{r_{c-k}}_{w^2_{r_{c-k}}}, f^{c+1}_{u_{c+1}}, \dots, f^{c+r}_{u_{c+r}}}\mathbb{P}_{f^{i_1}_{v_{i_1}}, \dots, f^{i_k}_{v_{i_k}}}.\end{aligned}$$

Similarly, we can show that the scheme ${}_{F_{c+1},\dots,F_{c+r}}\mathbf{P}_{F_1,\dots,F_c}$ also decomposes into a union of subschemes:

$$\begin{aligned}{}_{F_{c+1},\dots,F_{c+r}}\mathbf{P}_{F_1,\dots,F_c} = & \cup_{k=0\cdots c} \cup_{1 \leqslant i_1 < \cdots < i_k \leqslant c} \cup_{\substack{1 \leqslant v_{i_j} \leqslant p_{v_j} + q_{v_j} \\ j = 1 \dots k}} \\ & \cup_{\substack{\{r_1,\dots,r_{c-k}\} = \{1,\dots,c\} \setminus \{i_1,\dots,i_k\} \\ 1 \leqslant w^1_{r_l} < w^2_{r_l} \leqslant p_{r_l} + q_{r_l} \\ l = 1 \dots c-k}} \cup_{\substack{1 \leqslant u_j \leqslant p_j + q_j \\ j = c+1 \dots c+r}} \\ & {}_{f^{r_1}_{w^1_{r_1}}, f^{r_1}_{w^2_{r_1}}, \dots, f^{r_{c-k}}_{w^1_{r_{c-k}}}, f^{r_{c-k}}_{w^2_{r_{c-k}}}, f^{c+1}_{u_{c+1}}, \dots, f^{c+r}_{u_{c+r}}}\mathbf{P}_{f^{i_1}_{v_{i_1}}, \dots, f^{i_k}_{v_{i_k}}}.\end{aligned}$$

Note that, for each subscheme on the right hand side, the number of polynomials on the lower-left of '$\mathbf{P}$' is $\#_L = 2(c-k)+r$, and the number of polynomials on the lower-right is $\#_R = k$, whence $2\#_R + \#_L = 2c + r \geqslant N$. Now, applying Theorem 5.4, we can choose one $\{f_{\bullet}^{\bullet}\}_{\bullet,\bullet}$ such that the twisted Serre line bundle $\mathscr{O}_{\mathbf{P}(\Omega^1_{\mathbb{P}^N_{\mathbb{K}}})}(1) \otimes \pi_0^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-\heartsuit)$ is nef on each subscheme ${}_{f^{r_1}_{w^1_{r_1}}, f^{r_1}_{w^2_{r_1}}, \dots, f^{r_{c-k}}_{w^1_{r_{c-k}}}, f^{r_{c-k}}_{w^2_{r_{c-k}}}, f^{c+1}_{u_{c+1}}, \dots, f^{c+r}_{u_{c+r}}}\mathbf{P}_{f^{i_1}_{v_{i_1}}, \dots, f^{i_k}_{v_{i_k}}}$, and therefore is also nef on their union ${}_{F_{c+1},\dots,F_{c+r}}\mathbf{P}_{F_1,\dots,F_c}$. Since nefness is a very generic property in family, we conclude the proof. $\square$

6 Generalizations of Brotbek's symmetric differential forms

6.1 Preliminaries on symmetric differential forms in projective space

For three fixed integers $N \geqslant 2$, c, $r \geqslant 0$ such that $2c+r \geqslant N$ and $c+r \leqslant N-1$, for $c+r$ positive integers $d_1, \dots, d_{c+r}$, for all $i = 1 \dots c+r$, let $H_i \subset \mathbb{P}^N_{\mathbb{K}}$ be $c+r$ hypersurfaces defined by some degree d_i homogeneous polynomials $F_i \in \mathbb{K}[z_0, \dots, z_N]$. Let V be the intersection of the first c hypersurfaces:

$$V := H_1 \cap \cdots \cap H_c = \Big\{[z] \in \mathbb{P}^N_{\mathbb{K}} : F_i(z) = 0, \ \forall\, i = 1 \dots c\Big\}, \tag{25}$$

and let X be the intersection of all the $c+r$ hypersurfaces:

$$X := H_1 \cap \cdots \cap H_{c+r} = \Big\{[z] \in \mathbb{P}^N_{\mathbb{K}} : F_i(z) = 0, \ \forall\, i = 1 \dots c+r\Big\}. \tag{26}$$

It is well known that, for generic choices of $\{F_i\}_{i=1}^{c+r}$, the intersections $V = \cap_{i=1}^{c} H_i$ and $X = \cap_{i=1}^{c+r} H_i$ are both smooth complete, and we shall assume this henceforth. Set:

$$n := N - (c+r) = \mathsf{dim}\, X,$$

and observe that $1 \leqslant n \leqslant c$.

Let us denote by $\pi : \mathbb{K}^{N+1} \setminus \{0\} \to \mathbb{P}^N_{\mathbb{K}}$ the canonical projection. For every $k \in \mathbb{Z}$, over any Zariski open subset $U \subset \mathbb{P}^N_{\mathbb{K}}$, sections in $\mathsf{H}^0\big(U, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(k)\big)$ can be defined as regular functions $\widehat{f}$ on $\pi^{-1}(U)$ satisfying $\widehat{f}(\lambda z) = \lambda^k \widehat{f}(z)$, for all $z \in \pi^{-1}(U)$ and all $\lambda \in \mathbb{K}^{\times}$.

For the cone $\widehat{V} := \pi^{-1}(V) = \big\{z \in \mathbb{K}^{N+1} \setminus \{0\} : F_i(z) = 0, \forall\, i = 1 \dots c\big\}$ of V, we define the horizontal tangent bundle $\mathrm{T}_{\mathrm{hor}}\widehat{V}$ which has fibre at any point $z \in \widehat{V}$:

$$\mathrm{T}_{\mathrm{hor}}\widehat{V}\big|_z = \Big\{[\xi] \in \mathbb{K}^{N+1} \big/ \mathbb{K} \cdot z : dF_i\big|_z(\xi) = 0, \ \forall\, i = 1 \dots c\Big\}.$$

Its total space is:

$$\mathrm{T}_{\mathrm{hor}}\widehat{V} := \Big\{(z,[\xi]) \,:\, z\in\widehat{V},\ [\xi]\in\mathbb{K}^{N+1}\Big/\mathbb{K}\cdot z,\ dF_i\big|_z(\xi)=0,\ \forall\, i=1\ldots c\Big\}. \tag{27}$$

Thus the total tangent bundle T_V of V can be viewed as:

$$\mathrm{T}_V = \mathrm{T}_{\mathrm{hor}}\widehat{V}/\sim, \qquad \text{where } (z,[\xi])\sim(\lambda z,[\lambda\xi]),\ \forall\,\lambda\in\mathbb{K}^{\times}.$$

Let Ω_V be the dual bundle of T_V, and let $\Omega_{\mathrm{hor}}\widehat{V}$ be the dual bundle of $\mathrm{T}_{\mathrm{hor}}\widehat{V}$. For all $l\geqslant 1$ and for all $\heartsuit\in\mathbb{Z}$, we denote by $\mathsf{Sym}^l\,\Omega_V(\heartsuit) := \mathsf{Sym}^l\,\Omega_V\otimes\mathscr{O}_V(\heartsuit)$ the twisted symmetric cotangent bundle.

Proposition 6.1 *Let $Y\subset V$ be any regular subvariety. For every Zariski open set $U\subset Y$ together with its cone $\widehat{U}:=\pi^{-1}(U)$, there is a canonical injection:*

$$\mathsf{H}^0\big(U,\mathsf{Sym}^l\,\Omega_V(\heartsuit)\big)\hookrightarrow\mathsf{H}^0\big(\widehat{U},\mathsf{Sym}^l\,\Omega_{\mathrm{hor}}\widehat{V}\big),$$

whose image is the set of sections Φ satisfying:

$$\Phi\big(\lambda z,[\lambda\xi]\big)=\lambda^{\heartsuit}\,\Phi\big(z,[\xi]\big), \tag{28}$$

for all $z\in\widehat{U}$, for all $[\xi]\in\mathrm{T}_{\mathrm{hor}}\widehat{V}\big|_z$ and for all $\lambda\in\mathbb{K}^{\times}$. □

In future applications, $Y=X$ or $Y=X\cap\{z_{v_1}=0\}\cap\cdots\cap\{z_{v_\eta}=0\}$ for some vanishing coordinate indices $0\leqslant v_1<\cdots<v_\eta\leqslant N$.

6.2 Symmetric horizontal differential forms

Now, we introduce the *Fermat-type hypersurfaces* $H_i\subset\mathbb{P}^N_{\mathbb{K}}$ defined by some homogeneous polynomials F_i of the form:

$$F_i=\sum_{j=0}^{N}A_i^j\,z_j^{\lambda_j}\qquad(i=1\ldots c+r), \tag{29}$$

where $\lambda_0,\ldots,\lambda_N\geqslant 2$ are some positive integers and where $A_i^j\in\mathbb{K}[z_0,z_1,\ldots,z_N]$ are some homogeneous polynomials with $\deg A_i^j+\lambda_j=\deg F_i$ for $j=0\cdots N$.

Differentiating F_i for $i=1\ldots c$, we receive:

$$\mathrm{d}\,F_i=\sum_{j=0}^{N}\mathsf{B}_i^j\,z_j^{\lambda_j-1}\qquad(i=1\ldots c),$$

where $\mathsf{B}_i^j := z_j \,\mathrm{d}\, A_i^j + \lambda_j\, A_i^j \,\mathrm{d}\, z_j$. Now, set $\widetilde{\mathsf{A}}_i^j := A_i^j\, z_j$, so that $F_i = \sum_{j=0}^N \widetilde{\mathsf{A}}_i^j\, z_j^{\lambda_j - 1}$ has the same structure as $\mathrm{d}\, F_i$.

We denote the cone of X by:

$$\widehat{X} := \Big\{ z \in \mathbb{K}^{N+1} \setminus \{0\} \,:\; F_i(z) = 0,\; \forall\, i = 1 \dots c + r \Big\}.$$

For all $z \in \widehat{X}$ and for all $[\xi] \in \mathrm{T}_{\mathrm{hor}} \widehat{V}\big|_z$, by the definition (27) of $\mathrm{T}_{\mathrm{hor}} \widehat{V}$, we have:

$$\begin{cases} \sum_{j=0}^N \widetilde{\mathsf{A}}_i^j\, z_j^{\lambda_j - 1}(z) = 0 & (i = 1 \dots c+r), \\ \sum_{j=0}^N \mathsf{B}_i^j(z, \xi)\, z_j^{\lambda_j - 1}(z) = 0 & (i = 1 \dots c). \end{cases} \tag{30}$$

Now, we view these $c+r+c$ equations as a linear system with $N+1$ unknowns $z_0^{\lambda_0 - 1}, \dots, z_N^{\lambda_N - 1}$:

$$\mathsf{C} \begin{pmatrix} z_0^{\lambda_0 - 1} \\ \vdots \\ z_N^{\lambda_N - 1} \end{pmatrix} = 0, \qquad \text{where} \qquad \mathsf{C} := \begin{pmatrix} \widetilde{\mathsf{A}}_1^0 & \cdots & \widetilde{\mathsf{A}}_1^N \\ \vdots & & \vdots \\ \widetilde{\mathsf{A}}_{c+r}^0 & \cdots & \widetilde{\mathsf{A}}_{c+r}^N \\ \mathsf{B}_1^0 & \cdots & \mathsf{B}_1^N \\ \vdots & & \vdots \\ \mathsf{B}_c^0 & \cdots & \mathsf{B}_c^N \end{pmatrix}. \tag{31}$$

Next, let D be the upper $(c+r+n) \times (N+1) = N \times (N+1)$ submatrix of C. For every $j = 0 \cdots N$, let $\widehat{\mathsf{D}}_j$ denote the submatrix of D obtained by omitting the $(j+1)$th column, and let D_{j} denote the $(j+1)$th column of D. Introduce the affine $W_j := \{z_j \neq 0\} \subset \mathbb{P}_{\mathbb{K}}^N$ having cone $\widehat{W}_j := \pi^{-1}(W_j) \subset \mathbb{K}^{N+1} \setminus \{0\}$. Denote also $U_j := W_j \cap X$, whose cone is $\widehat{U}_j := \pi^{-1}(U_j) \subset \widehat{X}$. Lastly, let $\Omega_{\mathrm{hor}} \mathbb{K}^{N+1}$ be the dual bundle of $\mathrm{T}_{\mathrm{hor}} \mathbb{K}^{N+1}$, and let $\Omega\, \mathbb{K}^{N+1}$ be the cotangent bundle of $\mathbb{K}^{N+1}$.

Proposition 6.2 *For every $j = 0 \cdots N$, on the affine set $\widehat{W}_j$, the following affine symmetric horizontal differential n-form is well defined:*

$$\widehat{\omega}_j := \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det\big(\widehat{\mathsf{D}}_j\big) \;\in\; \mathsf{H}^0\Big(\widehat{W}_j,\, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}} \mathbb{K}^{N+1}\Big). \tag{32}$$

Proof It suffices to treat the case $j = 0$. First of all, we may view $\widehat{\omega}_0 \in \mathsf{H}^0\Big(\widehat{V}_0,\, \mathsf{Sym}^n\, \Omega\, \mathbb{K}^{N+1}\Big)$, and in order to show $\widehat{\omega}_0 \in \mathsf{H}^0\big(\widehat{V}_0,\, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}} \mathbb{K}^{N+1}\big)$, we only have to check that:

$$\widehat{\omega}_0(z, \xi + \mu z) = \widehat{\omega}_0(z, \xi) \qquad {\scriptstyle (\forall z \in \widehat{V}_0,\ \xi \in \mathbb{K}^{N+1},\ \mu \in \mathbb{K}^\times)}. \tag{33}$$

Coming back to the definitions of B_i^j and $\widetilde{\mathsf{A}}_i^j$, Euler's identity gives:

$$\mathsf{B}_i^j(z, \xi + \mu z) \ = \ \mathsf{B}_i^j(z, \xi) + \mu \left(\lambda_j + \deg A_i^j\right) \widetilde{\mathsf{A}}_i^j(z) \qquad {\scriptstyle (i = 1 \cdots c,\ j = 0 \cdots N)}.$$

Therefore, the matrix $\widehat{\mathsf{D}}_0\,(z, \xi + \lambda z)$ not only has the same first $c + r$ rows as the matrix $\widehat{\mathsf{D}}_0\,(z, \xi)$, but also for every $\ell = 1 \dots n$, the $(c + r + \ell)$th row of the former one is equal to the $(c + r + \ell)$th row of the latter one plus a multiple of the ℓth row. Therefore both matrices have the same determinant, which verifies (33). □

Cramer's Rule *In a commutative ring R, for all positive integers $N \geqslant 1$, let $A^0, A^1, \dots, A^N \in R^N$ be $(N + 1)$ column vectors, and suppose that $z_0, z_1, \dots, z_N \in R$ satisfy:*

$$A^0\, z_0 + A^1\, z_1 + \cdots + A^N\, z_N \ = \ \mathbf{0}.$$

Then for all $0 \leqslant i, j \leqslant N$, there hold the identities:

$$(-1)^j \det\left(A^0, \dots, \widehat{A^j}, \dots, A^N\right) z_i = (-1)^i \det\left(A^0, \dots, \widehat{A^i}, \dots, A^N\right) z_j.$$

□

We may thus glue horizontal symmetric differential forms.

Proposition 6.3 *The following $(N+1)$ affine symmetric horizontal differential n-forms:*

$$\widehat{\omega}_j := \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det\left(\widehat{\mathsf{D}}_j\right) \ \in \mathsf{H}^0\Big(\widehat{U}_j,\ \mathsf{Sym}^n\, \Omega_{\mathrm{hor}} \widehat{V}\Big) \qquad {\scriptstyle (j = 0 \dots N)}$$

glue together to make a symmetric horizontal differential n-form on $\widehat{X}$.

Proof Proposition 6.2 yields that, by restriction, each $\widehat{\omega}_j \in \mathsf{H}^0\Big(\widehat{U}_j,\ \mathsf{Sym}^n\, \Omega_{\mathrm{hor}} \widehat{V}\Big)$ is well defined. We now show that, for every $0 \leqslant j_1 < j_2 \leqslant N$, the two forms $\widehat{\omega}_{j_1}$ and $\widehat{\omega}_{j_2}$ glue together along $\widehat{U}_{j_1} \cap \widehat{U}_{j_2}$.

Indeed, recalling (31), granted that D consists of the first $(c + r + n)$ rows of C, applying the above Cramer's rule to all the $(N + 1)$ columns of D and the $(N + 1)$ values $z_0^{\lambda_0 - 1}, \dots, z_N^{\lambda_N - 1}$, we receive:

$$(-1)^{j_2} \det\left(\widehat{\mathsf{D}}_{j_2}\right) z_{j_1}^{\lambda_{j_1} - 1} = (-1)^{j_1} \det\left(\widehat{\mathsf{D}}_{j_1}\right) z_{j_2}^{\lambda_{j_2} - 1}.$$

When $z_{j_1} \neq 0$, $z_{j_2} \neq 0$, dividing by $z_{j_1}^{\lambda_{j_1}-1} z_{j_2}^{\lambda_{j_2}-1}$ on both sides, we conclude the proof. □

By permuting the indices, the preceding Proposition 6.3 generalizes to any $(c+r+n) \times (N+1)$ submatrix of C containing the upper $c+r$ rows. Indeed, for every $1 \leqslant j_1 < \cdots < j_n \leqslant c$, denote by $\mathsf{C}_{j_1,\ldots,j_n}$ the $(c+r+n) \times (N+1)$ submatrix of C consisting of the first upper $c+r$ rows and the rows $c+r+j_1, \ldots, c+r+j_n$. Also, for $j = 0 \cdots N$, let $\widehat{\mathsf{C}}_{j_1,\ldots,j_n;\, j}$ denote the submatrix of $\mathsf{C}_{j_1,\ldots,j_n}$ obtained by omitting the $(j+1)$th column.

Proposition 6.4 *The following $(N+1)$ affine symmetric horizontal differential n-forms:*

$$\widehat{\omega}_{j_1,\ldots,j_n;\, j} := \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det\big(\widehat{\mathsf{C}}_{j_1,\ldots,j_n;\, j}\big) \quad \in \mathsf{H}^0\Big(\widehat{U}_j,\ \mathsf{Sym}^n\, \Omega_{\mathrm{hor}}\, \widehat{V}\Big)$$
$$(j = 0 \cdots N)$$

glue together to make a global symmetric horizontal differential n-form $\widehat{\omega}_{j_1,\ldots,j_n}$ on $\widehat{X}$. □

6.3 Twisted symmetric differential forms

Now, applying Proposition 6.1 to the above symmetric forms, we thus generalize [2, p. 434, Lemma 4].

Proposition 6.5 *The symmetric horizontal differential n-form $\widehat{\omega}_{j_1,\ldots,j_n}$ is the image of a global twisted symmetric differential n-form:*

$$\omega_{j_1,\ldots,j_n} \in \mathsf{H}^0\big(X, \mathsf{Sym}^n\, \Omega_V(\heartsuit)\big)$$

under the canonical injection as a particular case of Proposition 6.1:

$$\mathsf{H}^0\big(X, \mathsf{Sym}^n\, \Omega_V(\heartsuit)\big) \hookrightarrow \mathsf{H}^0\big(\widehat{X}, \mathsf{Sym}^n\, \Omega_{\mathrm{hor}}\, \widehat{V}\big),$$

with the twisted degree:

$$\heartsuit := \sum_{p=1}^{c+r} \deg F_p + \sum_{q=1}^{n} \deg F_{j_q} - \sum_{j=0}^{N} \lambda_j + N + 1. \tag{34}$$

For all homogeneous polynomials $P \in \mathsf{H}^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\deg P)\big)$, by multiplication:

$$P\, \omega_{j_1,\ldots,j_n} \in \mathsf{H}^0\big(X, \mathsf{Sym}^n\, \Omega_V(\deg P + \heartsuit)\big).$$

□

Proof According to the criterion (28), we only need to check, for all $z \in \widehat{X}$, for all $[\xi] \in \mathrm{T}_{\mathrm{hor}} \widehat{V}\big|_z$ and for all $\lambda \in \mathbb{K}^\times$, that:

$$\widehat{\omega}_{j_1,\ldots,j_n}\big(\lambda z, [\lambda \xi]\big) = \lambda^{\heartsuit}\, \widehat{\omega}_{j_1,\ldots,j_n}\big(z, [\xi]\big), \tag{35}$$

which follows from straightforward computations. □

Now, let K be the $(c+r+c) \times (N+1)$ matrix whose first $c+r$ rows consist of all $(N+1)$ terms in the expressions of $F_1, \ldots, F_{c+r}$ in the exact order, i.e. the (i, j)th entry of K is:

$$\mathsf{K}_{i,j} := A_i^{j-1}\, z_{j-1}^{\lambda_{j-1}} \qquad (i = 1 \cdots c+r,\ j = 1 \cdots N+1), \tag{36}$$

and whose last c rows consist of all $(N+1)$ terms in the expressions of $\mathrm{d}\, F_1, \ldots, \mathrm{d}\, F_c$ in the exact order, i.e. the $(c+r+i, j)$th entry of K is:

$$\mathsf{K}_{c+r+i,j} := \mathrm{d}\,\big(A_i^{j-1}\, z_{j-1}^{\lambda_{j-1}}\big) \qquad (i = 1 \ldots c,\ j = 1 \ldots N+1).$$

The jth column K_j of K and the jth column C_j of C are proportional:

$$\mathsf{K}_j = \mathsf{C}_j\, z_{j-1}^{\lambda_{j-1}-1} \qquad (j = 1 \ldots N+1). \tag{37}$$

In later applications, we will be mainly interested in determining the base locus of $\omega_{j_1,\ldots,j_n}$ in the coordinates nonvanishing part $\{z_0 \cdots z_N \neq 0\}$, thus, using (37), we may compute the corresponding symmetric horizontal differential n-forms $\widehat{\omega}_{j_1,\ldots,j_n}$ as:

$$\begin{aligned}\widehat{\omega}_{j_1,\ldots,j_n;\,j} &= \frac{(-1)^j}{z_j^{\lambda_j-1}} \det\big(\widehat{\mathsf{C}}_{j_1,\ldots,j_n;\,j}\big) \\ &= \frac{(-1)^j}{z_0^{\lambda_0-1} \cdots z_N^{\lambda_N-1}} \det\big(\widehat{\mathsf{K}}_{j_1,\ldots,j_n;\,j}\big) \qquad (j = 0 \cdots N),\end{aligned} \tag{38}$$

where $\widehat{\mathsf{K}}_{j_1,\ldots,j_n;\,j}$ is defined as an analog of $\widehat{\mathsf{C}}_{j_1,\ldots,j_n;\,j}$.

6.4 Twisted symmetric differential forms with some vanishing coordinates

Investigating further the above construction of symmetric differential forms, for every integer $1 \leqslant \eta \leqslant n-1$, for any indices $0 \leqslant v_1 < \cdots < v_\eta \leqslant N$, by focusing on:

$${}_{v_1,\ldots,v_\eta}X := X \cap \{z_{v_1} = 0\} \cap \cdots \cap \{z_{v_\eta} = 0\},$$

we can also construct a series of twisted symmetric differential $(n-\eta)$-forms as follows.

For indices $1 \leqslant j_1 < \cdots < j_{n-\eta} \leqslant c$, denote by ${}_{v_1,\dots,v_\eta}\mathsf{C}_{j_1,\dots,j_{n-\eta}}$ the $(N-\eta)\times(N-\eta+1)$ submatrix of C determined by the first $c+r$ rows and the selected rows $c+r+j_1,\dots,c+r+j_{n-\eta}$ as well as the $(N-\eta+1)$ columns which are complement to the columns $v_1+1,\dots,v_\eta+1$. Next, for every index $j\in\{0,\dots,N\}\setminus\{v_1,\dots,v_\eta\}$, let ${}_{v_1,\dots,v_\eta}\widehat{\mathsf{C}}_{j_1,\dots,j_{n-\eta};\,j}$ denote the submatrix of ${}_{v_1,\dots,v_\eta}\mathsf{C}_{j_1,\dots,j_{n-\eta}}$ obtained by deleting the column which is originally contained in the $(j+1)$th column of C. Lastly, denote:

$$ {}_{v_1,\dots,v_\eta}U_j \;:=\; {}_{v_1,\dots,v_\eta}X\cap\{z_j\neq 0\}, $$

whose cone is ${}_{v_1,\dots,v_\eta}\widehat{U}_j := \pi^{-1}\big({}_{v_1,\dots,v_\eta}U_j\big)$.

Now, we have two very analogs of Propositions 6.4 and 6.5. Before, we write the $(N-\eta+1)$ remaining numbers of $\{0,\dots,N\}\setminus\{v_1,\dots,v_\eta\}$ in the ascending order $r_0<r_1<\cdots<r_{N-\eta}$.

Proposition 6.6 *For all $j=0\cdots N-\eta$, the following $(N+1-\eta)$ affine symmetric horizontal differential $(n-\eta)$-forms:*

$$ \begin{aligned} {}_{v_1,\dots,v_\eta}\widehat{\omega}_{j_1,\dots,j_{n-\eta};\,r_j} &:= \frac{(-1)^j}{z_{r_j}^{\lambda_{r_j}-1}}\det\big({}_{v_1,\dots,v_\eta}\widehat{\mathsf{C}}_{j_1,\dots,j_{n-\eta};\,r_j}\big) \\ &\in \mathsf{H}^0\Big({}_{v_1,\dots,v_\eta}\widehat{U}_{r_j},\ \mathsf{Sym}^{n-\eta}\,\Omega_{\mathrm{hor}}\widehat{V}\Big) \end{aligned} $$

glue together to make a symmetric horizontal differential $(n-\eta)$-form:

$$ {}_{v_1,\dots,v_\eta}\widehat{\omega}_{j_1,\dots,j_{n-\eta}} \in \mathsf{H}^0\Big({}_{v_1,\dots,v_\eta}\widehat{X},\ \mathsf{Sym}^{n-\eta}\,\Omega_{\mathrm{hor}}\widehat{V}\Big). $$

Proof Without loss of generality, we may assume that $(v_1,\dots,v_\eta)=(N-\eta+1,\dots,N)$ and that $(j_1,\dots,j_{n-\eta})=(1,\dots,n-\eta)$. Recalling (31), for all $z\in {}_{N-\eta+1,\dots,N}\widehat{X}$ and for all $[\xi]\in \mathrm{T}_{\mathrm{hor}}\widehat{V}\big|_z$, since $z_{N-\eta+1}=\cdots=z_N=0$, we have:

$$ {}_{N-\eta+1,\dots,N}\mathsf{C}_{1,\dots,n-\eta}\begin{pmatrix} z_0^{\lambda_0-1} \\ \vdots \\ z_{N-\eta}^{\lambda_{N-\eta}-1}\end{pmatrix} = 0. $$

Therefore, by mimicking the arguments of Propositions 6.2–6.4, we conclude the proof. $\square$

Proposition 6.7 *The symmetric horizontal differential $(n-\eta)$-form ${}_{v_1,\dots,v_\eta}\widehat{\omega}_{j_1,\dots,j_{n-\eta}}$ on ${}_{v_1,\dots,v_\eta}\widehat{X}$ is the image of a twisted symmetric differential $(n-\eta)$-form on ${}_{v_1,\dots,v_\eta}X$:*

$$
{}_{v_1,\dots,v_\eta}\omega_{j_1,\dots,j_{n-\eta}} \in \mathsf{H}^0\Big({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V\big({}_{v_1,\dots,v_\eta}\heartsuit_{j_1,\dots,j_{n-\eta}}\big)\Big)
$$

under the canonical injection:

$$
\begin{aligned}
&\mathsf{H}^0\Big({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V\big({}_{v_1,\dots,v_\eta}\heartsuit_{j_1,\dots,j_{n-\eta}}\big)\Big)\\
&\hookrightarrow \mathsf{H}^0\big({}_{v_1,\dots,v_\eta}\widehat{X}, \mathsf{Sym}^{n-\eta}\,\Omega_{\mathrm{hor}}\widehat{V}\big),
\end{aligned}
$$

with the twisted degree:

$$
\begin{aligned}
{}_{v_1,\dots,v_\eta}\heartsuit_{j_1,\dots,j_{n-\eta}} := \sum_{p=1}^{c+r}\deg F_p + \sum_{q=1}^{n-\eta}\deg F_{j_q}&\\
-\Big(\sum_{j=0}^{N}\lambda_j - \sum_{\mu=1}^{\eta}\lambda_{v_\mu}\Big) + (N-\eta) + 1.& \qquad (39)
\end{aligned}
$$

Furthermore, for all homogeneous polynomials $P \in \mathsf{H}^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\deg P)\big)$, by multiplication:

$$
\begin{aligned}
&P\ {}_{v_1,\dots,v_\eta}\omega_{j_1,\dots,j_{n-\eta}}\\
&\in \mathsf{H}^0\Big({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V\,\big(\deg P + {}_{v_1,\dots,v_\eta}\heartsuit_{j_1,\dots,j_{n-\eta}}\big)\Big).
\end{aligned}
$$

□

Remark 6.8 We thank an anonymous referee for providing an alternative algebraic construction, according to which Propositions 6.6 and 6.7 can be obtained as corollaries of Propositions 6.4 and 6.5 by introducing η new equations:

$$
F_{c+r+1} = z_{v_1},\ \dots,\ F_{c+r+\eta} = z_{v_\eta}.
$$

□

In our coming applications, we will be mainly interested in determining the base locus of ${}_{v_1,\dots,v_\eta}\omega_{j_1,\dots,j_{n-\eta}}$ in the coordinates nonvanishing part $\{z_{r_0}\cdots z_{r_{N-\eta}} \neq 0\}$, thus, using (37), we may compute the corresponding symmetric horizontal differential $(n-\eta)$-forms ${}_{v_1,\dots,v_\eta}\widehat{\omega}_{j_1,\dots,j_{n-\eta}}$ as:

$$
\begin{aligned}
{}_{v_1,\ldots,v_\eta}\widehat{\omega}_{j_1,\ldots,j_{n-\eta};\,r_j} &= \frac{(-1)^j}{z_{r_j}^{\lambda_{r_j}-1}} \det\left({}_{v_1,\ldots,v_\eta}\widehat{\mathsf{C}}_{j_1,\ldots,j_{n-\eta};\,r_j}\right) \\
&= \frac{(-1)^j}{z_{r_0}^{\lambda_{r_0}-1}\cdots z_{r_{N-\eta}}^{\lambda_{r_{N-\eta}}-1}} \det\left({}_{v_1,\ldots,v_\eta}\widehat{\mathsf{K}}_{j_1,\ldots,j_{n-\eta};\,r_j}\right) \\
&\quad (j=0\cdots N-\eta),
\end{aligned} \tag{40}
$$

where ${}_{v_1,\ldots,v_\eta}\widehat{\mathsf{K}}_{j_1,\ldots,j_{n-\eta};\,r_j}$ is defined as an analog of ${}_{v_1,\ldots,v_\eta}\widehat{\mathsf{C}}_{j_1,\ldots,j_{n-\eta};\,r_j}$.

The two identities (38), (40) will enable us to efficiently narrow the base loci of the obtained twisted symmetric differential forms, as the matrix K directly copies the original equations/differentials of the hypersurface polynomials $F_1, \ldots, F_{c+r}$.

7 Moving coefficients method (I)

7.1 Motivation and ideas

It would be natural to try to settle the Debarre Ampleness Conjecture by means of pure Fermat-type hypersurfaces $F_i = \sum_{j=0}^{N} A_i^j\, z_j^e$ with $\deg A_i^j = \epsilon \geqslant 1$, a strategy which was indeed successful in [2] for the case $4c \geqslant 3N - 2$.

However, one could soon realize that this approach would not work, simply because, for instance in the limiting case $N = 2c$ (with $r = 0$), one could only obtain a *single* symmetric form $\omega \in \mathsf{H}^0\big(X, \mathsf{Sym}^c\, \Omega_X(-\heartsuit)\big)$ for such intersection $X = H_1 \cap \cdots \cap H_c$, since $1 \leqslant j_1 < \cdots < j_n \leqslant c = n$ forces $j_1 = 1, \ldots, j_n = n$, whereas $\dim \mathbb{P}(\mathrm{T}_X) = 2 \dim X - 1 = N - 1 \gg 1$.

To overcome this difficulty, our core idea is to look for more 'flexible' hypersurface equations, such that they can be expressed as general Fermat-type polynomials *in several distinct ways*, while keeping negative twist for all the symmetric differential forms obtained by Proposition 6.5.

To be more precise, assume that $N = 2c$, $r = 0$, and that all hypersurface equations $F_1, \ldots, F_c$ have the same degree d. First, every F_i can have a lot of homogeneous monomials, and we arbitrarily gather them in $N + 1$ parts as $F_i = \sum_{j=0}^{N} F_i^j$. Next, for every $j = 0 \cdots N$, choose the maximum integer λ_j such that $z_j^{\lambda_j}$ divides $F_1^j, \ldots, F_c^j$, so F_i rewrites under the form $F_i = \sum_{j=0}^{N} A_i^j\, z_j^{\lambda_j}$. Hence by applying Proposition 6.5, we receive a symmetric differential form with twisted degree:

$$
\heartsuit = N\,\mathsf{d} - \sum_{j=0}^{N} \lambda_j + N + 1 \qquad \text{[use (34)]}.
$$

Lastly, we think on how to make $\heartsuit \leqslant -1$. Observe that, although each $\lambda_j \leqslant \mathtt{d}$, yet the total number of $\lambda_\bullet$ is $N+1 > N =$ the multiple number of $\mathtt{d}$, and it is this feature that leaves some room to design the *moving coefficients method* (MCM). Indeed, we can choose all but one $\lambda_j \approx \mathtt{d} \gg 1$, and let the remaining one $\lambda_\nu > \sum_{j\neq\nu}(\mathtt{d}-\lambda_j)+N+1$ be large enough, so as to ensure:

$$\heartsuit = -\lambda_\nu + \sum_{j\neq\nu}(\mathtt{d}-\lambda_j)+N+1 \approx -\lambda_\nu \ll -1. \tag{41}$$

Let us best illustrate this central idea of MCM by the following c hypersurface equations, each being a Fermat-type polynomial plus one *moving coefficient term*:

$$F_i = \sum_{j=0}^{N} A_i^j\, z_j^d + M_i\, z_0^{\lambda_0}\cdots z_N^{\lambda_N} \qquad (i=1\cdots c),$$

where A_i^j, M_i are some homogeneous polynomials of degree $\epsilon \geqslant 0$, and where $\lambda_0+\cdots+\lambda_N = d$ are some positive integers to be determined. For every $\nu = 0\cdots N$, associating the moving coefficient term $M_i\, z_0^{\lambda_0}\cdots z_N^{\lambda_N}$ to the term $A_i^\nu\, z_\nu^d$, we may rewrite:

$$F_i = \big(A_i^\nu\, z_\nu^d + M_i\, z_0^{\lambda_0}\cdots z_N^{\lambda_N}\big) + \sum_{j\neq\nu}^{N} A_i^j\, z_j^d \qquad (i=1\cdots c),$$

and thus by Proposition 6.5, we obtain a symmetric differential form with twisted degree $\heartsuit_\nu = -\lambda_\nu + N\epsilon + N + 1$. Therefore we require all $\lambda_\nu \geqslant N\epsilon + N + 1 + 1$ to make $\heartsuit_\nu \leqslant -1$.

7.2 Global moving coefficients method

Recalling Theorems 5.1, 5.4, we first fix positive integers $\blacktriangledown$, $\epsilon_1, \ldots, \epsilon_{c+r} \geqslant 1$. Skipping some thinking process which essentially roots in the above example, we construct the following $c+r$ hypersurface equations, each being a Fermat-type polynomial plus $N+1$ *moving coefficient terms*:

$$F_i := \sum_{j=0}^{N} A_i^j\, z_j^d + \sum_{k=0}^{N} M_i^k\, z_0^{\mu_k}\cdots \widehat{z_k^{\mu_k}}\cdots z_N^{\mu_k}\, z_k^{d-N\,\mu_k} \qquad (i=1\cdots c+r), \tag{42}$$

where all coefficients A_i^j, $M_i^k \in \mathbb{K}[z_0, \dots, z_N]$ are some degree ϵ_i homogeneous polynomials, and where all positive integers $\mu_\bullet, d$ are to be chosen by a certain *Algorithm*, so as to make all the symmetric differential forms to be obtained later have negative twisted degrees. For the moment, instead of making the Algorithm clear, we prefer to roughly summarize it as:

$$1 \leqslant \max\{\epsilon_i\}_{i=1\dots c+r} \ll \underbrace{\mu_0 \ll \cdots \ll \mu_N}_{\mu_\bullet \text{ grow exponentially}} \ll d. \tag{43}$$

There are two kinds of manipulations in MCM to transform the polynomials $F_1, \dots, F_{c+r}$ to Fermat-type. The first one is much like the above example, in the sense that we view all the $N+1$ moving coefficient terms as a whole. Precisely, for every $\nu = 0 \cdots N$, for all F_i, we associate all $(N+1)$ moving coefficient terms with the term $A_i^\nu z_\nu^d$ by rewriting:

$$F_i = \sum_{j \neq \nu} A_i^j z_j^d + \Big(A_i^\nu z_\nu^d + \sum_{k=0}^{N} M_i^k z_0^{\mu_k} \cdots \widehat{z_k^{\mu_k}} \cdots z_N^{\mu_k} z_k^{d - N\mu_k} \Big) \qquad (i = 1 \cdots c+r). \tag{44}$$

By (43), all terms in the bracket are divisible by $z_\nu^{\mu_0}$. Hence, applying Proposition 6.5, and choosing all multi-indices $1 \leqslant j_1 < j_2 < \cdots < j_n \leqslant c$, we can receive many negatively twisted jet differential forms. Indeed, that was our original approach in a previous long version [17]. Now, inspired by suggestions from an anonymous referee, we shall simplify our construction by constantly choosing $(j_1, \dots, j_n) = (1, \dots, n)$. In fact, in the limiting case $N = 2c$ it is indeed our only choice, and in the general case, this choice will turn out to substantially shorten the arguments of controlling the base loci. Thus we receive a twisted symmetric differential n-form:

$$\phi^\nu \in \mathsf{H}^0\big(X, \mathsf{Sym}^n\, \Omega_V(\heartsuit^\nu)\big),$$

whose twisted degree $\heartsuit^\nu$ according to formula (34) is negative:

$$\heartsuit^\nu = -\mu_0 + \sum_{i=1}^{c+r} \epsilon_i + \sum_{\ell=1}^{n} \epsilon_\ell + N + 1 \leqslant -n\,\blacktriangledown \qquad \text{[compare with (41), use (43)]}. \tag{45}$$

The second kind of manipulations is, for every $\tau = 0 \cdots N-1$, for every $\rho = \tau + 1 \cdots N$, for all F_i, to associate each of the first $(\tau+1)$ moving coefficient terms with the corresponding terms $A_i^\bullet z_\bullet^d$, to associate all the remaining $(N - \tau)$ moving coefficient terms with the term $A_i^\rho z_\rho^d$, hence to rewrite:

$$F_i = \sum_{k=0}^{\tau} \Big(A_i^k \, z_k^d + M_i^k \, z_0^{\mu_k} \cdots \widehat{z_k^{\mu_k}} \cdots z_N^{\mu_k} \, z_k^{d-N\,\mu_k} \Big) + \sum_{\substack{j=\tau+1 \\ j\neq\rho}}^{N} A_i^j \, z_j^d + \Big[A_i^\rho \, z_\rho^d + \sum_{j=\tau+1}^{N} M_i^j \, z_0^{\mu_j} \cdots \widehat{z_j^{\mu_j}} \cdots z_N^{\mu_j} \, z_j^{d-N\,\mu_j} \Big] \quad (i = 1 \dots c+r). \tag{46}$$

Each bracket in the first sum is divisible by the corresponding $z_k^{d-N\,\mu_k}$, and remembering (43), the last square bracket is divisible by $z_\rho^{\mu_{\tau+1}}$. Now applying Proposition 6.5 again, taking $(j_1, \dots, j_n) = (1, \dots, n)$, we receive a twisted symmetric differential n-form:

$$\psi^{\tau,\,\rho} \in \mathsf{H}^0\big(X, \mathsf{Sym}^n \, \Omega_V(\heartsuit^{\tau,\,\rho})\big),$$

whose twisted degree $\heartsuit^{\tau,\,\rho}$ according to formula (34) is negative:

$$\heartsuit^{\tau,\,\rho} = -\,\mu_{\tau+1} + \sum_{k=0}^{\tau} N\,\mu_k + \sum_{i=1}^{c+r} \epsilon_i + \sum_{\ell=1}^{n} \epsilon_\ell + N + 1 \leqslant -\,n \, \heartsuit \qquad \text{[compare with (41), use (43)]}. \tag{47}$$

8 Controlling the base locus

Recalling the claimed step (16), at first we shall determine the:

$$\text{Base Locus of } \Big\{ \phi^\nu, \psi^{\tau,\,\rho} \Big\}_{\nu,\,\tau,\,\rho} =: \mathsf{BS} \subset \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \times \mathbb{P}(\mathrm{T}_{\mathbb{P}_{\mathbb{K}}^N}). \tag{48}$$

8.1 Characterization of BS

For a fixed $\nu = 0 \cdots N$, by mimicking the construction of the upper $N \times (N+1)$ submatrix $\mathsf{K}_{1,\dots,n}$ of the matrix K defined in around (36), in accordance with the manipulation (44), we construct the $N \times (N+1)$ matrix K^ν analogously, i.e. by copying terms, differentials, and then we denote by $\widehat{\mathsf{K}}_j^\nu$ the $N \times N$ submatrix of K^ν obtained by deleting the $(j+1)$th colum.

Now, let us look at points $\big([A_\bullet^\bullet, M_\bullet^\bullet], [z, \xi]\big) \in \mathsf{BS}$ with $z_0 \cdots z_N \neq 0$. For the sake of readability, we will keep in mind the parameter $(A_\bullet^\bullet, M_\bullet^\bullet)$ without writing it. Recalling Propositions 6.4, 6.5, for every $\widehat{\phi}^\nu$ corresponding to ϕ^ν, we have:

$$0 = \widehat{\phi}_0^\nu(z,\xi) = \frac{(-1)^0}{z_0^\star \cdots z_N^\star} \det\big(\widehat{\mathsf{K}}_0^\nu\big)(z,\xi) \qquad \text{[use (38) for } j=0],$$

where all integers $\star$ are of no importance here. Indeed, we can drop the nonzero factor $\frac{(-1)^0}{z_0^\star \cdots z_N^\star}$:

$$\operatorname{rank}_{\mathbb{K}} \widehat{\mathsf{K}}_0^\nu(z,\xi) \leqslant N-1. \tag{49}$$

Analogously, for every $\tau = 0 \cdots N-1$ and $\rho = \tau+1 \ldots N$, we define the $N \times (N+1)$ matrix $\mathsf{K}^{\tau,\rho}$ in accordance with the manipulation (46), and we receive:

$$\operatorname{rank}_{\mathbb{K}} \mathsf{K}_0^{\tau,\rho}(z,\xi) \leqslant N-1, \tag{50}$$

where $\mathsf{K}_0^{\tau,\rho}$ is the $N \times N$ submatrix of $\mathsf{K}^{\tau,\rho}$ obtained by deleting the first column.

It is now time to clarify the (uniform) structures of the matrices K^ν, $\mathsf{K}^{\tau,\rho}$. First of all, we construct the $N \times (2N+2)$ ambient matrix M such that, for every $i = 1 \ldots c+r$, $j = 1 \ldots n$, its ith row copies the $(2N+2)$ terms of F_i in (42) in the exact order, and its $(c+r+j)$th row is the differential of the jth row. In order to distinguish the first $(N+1)$ 'dominant' columns from the last $(N+1)$ 'moving coefficients' columns, we write M as:

$$\mathsf{M} = \Big(\mathsf{A}_0 \mid \cdots \mid \mathsf{A}_N \mid \mathsf{B}_0 \mid \cdots \mid \mathsf{B}_N\Big).$$

For every $\nu = 0 \cdots N$, comparing (44) with (42), the matrix K^ν is nothing but:

$$\mathsf{K}^\nu = \Big(\mathsf{A}_0 \mid \cdots \mid \widehat{\mathsf{A}_\nu} \mid \cdots \mid \mathsf{A}_N \mid \mathsf{A}_\nu + \sum_{j=0}^N \mathsf{B}_j\Big), \tag{51}$$

where the last column is understood to appear in the 'omitted' column. Similarly, for every $\tau = 0 \cdots N-1$ and every $\rho = \tau+1 \ldots N$, comparing (46) with (42), the matrix $\mathsf{K}^{\tau,\rho}$ is nothing but:

$$\mathsf{K}^{\tau,\rho} = \Big(\mathsf{A}_0 + \mathsf{B}_0 \mid \cdots \mid \mathsf{A}_\tau + \mathsf{B}_\tau \mid \mathsf{A}_{\tau+1} \mid \cdots \mid \widehat{\mathsf{A}_\rho} \mid \cdots \mid \mathsf{A}_N \mid \mathsf{A}_\rho + \sum_{j=\tau+1}^N \mathsf{B}_j\Big),$$

hence:

$$\mathsf{K}_0^{\tau,\rho} = \Big(\mathsf{A}_1 + \mathsf{B}_1 \mid \cdots \mid \mathsf{A}_\tau + \mathsf{B}_\tau \mid \mathsf{A}_{\tau+1} \mid \cdots \mid \widehat{\mathsf{A}_\rho} \mid \cdots \mid \mathsf{A}_N \mid \mathsf{A}_\rho + \sum_{j=\tau+1}^N \mathsf{B}_j\Big). \tag{52}$$

Now, trying to get rid of both A_0 and B_0, we only consider the inequalities (50), and thus we introduce the subvariety:

$$\mathscr{M}_N \quad \subset \quad \mathsf{Mat}_{N\times 2N}(\mathbb{K}) \tag{53}$$

consisting of all $N \times 2N$ matrices $(\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_N \mid \beta_1 \mid \beta_2 \mid \cdots \mid \beta_N)$ such that:

(i) for every $\nu = 1 \dots N$, there holds the rank inequality:

$$\mathsf{rank}_{\mathbb{K}} \left\{\alpha_1, \dots, \widehat{\alpha_\nu}, \dots, \alpha_N, \alpha_\nu + (\beta_1 + \beta_2 + \cdots + \beta_N)\right\} \leqslant N - 1; \tag{54}$$

(ii) for every $\tau = 1 \dots N-1$, for every $\rho = \tau + 1 \cdots N$, there holds:

$$\begin{aligned}&\mathsf{rank}_{\mathbb{K}} \left\{\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \dots, \widehat{\alpha_\rho}, \dots, \alpha_N,\right.\\ &\left.\alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_N)\right\} \leqslant N - 1.\end{aligned} \tag{55}$$

Denoting by $\widehat{\mathsf{M}}^0$ the $N \times 2N$ submatrix of M obtained by deleting the columns A_0, B_0, we thus characterize the base locus over coordinates nonvanishing part by:

Proposition 8.1 *For every point* $\left([A_{\bullet}^{\bullet}, M_{\bullet}^{\bullet}], [z, \xi]\right) \in \mathsf{BS}$ *with* $z_0 \cdots z_N \neq 0$, *the defining equations* $F_1(z) = \cdots = F_{c+r}(z) = 0$, $dF_1\,(z, \xi) = \cdots = dF_n\,(z, \xi) = 0$ *read as:*

$$(\mathsf{A}_0 + \cdots + \mathsf{A}_N + \mathsf{B}_0 + \cdots + \mathsf{B}_N)\,(z, \xi) = 0. \tag{56}$$

Moreover, concerning the algebraic dependence of $\mathsf{A}_1, \dots, \mathsf{A}_N, \mathsf{B}_1, \dots, \mathsf{B}_N$, *one has:*

$$\widehat{\mathsf{M}}^0(z, \xi) \in \mathscr{M}_N. \tag{57}$$

□

8.2 Emptiness of the base locus

In order to estimate the dimension of the base locus over coordinates nonvanishing part:

$$\mathsf{BS}^0 := \mathsf{BS} \cap \left(\mathbb{P}_{\mathbb{K}}^{\blacklozenge} \times \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})\right),$$

where $\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) = \mathbb{P}(\mathrm{T}_{\mathbb{P}^N}) \cap \{z_0 \dots z_N \neq 0\}$, we now claim the following *Core Lemma* in advance, whose proof will constitute the most technical part of this article (see Sect. 11 below).

Core Lemma *For every $N \geqslant 2$, there holds the codimension estimate:*

$$\mathsf{codim}\, \mathscr{M}_N = N. \qquad \square$$

Proposition 8.2 *One has the dimension estimate:*

$$\mathsf{dim}\, \mathsf{BS}^0 < \mathsf{dim}\, \mathbb{P}_{\mathbb{K}}^{\blacklozenge}. \tag{58}$$

Proof Let π_1, π_2 be the two canonical projections of $\mathbb{P}_{\mathbb{K}}^{\blacklozenge} \times \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})$ onto two components. For every $\zeta = [z, \xi] \in \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})$, Proposition 8.1 forces all parameters

$$\big[A_{\bullet}^{\bullet}, M_{\bullet}^{\bullet}\big] \times \{\zeta\} \in \pi_2^{-1}(\zeta) \cap \mathsf{BS}^0 \subset \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \times \{\zeta\}$$

to satisfy the Eqs. (56) and (57). Note that, thanks to the Lemma 10.1 below, the linear evaluation map at (z, ξ):

$$\big(\mathsf{A}_0 \,|\, \cdots \,|\, \mathsf{A}_N \,|\, \mathsf{B}_0 \,|\, \cdots \,|\, \mathsf{B}_N\big) : \mathbb{K}^{\blacklozenge+1} \longrightarrow \mathsf{Mat}_{N\times 2(N+1)}(\mathbb{K})$$

is surjective, because we have the nonvanishing:

$$z_j^d \neq 0, \quad z_0^{\mu_k} \cdots \widehat{z_k^{\mu_k}} \cdots z_N^{\mu_k}\, z_k^{d-N\,\mu_k} \neq 0 \qquad (j=0\cdots N,\ k=0\cdots N).$$

Therefore, all such parameters $\big(A_{\bullet}^{\bullet}, M_{\bullet}^{\bullet}\big) \in \mathbb{K}^{\blacklozenge+1} \setminus \{0\}$ together with $0 \in \mathbb{K}^{\blacklozenge+1}$ constitute a subvariety of $\mathbb{K}^{\blacklozenge+1}$ having codimension $\geqslant N + N = 2N$, because the equation (56) is clearly independent of (57), which, by the Core Lemma, contributes codimension N. Therefore, using the Theorem 10.4 below, we see that

$$\pi_2^{-1}(\zeta) \cap \mathsf{BS}^0 \subset \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \times \{\zeta\}$$

has codimension $\geqslant 2N$. Lastly, applying Theorem 10.2 with respect to $\pi_2 \colon \mathsf{BS}^0 \longrightarrow \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})$, we receive:

$$\begin{aligned}
\mathsf{dim}\, \mathsf{BS}^0 &\leqslant \mathsf{dim}\, \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) + \mathsf{max}_\zeta\, \mathsf{dim}\, \big(\pi_2^{-1}(\zeta) \cap \mathsf{BS}^0\big) \\
&\leqslant \mathsf{dim}\, \mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) + \mathsf{max}_\zeta \Big(\mathsf{dim}\, \big(\mathbb{P}_{\mathbb{K}}^{\blacklozenge} \times \{\zeta\}\big) - 2N\Big) \\
&= (2N-1) + (\mathsf{dim}\, \mathbb{P}_{\mathbb{K}}^{\blacklozenge} - 2N) < \mathsf{dim}\, \mathbb{P}_{\mathbb{K}}^{\blacklozenge},
\end{aligned}$$

whence we finish the proof. $\square$

Now, thanks to the dimension estimate (58), by the projection $\pi_1 \colon \mathsf{BS}^0 \to \mathbb{P}_{\mathbb{K}}^{\blacklozenge}$, we receive

Proposition 8.3 *There exists a proper algebraic subvariety $\Sigma \subsetneqq \mathbb{P}_{\mathbb{K}}^{\blacklozenge}$ such that, for every parameter $t = \big[A_{\bullet}^{\bullet}, M_{\bullet}^{\bullet}\big] \in \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \setminus \Sigma$, the corresponding base locus BS_t is empty over the coordinates nonvanishing part $\{z_0 \cdots z_N \neq 0\}$.* □

9 Moving coefficients method (II)

9.1 Obstacles of MCM when some coordinates vanish

It would be desirable to find one parameter t such that $\mathsf{BS}_t \cap \{z_0 \cdots z_N = 0\}$ is discrete or empty, whence the semicontinuity theorem would guarantee the same property for all generic t. Then, together with the above Proposition 8.3, we would prove the Debarre Ampleness Conjecture! However, this is impossible because, for instance when $z_N = 0$, we can check that all the obtained global symmetric differential forms either vanish or become the same one, up to a scalar of the form $z_0^{\bullet} \cdots z_{N-1}^{\bullet}$. Moreover, all hypersurface equations in (42) become exactly Fermat-type $F_i = \sum_{j=0}^{N-1} A_i^j \, z_j^d$, so even with the further help of Propositions 6.6, 6.7, we cannot construct enough negatively twisted symmetric forms. Nevertheless, supposing at this moment that we still have abundant moving coefficient terms:

$$F_i\big|_{z_N=0} \approx \sum_{j=0}^{N-1} A_i^j \, z_j^d + \sum_{k=0}^{N-1} M_i^k \, z_0^{\mu_k} \cdots \widehat{z_k^{\mu_k}} \cdots z_{N-1}^{\mu_k} \, z_k^{d-(N-1)\,\mu_k},$$

we may then continue to move on by adapting MCM …Yet when further coordinates vanish, the same obstacle might appear …

9.2 Refined construction of hypersurfaces for MCM

The above reasoning process leads us to construct the following $c+r$ refined homogeneous polynomials $F_1, \ldots, F_{c+r}$, each being the sum of a *dominant* Fermat-type polynomial plus an ‘army’ of *moving coefficient* terms:

$$F_i = \sum_{j=0}^{N} A_i^j \, z_j^d + \sum_{l=c+r+1}^{N} \sum_{0 \leqslant j_0 < \cdots < j_l \leqslant N} \times \sum_{k=0}^{l} M_i^{j_0, \ldots, j_l; j_k} \, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}} \, z_{j_k}^{d-l\mu_{l,k}} \qquad (i = 1 \cdots c+r), \tag{59}$$

where all coefficients $A_i^{\bullet}, M_i^{\bullet;\bullet} \in \mathbb{K}[z_0, \dots, z_N]$ are some degree $\epsilon_i \geqslant 1$ homogeneous polynomials, and where all positive integers $\mu^{l,k}, d$ are to be chosen by a certain *Algorithm*, which is designed to make all the symmetric differential forms to be obtained later have *negative twisted degrees*. For the moment, we just roughly summarize the Algorithm as:

$$\begin{aligned} 1 \leqslant \max\{\epsilon_i\}_{i=1\dots c+r} &\ll \underbrace{\mu_{c+r+1,0} \ll \cdots \ll \mu_{c+r+1,c+r+1}}_{\mu_{c+r+1,\bullet} \text{ grow exponentially}} \\ &\ll \cdots\cdots \ll \underbrace{\mu_{N,0} \ll \cdots \ll \mu_{N,N}}_{\mu_{N,\bullet} \text{ grow exponentially}} \ll d, \end{aligned} \tag{60}$$

and we will state it explicitly in Sect. 12.1 below when it is really necessary.

The above construction fulfills the claimed **Step 1** in Sect. 5.3. Now, we illustrate how to transform (59) into the form (42), so as to apply MCM that we have developed in Sect. 7.

First, we rewrite the polynomial F_i by extracting the terms for which $l = N$:

$$\begin{aligned} F_i = \sum_{j=0}^{N} A_i^j\, z_j^d + \sum_{l=c+r+1}^{N-1} \sum_{0 \leqslant j_0 < \cdots < j_l \leqslant N} \\ \times \sum_{k=0}^{l} M_i^{j_0,\dots,j_l;\,j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}}\, z_{j_k}^{d-l\mu_{l,k}} + \\ + \sum_{k=0}^{N} M_i^{0,\dots,N;k}\, z_0^{\mu_{N,k}} \cdots \widehat{z_k^{\mu_{N,k}}} \cdots z_N^{\mu_{N,k}}\, z_k^{d-N\,\mu_{N,k}} \qquad (i = 1 \cdots c+r). \end{aligned}$$

Next, we associate each term $M_i^{j_0,\dots,j_l;\,j_k}\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}}\, z_{j_k}^{d-l\mu_{l,k}}$ in the second sums with the corresponding term $A_i^{j_k}\, z_{j_k}^d$ in the first sum by rewriting F_i as:

$$F_i = \sum_{j=0}^{N} C_i^j\, z_j^{d-\delta_N} + \sum_{k=0}^{N} M_i^{0,\dots,N;k}\, z_0^{\mu_{N,k}} \cdots \widehat{z_k^{\mu_{N,k}}} \cdots z_N^{\mu_{N,k}}\, z_k^{d-N\,\mu_{N,k}} \qquad (i = 1 \cdots c+r), \tag{61}$$

where $\delta_N := (N-1)\,\mu_{N-1,N-1}$ and C_i^j are uniquely determined by gathering:

$$C_i^j\, z_j^{d-\delta_N} = A_i^j\, z_j^d + \sum_{l=c+r+1}^{N-1} \sum_{\substack{0 \leqslant j_0 < \cdots < j_l \leqslant N \\ j_k = j \text{ for some } 0 \leqslant k \leqslant l}} M_i^{j_0,\dots,j_l;\,j_k}$$
$$\times\, z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}}\, z_{j_k}^{d-l\mu_{l,k}},$$

namely, after dividing out the common factor $z_j^{d-\delta_N}$—guaranteed by (60)—of both sides above:

$$C_i^j := A_i^j\, z_j^{\delta_N} + \sum_{l=c+r+1}^{N-1} \sum_{\substack{0 \leqslant j_0 < \cdots < j_l \leqslant N \\ j_k = j \text{ for some } 0 \leqslant k \leqslant l}} M_i^{j_0,\dots,j_l;\,j_k}$$
$$\times z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}}\, z_{j_k}^{\delta_N - l\mu_{l,k}}.$$

Note that by the Algorithm (60), we now have the rough estimates:

$$\begin{aligned} \deg\, C_i^j &= \epsilon_i + \delta_N = \epsilon_i + (N-1)\,\mu_{N-1,N-1} \\ &\ll\; \mu_{N,0} \;\ll\; \cdots \;\ll\; \mu_{N,N} \ll\; d, \end{aligned} \tag{62}$$

with the same growth structure as the previous Algorithm (43)!

Thus, we are able to adapt the MCM in Sect. 7.2 to produce a series of negatively twisted symmetric differential forms. Indeed, using the same kind of manipulations and the corresponding notation, for every $\nu = 0 \dots N$, we receive:

$$\phi^\nu \;\in\; \mathsf{H}^0\big(X, \mathsf{Sym}^n\, \Omega_V(\heartsuit^\nu)\big),$$

with negative twisted degree:

$$\heartsuit^\nu = -\,\mu_{N,0} + N\,\delta_N + \sum_{i=1}^{c+r} \epsilon_i + \sum_{\ell=1}^{n} \epsilon_\ell + N + 1 \;\leqslant\; -n\,\blacktriangledown \qquad \text{[compare with (45), use (62)]}. \tag{63}$$

Also, for every $\tau = 0 \cdots N-1$, for every $\rho = \tau + 1 \cdots N$, we receive:

$$\psi^{\tau,\,\rho} \;\in\; \mathsf{H}^0\big(X, \mathsf{Sym}^n\, \Omega_V(\heartsuit^{\tau,\,\rho})\big),$$

with negative twisted degree:

$$\heartsuit^{\tau,\,\rho} = -\,\mu_{N,\tau+1} + \sum_{k=0}^{\tau} N\,\mu_{N,k} + (N-\tau-1)\,\delta_N + \sum_{i=1}^{c+r} \epsilon_i + \sum_{\ell=1}^{n} \epsilon_\ell + N + 1 \;\leqslant\; -\,n\,\blacktriangledown \qquad \text{[see (47), use (62)]}. \tag{64}$$

Thus, we complete the claimed construction (13).

To respect the coherence of notation, we will continue to denote by $\mathbb{P}_{\mathbb{K}}^{\blacklozenge}$ the projective parameter space of $F_1, \dots, F_{c+r}$ in (59), and we will also use the same notation as in Sect. 8, and then by exactly the same arguments, everything goes on smoothly, and finally we get the analogue Proposition 8.3, which is our claimed estimate (19). Similar details can be also found in the proof of the Proposition 9.1 below.

9.3 MCM with vanishing coordinates

For every $1 \leqslant \eta \leqslant n-1$, for every $0 \leqslant v_1 < \cdots < v_\eta \leqslant N$, consider the intersection of X with the η coordinate hyperplanes:

$${}_{v_1,\dots,v_\eta}X := X \cap \{z_{v_1} = 0\} \cap \cdots \cap \{z_{v_\eta} = 0\}.$$

Recalling the claimed step (14), we now investigate further the MCM to construct a series of:

$$\omega_\ell \in \mathsf{H}^0\left({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V(\heartsuit_\ell)\right) \qquad (\ell = 1 \dots \circledast),$$

with all $\heartsuit_\ell \leqslant -(n-\eta)\,\blacktriangledown$, as follows.

First, we write the complement elements of $\{v_1, \dots, v_\eta\}$ in $\{0, \dots, N\}$ as $r_0 < \cdots < r_{N-\eta}$. Observe that in Propositions 6.6, 6.7, the polynomial terms in $F_1, \dots, F_{c+r}$ involving $z_{v_1}, \dots, z_{v_\eta}$ eventually play no role—because they just vanish. Thus, we may decompose F_i into two parts:

$$F_i = \sum_{j=0}^{N-\eta} A_i^{r_j}\, z_{r_j}^d + \sum_{l=c+r+1}^{N-\eta} \sum_{0\leqslant j_0<\cdots<j_l\leqslant N-\eta} \sum_{k=0}^{l} M_i^{r_{j_0},\dots,r_{j_l};r_{j_k}} \times z_{r_{j_0}}^{\mu_{l,k}} \cdots \widehat{z_{r_{j_k}}^{\mu_{l,k}}} \cdots z_{r_{j_l}}^{\mu_{l,k}}\, z_{r_{j_k}}^{d-l\mu_{l,k}} + \underbrace{+\,(\text{Residue Terms})_i^{v_1,\dots,v_\eta}}_{\text{negligible}} \qquad (i = 1 \cdots c+r), \tag{65}$$

where the first part (1st line above) only involves the variables $z_{r_0}, \dots, z_{r_{N-\eta}}$, and where the second part (2nd line above) collects all terms involving at least one of $z_{v_1}, \dots, z_{v_\eta}$.

Next, before applying Propositions 6.6, 6.7 to (65), it is appropriate to write:

$$\begin{aligned} F_i = \sum_{j=0}^{N-\eta} A_i^{r_j}\, z_{r_j}^d + \sum_{l=c+r+1}^{N-\eta} \sum_{0\leqslant j_0<\cdots<j_l\leqslant N-\eta} \\ \times \sum_{k=0}^{l} M_i^{r_{j_0},\dots,r_{j_l};r_{j_k}}\, z_{r_{j_0}}^{\mu_{l,k}} \cdots \widehat{z_{r_{j_k}}^{\mu_{l,k}}} \cdots z_{r_{j_l}}^{\mu_{l,k}}\, z_{r_{j_k}}^{d-l\mu_{l,k}} \qquad (i = 1\cdots c+r). \end{aligned} \tag{66}$$

Now, we see that the above (66) has the same structure as (59), in the sense of replacing:

$$N \longmapsto N-\eta, \qquad \{0,\dots,N\} \longmapsto \{r_0,\dots,r_{N-\eta}\}.$$

Thus, we are able to adapt the MCM in the preceding Sect. 9.2 to produce negatively twisted symmetric differential forms.

Indeed, using the same kind of manipulations and the corresponding notation, fixing the multi-indices $(j_1,\dots,j_{n-\eta}) = (1,\dots,n-\eta)$, for every $\nu = 0\cdots N-\eta$, we receive:

$${}_{v_1,\dots,v_\eta}\phi^\nu \in \Gamma\big({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V\,({}_{v_1,\dots,v_\eta}\heartsuit^\nu)\big),$$

with negative twisted degree:

$$\begin{aligned} {}_{v_1,\dots,v_\eta}\heartsuit^\nu = -\,\mu_{N-\eta,0} + (N-\eta)\,\delta_{N-\eta} + \sum_{i=1}^{c+r} \epsilon_i \\ + \sum_{\ell=1}^{n-\eta} \epsilon_\ell + (N-\eta)+1 \;\leqslant\; -(n-\eta)\,\blacklozenge \\ \text{[compare with (63)]}, \end{aligned} \tag{67}$$

where $\delta_{c+r+1} \geqslant \max\{\epsilon_i\}_{i=1\dots c+r}$ and where for $N-\eta > c+r+1$ we define $\delta_{N-\eta}$ by replacing:

$$(N-1)\,\mu_{N-1,N-1} = \delta_N \;\longmapsto\; \delta_{N-\eta} := (N-\eta-1)\,\mu_{N-\eta-1,N-\eta-1}.$$

Also, for every $\tau = 0\cdots N-\eta-1$, for every $\rho = \tau+1\cdots N-\eta$, we receive:

$${}_{v_1,\dots,v_\eta}\psi^{\tau,\rho} \in \Gamma\big({}_{v_1,\dots,v_\eta}X, \mathsf{Sym}^{n-\eta}\,\Omega_V\,({}_{v_1,\dots,v_\eta}\heartsuit^{\tau,\rho})\big),$$

with negative twisted degree:

$$
\begin{aligned}
{}_{v_1,\dots,v_\eta}\heartsuit^{\tau,\,\rho} &= -\,\mu_{N-\eta,\tau+1} + \sum_{k=0}^{\tau} (N-\eta)\,\mu_{N-\eta,k} + (N-\eta-\tau-1)\,\delta_{N-\eta} \\
&\quad + \sum_{i=1}^{c+r} \epsilon_i + \sum_{\ell=1}^{n-\eta} \epsilon_\ell + (N-\eta) + 1 \;\leqslant\; -\,(n-\eta)\,\blacktriangledown. \qquad (68)
\end{aligned}
$$

The above construction fulfills our claimed step (14). So now, we complete the $\mathsf{Step\ 2}$ in Sect. 5.3.

To prove the claimed estimate (18), we now determine the:

$$
\begin{aligned}
&\text{Base Locus of } \{{}_{v_1,\dots,v_\eta}\phi^{\nu},\ {}_{v_1,\dots,v_\eta}\psi^{\tau,\,\rho}\}_{\nu,\,\tau,\,\rho} \\
&=:\ {}_{v_1,\dots,v_\eta}\mathsf{BS} \ \subset\ \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \times {}_{v_1,\dots,v_\eta}\mathbb{P}(\mathrm{T}_{\mathbb{P}_{\mathbb{K}}^N}), \qquad (69)
\end{aligned}
$$

where:

$$
{}_{v_1,\dots,v_\eta}\mathbb{P}(\mathrm{T}_{\mathbb{P}^N}) := \mathbb{P}(\mathrm{T}_{\mathbb{P}^N}) \cap \{z_{v_1} = \cdots = z_{v_\eta} = 0\}.
$$

Employing the same arguments, here is an analog of Proposition 8.3.

Proposition 9.1 *There exists some proper subvariety* ${}_{v_1,\dots,v_\eta}\Sigma \subsetneqq \mathbb{P}_{\mathbb{K}}^{\blacklozenge}$ *such that, for every parameter* $t \in \mathbb{P}_{\mathbb{K}}^{\blacklozenge} \setminus {}_{v_1,\dots,v_\eta}\Sigma$*, the base locus* ${}_{v_1,\dots,v_\eta}\mathsf{BS}_t := {}_{v_1,\dots,v_\eta}\mathsf{BS} \cap \pi_1^{-1}(t)$ *is empty over the coordinates nonvanishing part* $\{z_{r_0} \cdots z_{r_{N-\eta}} \neq 0\}$.

Proof Without loss of generality, we may assume that $(v_1, \dots, v_\eta) = (N - \eta + 1, \dots, N)$.

Firstly, by mimicking the argument before Proposition 8.1, we can construct a similar $N \times 2(N+1-\eta)$ matrix:

$$
\Big(\mathsf{A}_0 \mid \cdots \mid \mathsf{A}_{N-\eta} \mid \mathsf{B}_0 \mid \cdots \mid \mathsf{B}_{N-\eta}\Big)
$$

so that, for every point $\big(t, [z, \xi]\big) \in {}_{N-\eta+1,\dots,N}\mathsf{BS}_t$ with $z_0 \cdots z_{N-\eta} \neq 0$, the defining equations

$$
F_1(z) = \cdots = F_{c+r}(z) = 0, \qquad d F_1\,(z, \xi) = \cdots = d F_n\,(z, \xi) = 0
$$

now read as:

$$
(\mathsf{A}_0 + \cdots + \mathsf{A}_{N-\eta} + \mathsf{B}_0 + \cdots + \mathsf{B}_{N-\eta})\,(z, \xi) = 0, \qquad (70)
$$

and also that, concerning the algebraic dependence of $\mathsf{A}_1, \ldots, \mathsf{A}_{N-\eta}, \mathsf{B}_1, \ldots, \mathsf{B}_{N-\eta}$, one has:

$$\widehat{\mathsf{M}}^{\eta}(z, \xi) \in \mathscr{M}_{N-\eta}, \tag{71}$$

where $\widehat{\mathsf{M}}^{\eta}$ is the upper $(N-\eta) \times 2(N-\eta+1)$ submatrix of $\big(\mathsf{A}_1 \mid \cdots \mid \mathsf{A}_{N-\eta} \mid \mathsf{B}_1 \mid \cdots \mid \mathsf{B}_{N-\eta}\big)$.

Next, we estimate the dimension of the base locus ${}_{N-\eta+1,\ldots,N}\mathsf{BS}^0 := {}_{N-\eta+1,\ldots,N}\mathsf{BS} \cap \{z_0 \cdots z_{N-\eta} \neq 0\}$ over the coordinates nonvanishing part. Notice that, by our construction and by Lemma 10.1 below, the linear evaluation map at (z, ξ):

$$\big(\mathsf{A}_0 \,|\, \cdots \,|\, \mathsf{A}_{N-\eta} \,|\, \mathsf{B}_0 \,|\, \cdots \,|\, \mathsf{B}_{N-\eta}\big) : \ \mathbb{K}^{\blacklozenge+1} \longrightarrow \mathsf{Mat}_{N \times 2(N+1-\eta)}(\mathbb{K})$$

is surjective, because in (66) we have the nonvanishing terms:

$$z_j^d \neq 0, \quad z_{j_0}^{\mu_{l,k}} \cdots \widehat{z_{j_k}^{\mu_{l,k}}} \cdots z_{j_l}^{\mu_{l,k}} z_{j_k}^{d-l\mu_{l,k}} \neq 0$$
$$(j = 0 \ldots N-\eta, l = 1 \cdots N-\eta, k = 0 \ldots l, 0 \leqslant j_0 < \cdots < j_k \leqslant N-\eta).$$

Recall the Core Lemma: $\mathsf{codim} \mathscr{M}_{N-\eta} = N - \eta$. Thus, by the same argument as that of Proposition 8.2, we obtain the dimension estimate:

$$\begin{aligned} \dim {}_{N-\eta+1,\ldots,N}\mathsf{BS}^0 &\leqslant \dim {}_{N-\eta+1,\ldots,N}\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N}) \\ &\quad + \max_{\zeta} \dim \big(\pi_2^{-1}(\zeta) \cap {}_{N-\eta+1,\ldots,N}\mathsf{BS}^0\big) \\ &\leqslant \big(2N - 1 - \eta\big) + \big(\dim \mathbb{P}_{\mathbb{K}}^{\blacklozenge} - N - (N - \eta)\big) \\ &= \dim \mathbb{P}_{\mathbb{K}}^{\blacklozenge} - 1 \ < \ \dim \mathbb{P}_{\mathbb{K}}^{\blacklozenge}. \end{aligned}$$

where in the first line above ζ is running through all points of ${}_{N-\eta+1,\ldots,N}\mathring{\mathbb{P}}(\mathrm{T}_{\mathbb{P}^N})$. Hence we conclude the proof. □

Thus, we finish the claimed **Step 3** in Sect. 5.3. Now, the proof of Theorem 5.5 is complete, and so are the proofs of Theorems 5.4, 5.1.

10 Basic technical preparations

10.1 Surjectivity of evaluation maps

For every $N \geqslant 1$, denote:

$$\mathscr{A}(\mathbb{K}^{N+1}) := \mathbb{K}[z_0, \ldots, z_N].$$

For every $\lambda \geqslant 1$, the $\mathbb{K}$-linear space spanned by all degree λ homogeneous polynomials is:

$$\mathscr{A}_\lambda(\mathbb{K}^{N+1}) \subset \mathscr{A}(\mathbb{K}^{N+1}).$$

For every $g \in \mathscr{A}(\mathbb{K}^{N+1})$, for every $z \in \mathbb{K}^{N+1}$, denote by $(g \cdot v)_z$ the $\mathbb{K}$-linear evaluation map:

$$(g \cdot v)_z : \mathscr{A}(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}, \qquad f \longmapsto (gf)(z),$$

and for every $\xi \in \mathrm{T}_z\mathbb{K}^{N+1} \cong \mathbb{K}^{N+1}$, denote by $d_z(g\cdot)(\xi)$ the $\mathbb{K}$-linear differential evaluation map:

$$d_z(g\cdot)(\xi) : \mathscr{A}(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}, \qquad f \longmapsto d\,(gf)\big|_z(\xi).$$

Lemma 10.1 *For all $\lambda \geqslant 1$, for all $g \in \mathscr{A}(\mathbb{K}^{N+1})$, at any $z \in \mathbb{K}^{N+1} \setminus \{0\}$ with $g(z) \neq 0$, for every $\xi \in \mathbb{K}^{N+1} \setminus \mathbb{K} \cdot z$, the $\mathbb{K}$-linear map:*

$$\begin{pmatrix} (g \cdot v)_z \\ d_z(g\cdot)(\xi) \end{pmatrix} : \mathscr{A}_\lambda(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}^2$$

is surjective.

A similar result was obtained in [2, p. 444, Claim 3].

Proof By Leibniz's rule, we have:

$$\begin{pmatrix} (g \cdot v)_z \\ d_z(g\cdot)(\xi) \end{pmatrix} = \underbrace{\begin{pmatrix} g(z) & 0 \\ d\,g\big|_z(\xi) & g(z) \end{pmatrix}}_{\text{invertible, since } g(z) \neq 0} \begin{pmatrix} v_z \\ d_z(\xi) \end{pmatrix}, \tag{72}$$

thus we only need to prove the claim for the $\mathbb{K}$-linear map $(v_z, d_z(\xi))^T$ corresponding to $g = 1$.

Up to a linear transformation of $\mathbb{K}^{N+1}$, we may assume that $z = (1, 0, \dots, 0)$ and $\xi = (0, \dots, 0, 1)$. Now, computing that:

$$\begin{pmatrix} v_z \\ d_z(\xi) \end{pmatrix} z_0^\lambda = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} v_z \\ d_z(\xi) \end{pmatrix} z_0^{\lambda-1} z_N = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we conclude the proof. □

10.2 Fibre dimension estimates

The following classical theorem will prove fundamental in dimensional estimations of all base loci in this article.

Theorem 10.2 *Let X, Y be two $\mathbb{K}$-varieties, and let $f\colon X \to Y$ be a morphism. Then one has:*

$$\dim X \leqslant \dim Y + \max_{y \in Y} \dim f^{-1}(y).$$

□

10.3 Classical codimension formulas for determinantal ideals

For all $p, q \geqslant 1$, denote by:

$$\mathsf{Mat}_{p\times q}(\mathbb{K}) = \mathbb{K}^{p\times q}$$

the space of all $p \times q$ matrices with entries in $\mathbb{K}$. For every $0 \leqslant \ell \leqslant \max\{p, q\}$, denote by:

$$\Sigma_\ell^{p,q} \subset \mathsf{Mat}_{p\times q}(\mathbb{K})$$

all $p \times q$ matrices with rank $\leqslant \ell$.

Theorem 10.3 *There holds the codimension formula:*

$$\operatorname{codim} \Sigma_\ell^{p,q} = \max\big\{(p-\ell)\,(q-\ell),\ 0\big\}.$$

□

10.4 Affine cones preserve codimensions

In our practice, it would be more convenient to count dimension in an affine space rather than in a projective space.

Theorem 10.4 *Let $\pi\colon \mathbb{K}^{N+1} \to \mathbb{P}^N_{\mathbb{K}}$ be the canonical projection, and let $Y \subset \mathbb{P}^N_{\mathbb{K}}$ be a nonempty algebraic set defined by a homogeneous ideal $I \subset \mathbb{K}[z_0, \ldots, z_N]$. Denote by $C(Y)$ the affine cone:*

$$C(Y) := \pi^{-1}(Y) \cup \{\mathbf{0}\} \subset \mathbb{K}^{N+1}.$$

Then $C(Y)$ is an algebraic set in $\mathbb{K}^{N+1}$, and it preserves codimension:

$$\operatorname{codim} C(Y) = \operatorname{codim} Y.$$

□

11 The engine of MCM

11.1 Core codimension formulas

To prove the Core Lemma, by induction on positive integers $p \geqslant 2$ and $0 \leqslant \ell \leqslant p$, we estimate the codimension ${}_{\ell}C_p$ of:

$$ {}_{\ell}\mathbf{X}_p \subset \mathsf{Mat}_{p\times 2p}(\mathbb{K}) \tag{73} $$

which consists of $p \times 2p$ matrices $\mathsf{X}_p = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p)$ such that:

(i) the first p column vectors have rank:

$$ \mathsf{rank}_{\mathbb{K}} \left\{\alpha_1, \ldots, \alpha_p\right\} \leqslant \ell; \tag{74} $$

(ii) for every $\nu = 1 \ldots p$, there holds:

$$ \mathsf{rank}_{\mathbb{K}} \left\{\alpha_1, \ldots, \widehat{\alpha_\nu}, \ldots, \alpha_p, \alpha_\nu + (\beta_1 + \cdots + \beta_p)\right\} \leqslant p - 1; \tag{75} $$

(iii) for every $\tau = 1 \ldots p-1$, for every $\rho = \tau + 1 \cdots p$, there holds:

$$ \begin{aligned} \mathsf{rank}_{\mathbb{K}} &\left\{\alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \ldots, \widehat{\alpha_\rho}, \ldots, \alpha_p, \alpha_\rho \right. \\ &\left. +(\beta_{\tau+1} + \cdots + \beta_p)\right\} \leqslant p - 1. \end{aligned} \tag{76} $$

Noting that our target variety $\mathscr{M}_N$ in (53) is nothing but ${}_N\mathbf{X}_N$, hence the Core Lemma reads as ${}_NC_N = N$.

Let us start with the easy case $\ell = 0$.

Proposition 11.1 *For every integer $p \geqslant 2$, the codimension value ${}_{\ell}C_p$ for $\ell = 0$ is:*

$$ {}_0C_p = p^2 + 1. \tag{77} $$

Proof Firstly, (i) is equivalent to $\alpha_1 = \cdots = \alpha_p = \mathbf{0}$, contributing codimension p^2. Hence (ii) holds trivially, and the only nontrivial inequality in (iii) is:

$$ \mathsf{rank}_{\mathbb{K}} \left\{\mathbf{0} + \beta_1, \ldots, \mathbf{0} + \beta_p\right\} \leqslant p - 1, $$

which contributes one more codimension. □

For the general case $\ell = 1 \ldots p$, we will use Gaussian eliminations and do inductions on p, ℓ. Firstly, we need

Lemma 11.2 *Let W be a $\mathbb{K}$-vector space, and let $p \geqslant 1$ be a positive integer. For any $p+1$ vectors $\alpha_1, \dots, \alpha_p, \beta \in W$, the rank restriction:*

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \widehat{\alpha_\nu}, \dots, \alpha_p, \alpha_\nu + \beta\} \leqslant p-1 \qquad {\scriptstyle (\nu = 1 \cdots p)}, \tag{78}$$

is equivalent to:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p, \beta\} \leqslant p-1,$$

or

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p\} = p, \qquad (\alpha_1 + \cdots + \alpha_p) + \beta = \mathbf{0}.$$

Proof The "if" direction being immediate, we only prove the "only if" part.

If $\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p\} \leqslant p-2$, then $\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p, \beta\} \leqslant p-1$. If $\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p\} = p-1$, then (78) implies that $\beta \in \mathsf{Span}_{\mathbb{K}} (\alpha_1, \dots, \alpha_p)$, hence $\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p, \beta\} \leqslant p-1$. Lastly, if $\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p\} = p$, then (78) implies that:

$$(\alpha_1 + \cdots + \alpha_p) + \beta \in \mathsf{Span}_{\mathbb{K}} \{\alpha_1, \dots, \widehat{\alpha_\nu}, \dots, \alpha_p\} \qquad {\scriptstyle (\nu = 1 \cdots p)}.$$

Since $\alpha_1, \dots, \alpha_p$ are linearly independent, this yields that $(\alpha_1 + \cdots + \alpha_p) + \beta = \mathbf{0}$. □

Now, we count the codimension of the exceptional locus of Gaussian eliminations.

Proposition 11.3 *For every integer $p \geqslant 2$, the codimensions ${}_{\ell}C_p^0$ of the subvarieties:*

$$\{\alpha_1 + \beta_1 = \mathbf{0}\} \cap {}_{\ell}\mathbf{X}_p \subset \mathsf{Mat}_{p \times 2p}(\mathbb{K})$$

read according to the values of ℓ as:

$${}_{\ell}C_p^0 = \begin{cases} p+2 & (\ell = p-1,\ p), \\ p + (p-\ell)^2 & (\ell = 0 \cdots p-2). \end{cases}$$

Proof The condition $\alpha_1 + \beta_1 = \mathbf{0}$ contributes codimension p. Now, the inequality (76) in (iii) trivially holds, and the restriction (75), thanks to the lemma 11.2, is equivalent to:

$$\mathsf{rank}_{\mathbb{K}} \{\alpha_1, \dots, \alpha_p, \beta_1 + \cdots + \beta_p\} \leqslant p-1, \tag{79}$$

or

$$\mathsf{rank}_{\mathbb{K}}\,\{\alpha_1,\dots,\alpha_p\}\;=\;p,\qquad (\alpha_1+\cdots+\alpha_p)+(\beta_1+\cdots+\beta_p)\;=\;\mathbf{0}. \tag{80}$$

Since $\alpha_1+\beta_1=0$, adding the first column vector of (79) to the last one, we get:

$$\mathsf{rank}_{\mathbb{K}}\,\{\alpha_1,\dots,\alpha_p,\beta_2+\cdots+\beta_p\}\;\leqslant\;p-1,$$

which, by Theorem 10.3, contributes codimension 2. Similarly, (80) is equivalent to:

$$\mathsf{rank}_{\mathbb{K}}\,\{\alpha_1,\dots,\alpha_p\}\;=\;p,\qquad (\alpha_2+\cdots+\alpha_p)+(\beta_2+\cdots+\beta_p)\;=\;\mathbf{0},$$

contributing codimension p. Therefore, when $\ell=p-1$ or $\ell=p$, we obtain that:

$${}_{p-1}C_p^0=p+2,\qquad {}_pC_p^0=\mathsf{min}\,\{p+2,\;p+p\}=p+2.$$

When $\ell=0\cdots p-2$, the condition (75) follows trivially after (74), which, by Theorem 10.3, contributes codimension $(p-\ell)^2$. Combined with the codimension p contributed by the independent equation $\alpha_1+\beta_1=\mathbf{0}$, this finishes the proof. □

Next, we claim the following *Codimension Induction Formulas*, the proof of which will appear in Sect. 11.4. To make sense of ${}_{\ell-2}C_{p-1}$ in (83) when $\ell=1$, we henceforth agree ${}_{-1}C_{p-1}:=\infty$.

Proposition 11.4 *For all $p\geqslant 3$, there hold the codimension induction formulas:*

$$\begin{aligned}
{}_pC_p &= \mathsf{min}\left\{p,\;\;{}_{p-1}C_p\right\} \qquad {\scriptstyle(\ell=p)}, \qquad\qquad (81)\\
{}_{p-1}C_p &\geqslant \mathsf{min}\left\{{}_{p-1}C_p^0,\;\;{}_{p-1}C_{p-1}+2,\;\;{}_{p-2}C_{p-1}+1,\;\;{}_{p-3}C_{p-1}\right\}\\
&\qquad {\scriptstyle(\ell=p-1)}, \qquad\qquad (82)\\
{}_\ell C_p &\geqslant \mathsf{min}\left\{{}_\ell C_p^0,\;\;{}_\ell C_{p-1}+2(p-\ell)-1,\right.\\
&\qquad \left.{}_{\ell-1}C_{p-1}+(p-\ell),\;\;{}_{\ell-2}C_{p-1}\right\} \qquad {\scriptstyle(\ell=1\cdots p-2)}. \qquad\qquad (83)
\end{aligned}$$

Now, we can estimate the codimension values ${}_\ell C_p$ iteratively to deduce the crucial

Proposition 11.5 *For all $p\geqslant 2$, there hold:*

$${}_pC_p=p,\qquad {}_\ell C_p\;\geqslant\;\ell+(p-\ell)^2+1 \qquad {\scriptstyle(\ell=0\cdots p-1)}. \tag{84}$$

In particular, the Core Lemma follows.

Proof We only need to verify the initial data for $p = 2$:

$$ {}_0C_2 = 5, \qquad {}_1C_2 = 3, \qquad {}_2C_2 = 2, $$

the remaining cases $p \geqslant 3$ being easy to check by straightforward induction using (81)–(83).

Recalling formulas (77) and (81), we only need to prove ${}_1C_2 = 3$.

For every matrix $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in {}_1\mathbf{X}_2 \setminus {}_0\mathbf{X}_2$, we have the following 4 defining equations: $\mathsf{rank}_{\mathbb{K}}\big\{\alpha_1, \alpha_2\big\} = 1$, $\mathsf{rank}_{\mathbb{K}}\big\{\alpha_1 + (\beta_1 + \beta_2), \alpha_2\big\} \leqslant 1$, $\mathsf{rank}_{\mathbb{K}}\big\{\alpha_1, \alpha_2 + (\beta_1 + \beta_2)\big\} \leqslant 1$, $\mathsf{rank}_{\mathbb{K}}\big\{\alpha_1 + \beta_1, \alpha_2 + \beta_2\big\} \leqslant 1$. Without loss of generality, we may assume that $\alpha_1 \neq \mathbf{0}$. Then the first equation yields $\alpha_2 \in \mathbb{K} \cdot \alpha_1$, and the third one implies $\alpha_2 + (\beta_1 + \beta_2) \in \mathbb{K} \cdot \alpha_1$, whence by subtracting we receive $\beta_1 + \beta_2 \in \mathbb{K} \cdot \alpha_1$. Hence $\alpha_1 + \alpha_2 + (\beta_1 + \beta_2) \in \mathbb{K} \cdot \alpha_1$. Now, the fourth equation implies $\mathsf{rank}_{\mathbb{K}}\big\{\alpha_1 + \alpha_2 + (\beta_1 + \beta_2),\ \alpha_2 + \beta_2\big\} \leqslant 1$, therefore, there are two possible situations, the first one being:

$$ \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = \mathbf{0}, \tag{85} $$

and the second one being:

$$ \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \ \in\ \mathbb{K} \cdot \alpha_1 \setminus \{\mathbf{0}\}, \qquad \alpha_2 + \beta_2 \ \in\ \mathbb{K} \cdot \alpha_1. $$

In either case, together with the known restrictions $\alpha_2, \beta_1 + \beta_2 \in \mathbb{K} \cdot \alpha_1$, the total restrained codimension is $\geqslant 3$. Therefore ${}_1C_2 \geqslant 3$. Lastly, noting that ${}_1\mathbf{X}_2$ contains a subvariety defined by the two equations:

$$ \mathsf{rank}_{\mathbb{K}}\, \big\{\alpha_1, \alpha_2\big\} \leqslant 1, \quad \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = \mathbf{0}, $$

whose codimension is exactly 3, we conclude that ${}_1C_2 = 3$. □

11.2 Gaussian eliminations

Following the notation in (73), we denote by:

$$ \mathsf{X}_p = (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p) $$

the coordinate columns of $\mathsf{Mat}_{p \times 2p}(\mathbb{K})$, where each of the first p columns explicitly writes as:

$$ \alpha_i = (\alpha_{1,i}, \dots, \alpha_{p,i})^{\mathrm{T}}, $$

and where each of the last p columns explicitly writes as:

$$ \beta_i = (\beta_{1,i}, \dots, \beta_{p,i})^{\mathrm{T}}. $$

First, observing the structures of the matrices in (75), (76):

$$
\begin{aligned}
\mathsf{X}_p^{0,\nu} &:= \big(\alpha_1 \mid \cdots \mid \underline{\widehat{\alpha_\nu}} \mid \cdots \mid \alpha_p \mid \underline{\alpha_\nu + (\beta_1 + \cdots + \beta_p)}\big),\\
\mathsf{X}_p^{\tau,\rho} &:= \big(\alpha_1 + \beta_1 \mid \cdots \mid \alpha_\tau + \beta_\tau \mid \alpha_{\tau+1}\\
&\qquad \mid \cdots \mid \underline{\widehat{\alpha_\rho}} \mid \cdots \mid \alpha_p \mid \underline{\alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_p)}\big),
\end{aligned}
$$

where the second underlined columns are understood to appear in the first underlined places, we realize that they have the uniform shapes:

$$
\mathsf{X}_p^{0,\nu} = \mathsf{X}_p\, \mathsf{I}_p^{0,\nu}, \qquad \mathsf{X}_p^{\tau,\rho} = \mathsf{X}_p\, \mathsf{I}_p^{\tau,\rho}, \tag{86}
$$

where the $2p \times p$ matrices $\mathsf{I}_p^{0,\nu}$ explicitly read as:

$$
\begin{array}{c}
\overbrace{\hphantom{1 \qquad \ddots \qquad 1}}^{p}\\
\left(\begin{array}{ccc}
1 & & \\
 & \ddots & \\
 & & 1 \\
\hdashline
 & 1 & \\
 & \vdots & \\
 & 1 &
\end{array}\right)
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\}p\\
\left.\vphantom{\begin{array}{c}1\\ \vdots\\ 1\end{array}}\right\}p
\end{array}\\
\uparrow\ \nu\text{-th column}
\end{array}
$$

the upper $p \times p$ submatrix being the identity, the lower $p \times p$ submatrix being zero except its νth column being a column of 1, and where lastly, the $2p \times p$ matrices $\mathsf{I}_p^{\tau,\rho}$ explicitly read as:

$$
\begin{array}{c}
\overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxx}}^{p}\\
\left(\begin{array}{ccc|ccc}
1 & & & & & \\
 & \ddots & & & & \\
 & & 1 & & & \\
 & & & 1 & & \\
 & & & & \ddots & \\
 & & & & & 1 \\ \hline
1 & & & & & \\
 & \ddots & & & & \\
 & & 1 & & & \\ \hline
 & & & & 1 & \\
 & & & & \vdots & \\
 & & & & 1 &
\end{array}\right)
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\}p\\
\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}\tau\\
\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}p-\tau
\end{array}\\
\begin{array}{cc}
\uparrow & \uparrow\\
\tau\text{-th column} & \rho\text{-th column}
\end{array}
\end{array}
$$

the upper $p \times p$ submatrix being the identity, the lower $p \times p$ submatrix being zero except τ copies of 1 in the beginning diagonal and $p - \tau$ copies of 1 at the end of the ρth column.

Next, observe that all matrices $\mathsf{X}^{\tau,\,\rho}$ have the same first column:

$$\alpha_1 + \beta_1 = (\,\alpha_{1,1} + \beta_{1,1} \mid \cdots \mid \alpha_{p,1} + \beta_{p,1})^{\mathrm{T}}.$$

Therefore, when $\alpha_{1,1}+\beta_{1,1} \neq 0$, operating Gaussian eliminations by means of the matrix:

$$\mathsf{G} := \begin{pmatrix} 1 & & & \\ -\frac{\alpha_{2,1}+\beta_{2,1}}{\alpha_{1,1}+\beta_{1,1}} & 1 & & \\ \vdots & & \ddots & \\ -\frac{\alpha_{p,1}+\beta_{p,1}}{\alpha_{1,1}+\beta_{1,1}} & & & 1 \end{pmatrix}, \tag{87}$$

these matrices $\mathsf{X}^{\tau,\,\rho}$ become simpler:

$$\mathsf{G}\,\mathsf{X}_p^{\tau,\,\rho} = \begin{pmatrix} \alpha_{1,1}+\beta_{1,1} & \bullet & \cdots & \bullet \\ 0 & \star & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \star & \cdots & \star \end{pmatrix}, \tag{88}$$

where we will soon see the $(p-1)\times(p-1)$ star submatrices enjoy amazing structural properties.

Observation 11.6 *Let $p \geqslant 1$ be a positive integer, let A be a $p \times 2p$ matrix, let B be a $2p \times p$ matrix such that both its $1st$, $(p+1)th$ rows are $(1, 0, \ldots, 0)$. Then there holds:*

$$(A\,B)' = A''\,B''',$$

where $(A\,B)'$ means the $(p-1)\times(p-1)$ matrix obtained by deleting the first row and column of $A\,B$, and where A'' means the $(p-1)\times 2(p-1)$ matrix obtained by deleting the first row and the columns $1, p+1$ of A, and where B''' means the $2(p-1)\times(p-1)$ matrix obtained by deleting the first column and the rows $1, p+1$ of B. □

Observation 11.7 *For all $p \geqslant 3$, $\tau = 1\ldots p-1$, $\rho = \tau+1\ldots p$, the matrices $\mathsf{I}_p^{\tau,\,\rho}$ transform to $\mathsf{I}_{p-1}^{\tau-1,\rho-1}$ after deleting the first column and the rows $1, p+1$, i.e. $\mathsf{I}_{p-1}^{\tau-1,\rho-1} = (\mathsf{I}_p^{\tau,\,\rho})'''$.* □

Now, thanks to the above two Observations, noting that:

$$\mathsf{G}\,\mathsf{X}_p^{\tau,\,\rho} = \mathsf{G}\left(\mathsf{X}_p\,\mathsf{I}_p^{\tau,\,\rho}\right) = \left(\mathsf{G}\,\mathsf{X}_p\right)\mathsf{I}_p^{\tau,\,\rho},$$

denoting $\mathsf{X}_p^{\mathsf{G}} := (\mathsf{G}\,\mathsf{X}_p)''$, the $(p-1)\times(p-1)$ star submatrices in (88) thus have the forms:

$$\begin{pmatrix} \star & \cdots & \star \\ \vdots & \ddots & \vdots \\ \star & \cdots & \star \end{pmatrix} = \mathsf{X}_p^{\mathsf{G}}\,\mathsf{I}_{p-1}^{\tau-1,\,\rho-1}. \tag{89}$$

Comparing (89) and (86), we immediately see that the star submatrices have the same structures as $\mathsf{X}_p^{0,\nu}$, $\mathsf{X}_p^{\tau,\,\rho}$.

11.3 Study of the morphism of left-multiplication by G

Let us denote by:

$$D(\alpha_{1,1}+\beta_{1,1}) \quad \subset \quad \mathsf{Mat}_{p\times 2p}(\mathbb{K})$$

the Zariski open set defined by $\alpha_{1,1}+\beta_{1,1}\neq 0$. Now, consider the morphism of left-multiplication by the regular function matrix G:

$$\begin{aligned} L_{\mathsf{G}}\colon \quad D(\alpha_{1,1}+\beta_{1,1}) &\longrightarrow D(\alpha_{1,1}+\beta_{1,1}) \\ \mathsf{X}_p &\longmapsto \mathsf{G}\,\mathsf{X}_p. \end{aligned}$$

Note that it is not surjective, because direct computation shows that its image lies in the variety:

$$\cap_{i=2}^{p}\,\{\alpha_{i,1}+\beta_{i,1}=0\}.$$

In order to compensate this loss of surjectivity, combining with:

$$\begin{aligned} ¤\colon \quad D(\alpha_{1,1}+\beta_{1,1}) &\longrightarrow \mathsf{Mat}_{(p-1)\times 1}(\mathbb{K}) \\ \mathsf{X}_p &\longmapsto (\alpha_{2,1}+\beta_{2,1}\mid\cdots\mid\alpha_{p,1}+\beta_{p,1})^{\mathrm{T}}, \end{aligned}$$

we construct a morphism:

$$\begin{aligned} &L_{\mathsf{G}}\oplus ¤\colon \quad D(\alpha_{1,1}+\beta_{1,1}) \\ &\longrightarrow \big(\cap_{\ell=2}^{p}\,\{\alpha_{l,1}+\beta_{l,1}=0\}\,\cap\, D(\alpha_{1,1}+\beta_{1,1})\big)\ \oplus\ \mathsf{Mat}_{(p-1)\times 1}(\mathbb{K}), \end{aligned}$$

which turns out to be an isomorphism. Indeed, denoting the right-hand-side target space by ✠, it has the inverse morphism:

$$\begin{aligned} ✠ &\longrightarrow D(\alpha_{1,1}+\beta_{1,1}) \\ \mathsf{Y}\,\oplus\,(s_2,\dots,s_p)^{\mathsf{T}} &\longmapsto {}^{-1}\mathsf{G}\cdot\mathsf{Y}, \end{aligned}$$

where the regular function matrix ${}^{-1}\mathsf{G}$ is the 'inverse' of the matrix G in (87):

$$\begin{pmatrix} 1 & & & \\ \frac{s_2}{\alpha_{1,1}+\beta_{1,1}} & 1 & & \\ \vdots & & \ddots & \\ \frac{s_p}{\alpha_{1,1}+\beta_{1,1}} & & & 1 \end{pmatrix}. \tag{90}$$

Now, denote by $\pi_p\colon \mathsf{Mat}_{p\times 2p}(\mathbb{K}) \longrightarrow \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K})$ the projection obtained by deleting the first row and the columns 1, $p+1$. Denote also $\mathscr{L}_{\mathsf{G}} := \pi_p \circ L_{\mathsf{G}}$.

Recalling the end of Sect. 11.2, we have $\mathsf{X}_p^{\mathsf{G}} = \mathscr{L}_{\mathsf{G}}(\mathsf{X}_p)$, and we receive the following key

Observation 11.8 *For every $p \geqslant 3$, for every $\ell = 1 \dots p-1$, the image of the variety:*

$$ {}_{\ell}\mathbf{X}_p \,\cap\, D(\alpha_{1,1}+\beta_{1,1}) \,\subset\, D(\alpha_{1,1}+\beta_{1,1}) $$

under the map:

$$ \mathscr{L}_{\mathsf{G}}\colon \quad D(\alpha_{1,1}+\beta_{1,1}) \longrightarrow \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}) $$

is contained in the variety:

$$ {}_{\ell}\mathbf{X}_{p-1} \quad\subset\quad \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}). $$

□

11.4 Proof of Proposition 11.4

First, applying Lemma 11.2, we receive:

Corollary 11.9 *For every $p \geqslant 2$, the difference of the varieties:*

$$ {}_{p}\mathbf{X}_p \setminus {}_{p-1}\mathbf{X}_p \quad\subset\quad \mathsf{Mat}_{p\times 2p}(\mathbb{K}) $$

is exactly the quasi-variety:

$$ \Big\{\alpha_1+\cdots+\alpha_p+\beta_1+\cdots+\beta_p=\mathbf{0}\Big\} \;\cap\; \Big\{\mathsf{rank}_{\mathbb{K}}\,\{\alpha_1,\dots,\alpha_p\}=p\Big\}, $$

whose codimension is p. □

Now, we give a complete proof of Proposition 11.4.

Proof of (81) This is a direct consequence of the above corollary. □

Proof of (82) By Observation 11.8, under the map $\mathscr{L}_{\mathsf{G}}\colon D(\alpha_{1,1}+\beta_{1,1}) \longrightarrow \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K})$, the image of the variety ${}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1}+\beta_{1,1})$ is contained in the variety ${}_{p-1}\mathbf{X}_{p-1} \subset \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K})$.

Now, we decompose the variety ${}_{p-1}\mathbf{X}_{p-1}$ into three pieces:

$$ {}_{p-1}\mathbf{X}_{p-1} \;=\; {}_{p-3}\mathbf{X}_{p-1} \cup \big({}_{p-2}\mathbf{X}_{p-1} \setminus {}_{p-3}\mathbf{X}_{p-1}\big) \cup \big({}_{p-1}\mathbf{X}_{p-1} \setminus {}_{p-2}\mathbf{X}_{p-1}\big), \tag{91} $$

where each matrix $(\alpha_1, \ldots, \alpha_{p-1}, \beta_1, \ldots, \beta_{p-1})$ in the $\underline{\text{first}}_1$ (resp. $\underline{\text{second}}_2$, $\underline{\text{third}}_3$) piece satisfies:

$$\mathsf{rank}_{\mathbb{K}}\,(\alpha_1, \ldots, \alpha_{p-1}) \underline{\leqslant\ p-3}_1 \qquad (\text{resp.} \underline{=\ p-2}_2\,, \underline{=\ p-1}_3). \tag{92}$$

Pulling back (91) by the map $\mathscr{L}_{\mathsf{G}}$, we see that ${}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1} + \beta_{1,1})$ is contained in:

$$\mathscr{L}_{\mathsf{G}}^{-1}\,({}_{p-3}\mathbf{X}_{p-1}) \cup \mathscr{L}_{\mathsf{G}}^{-1}\big({}_{p-2}\mathbf{X}_{p-1} \setminus {}_{p-3}\mathbf{X}_{p-1}\big) \\ \cup \mathscr{L}_{\mathsf{G}}^{-1}\big({}_{p-1}\mathbf{X}_{p-1} \setminus {}_{p-2}\mathbf{X}_{p-1}\big). \tag{93}$$

Firstly, for every point in the first piece $\mathsf{Y} \in {}_{p-3}\mathbf{X}_{p-1}$, using the commutative diagram:

$$\begin{tikzcd} D(\alpha_{1,1} + \beta_{1,1}) \arrow[r, "L_{\mathsf{G}} \oplus \text{¤}"] \arrow[dr, "\mathscr{L}_{\mathsf{G}}"'] & \maltese \arrow[d, "\pi_p \oplus \mathbf{0}"] \\ & \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}), \end{tikzcd} \tag{94}$$

where the horizontal map is an isomorphism, we obtain the fibre isormorphism:

$$\begin{aligned} \mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) &\cong (\pi_p \oplus \mathbf{0})^{-1}(\mathsf{Y}) \cap (L_{\mathsf{G}} \oplus \text{¤})\, D(\alpha_{1,1} + \beta_{1,1}) \\ &= \big(\pi_p^{-1}(\mathsf{Y}) \cap L_{\mathsf{G}}\, D(\alpha_{1,1} + \beta_{1,1})\big) \ \oplus\ \mathsf{Mat}_{(p-1)\times 1}(\mathbb{K}) \\ &= \Big(\pi_p^{-1}(\mathsf{Y}) \cap \big(\cap_{\ell=2}^{p}\ \{\alpha_{l,1} + \beta_{l,1} = 0\} \ \cap\ D(\alpha_{1,1} + \beta_{1,1})\big)\Big) \\ &\oplus\ \mathsf{Mat}_{(p-1)\times 1}(\mathbb{K}), \end{aligned}$$

thus we receive that:

$$\begin{aligned} \dim \mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) &= \big(\dim \ker \pi_p - (p-1)\big) + \dim \mathsf{Mat}_{(p-1)\times 1}(\mathbb{K}) \\ &= \dim \mathsf{Mat}_{p\times 2p}(\mathbb{K}) - \dim \mathsf{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}). \end{aligned}$$

Now, applying Theorem 10.2 to the regular map $\mathscr{L}$ restricted to:

$$\mathbf{I} := \mathscr{L}_{\mathsf{G}}^{-1}\,({}_{p-3}\mathbf{X}_{p-1}) \ \subset\ \mathsf{Mat}_{p\times 2p}(\mathbb{K}),$$

we receive the codimension estimate:

$$\mathsf{codim}\,\mathbf{I} \ \geqslant\ \mathsf{codim}\ {}_{p-3}\mathbf{X}_{p-1}. \tag{95}$$

Secondly, for every point in the second piece $\mathsf{Y} \in {}_{p-2}\mathbf{X}_{p-1} \setminus {}_{p-3}\mathbf{X}_{p-1}$, for every matrix:

$$(\alpha_1 \mid \cdots \mid \alpha_p \mid \beta_1 \mid \cdots \mid \beta_p) \;\in\; \mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \cap \big({}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1} + \beta_{1,1})\big) \\ =: \clubsuit$$

the bottom $(p-1) \times (p-1)$ submatrix J of $\big(\alpha_2 \mid \cdots \mid \alpha_p\big)$ is exactly the left $(p-1) \times (p-1)$ submatrix of Y, hence $\mathsf{rank}_{\mathbb{K}}\, J = p-2$. Consequently, the condition $\mathsf{rank}_{\mathbb{K}}\, \big(\alpha_1 \mid \cdots \mid \alpha_p\big) \leqslant p-1$ is non-trivial, whence guarantees that $\clubsuit \subset \pi_p^{-1}(\mathsf{Y})$ has codimension $\geqslant 1$. Hence, applying Theorem 10.2 to the map $\mathscr{L}_{\mathsf{G}}$ restricted to:

$$\mathbf{II} := \mathscr{L}_{\mathsf{G}}^{-1}\big({}_{p-2}\mathbf{X}_{p-1} \setminus {}_{p-3}\mathbf{X}_{p-1}\big) \cap \big({}_{p-1}\mathbf{X}_p \\ \cap D(\alpha_{1,1} + \beta_{1,1})\big) \;\subset\; \mathsf{Mat}_{p \times 2p}(\mathbb{K}),$$

we receive the codimension estimate:

$$\begin{aligned} \mathsf{codim}\, \mathbf{II} &\geqslant \mathsf{codim}\, \big({}_{p-2}\mathbf{X}_{p-1} \setminus {}_{p-3}\mathbf{X}_{p-1}\big) + 1 \\ &\geqslant \mathsf{codim}\, {}_{p-2}\mathbf{X}_{p-1} + 1. \end{aligned} \tag{96}$$

Thirdly, for every point in the third piece $\mathsf{Y} \in {}_{p-1}\mathbf{X}_{p-1} \setminus {}_{p-2}\mathbf{X}_{p-1}$, using the commutative diagram (94), we have the isomorphism:

$$\mathscr{L}_{\mathsf{G}}^{-1}(\mathsf{Y}) \cap \big({}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1} + \beta_{1,1})\big) \\ \cong (\pi_p \oplus \mathbf{0})^{-1}(\mathsf{Y}) \cap (L_{\mathsf{G}} \oplus ¤)\, \big({}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1} + \beta_{1,1})\big) =: \spadesuit.$$

For every element $(\alpha_1 \mid \cdots \mid \alpha_p \mid \beta_1 \mid \cdots \mid \beta_p) \oplus (s_2, \ldots, s_p)^{\mathsf{T}} \in \spadesuit$, we have:

$$\mathsf{rank}_{\mathbb{K}}\, (\alpha_1, \ldots, \alpha_p) \;\leqslant\; p-1.$$

Moreover, since the rank inequalities (75) are preserved under the map L_{G}, taking $\nu = 1$ we receive:

$$\mathsf{rank}_{\mathbb{K}}\, \big(\alpha_1 + (\beta_1 + \cdots + \beta_p), \alpha_2, \ldots, \alpha_p\big) \;\leqslant\; p-1.$$

Noting that the bottom $(p-1) \times (p-1)$ submatrix of $(\alpha_2 \mid \cdots \mid \alpha_p)$ is the left $(p-1) \times (p-1)$ submatrix of Y, having full rank, we receive $\mathsf{rank}_{\mathbb{K}}(\alpha_2, \ldots, \alpha_p) = p-1$, and hence we obtain:

$$\mathsf{rank}_{\mathbb{K}}\, (\beta_1 + \cdots + \beta_p, \alpha_1, \alpha_2, \ldots, \alpha_p) \;=\; p-1.$$

Therefore, we get 2 conditions which are easily seen to be independent from the original equations $\alpha_{1,j} + \beta_{1,j} = 0$ for $j = 2 \cdots p$. Thus $\spadesuit \cap (\pi_p \oplus \mathbf{0})^{-1}(\mathsf{Y}) \subset (\pi_p \oplus \mathbf{0})^{-1}(\mathsf{Y})$ has codimension $\geqslant 2$. Now, applying once again Theorem 10.2 to the map $\mathscr{L}_{\mathsf{G}}$ restricted to:

$$\begin{aligned}\mathbf{III} := \mathscr{L}_{\mathsf{G}}^{-1}\big({}_{p-1}\mathbf{X}_{p-1} \setminus {}_{p-2}\mathbf{X}_{p-1}\big) \cap \big({}_{p-1}\mathbf{X}_p \\ \cap D(\alpha_{1,1} + \beta_{1,1})\big) \ \subset \ \mathsf{Mat}_{p\times 2p}(\mathbb{K}),\end{aligned}$$

we receive the codimension estimate:

$$\begin{aligned}\mathsf{codim}\,\mathbf{III} &\geqslant \mathsf{codim}\,\big({}_{p-1}\mathbf{X}_{p-1} \setminus {}_{p-2}\mathbf{X}_{p-1}\big) + 2 \\ &\geqslant \mathsf{codim}\,{}_{p-1}\mathbf{X}_{p-1} + 2.\end{aligned} \tag{97}$$

Summarizing (93), (95), (96), (97), we receive the codimension estimate:

$$\begin{aligned}&\mathsf{codim}\,{}_{p-1}\mathbf{X}_p \cap D(\alpha_{1,1} + \beta_{1,1}) \\ &\quad \geqslant \mathsf{min}\,\big\{\mathsf{codim}\,{}_{p-3}\mathbf{X}_{p-1},\ \mathsf{codim}\,{}_{p-2}\mathbf{X}_{p-1} + 1,\ \mathsf{codim}\,{}_{p-1}\mathbf{X}_{p-1} + 2\big\}.\end{aligned}$$

By permuting the indices, we know that all:

$${}_{p-1}\mathbf{X}_p \cap D(\alpha_{i,1} + \beta_{i,1}) \ \subset \ \mathsf{Mat}_{p\times 2p}(\mathbb{K}) \qquad (i = 1 \cdots p)$$

have the same codimension, and so does their union:

$$\begin{aligned}&{}_{p-1}\mathbf{X}_p \cap D(\alpha_1 + \beta_1) \\ &\quad = \cup_{i=1}^{p}\big({}_{p-1}\mathbf{X}_p \cap D(\alpha_{i,1} + \beta_{i,1})\big) \ \subset \ \mathsf{Mat}_{p\times 2p}(\mathbb{K}).\end{aligned}$$

Finally, taking codimension on both sides of:

$${}_{p-1}\mathbf{X}_p = \big({}_{p-1}\mathbf{X}_p \cap \{\alpha_1 + \beta_1 = 0\}\big) \cup \big({}_{p-1}\mathbf{X}_p \cap D(\alpha_1 + \beta_1)\big),$$

Proposition 11.3 and the preceding estimate conclude the proof. □

Proof of (83) If $\ell \geqslant 2$, decompose the variety ${}_{\ell}\mathbf{X}_{p-1}$ into three pieces:

$${}_{\ell}\mathbf{X}_{p-1} = {}_{\ell-2}\mathbf{X}_{p-1} \cup \big({}_{\ell-1}\mathbf{X}_{p-1} \setminus {}_{\ell-2}\mathbf{X}_{p-1}\big) \cup \big({}_{\ell}\mathbf{X}_{p-1} \setminus {}_{\ell-1}\mathbf{X}_{p-1}\big);$$

and if $\ell = 1$, decompose the variety ${}_{\ell}\mathbf{X}_{p-1}$ into two pieces:

$${}_{1}\mathbf{X}_{p-1} = {}_{0}\mathbf{X}_{p-1} \cup \big({}_{1}\mathbf{X}_{p-1} \setminus {}_{0}\mathbf{X}_{p-1}\big).$$

Now, by mimicking the preceding proof, everything goes on smoothly. □

12 Algorithm of MCM and estimates on hypersurface degrees

12.1 A must design

Given the integer $\heartsuit \geqslant 1$ in Theorem 5.1, recalling the Step 2 in Sect. 5.3, we naturally carry out the *Algorithm* of MCM as follows.

The procedure is to first construct $\mu^{l,k}$ in a lexicographic order with respect to indices (l,k), for $l = c+r+1 \cdots N$, $k = 0 \cdots l$, along with a set of positive integers δ_l.

For simplicity, we start by setting:

$$\delta_{c+r+1} \geqslant \max\{\epsilon_1, \ldots, \epsilon_{c+r}\}. \tag{98}$$

For every integer $l = c + r + 1 \cdots N$, in this step, we begin with choosing $\mu_{l,0}$ that satisfies:

$$\text{[see (67), (63)]} \qquad \mu_{l,0} \geqslant l\,\delta_l + l\,(\delta_{c+r+1} + 1) + 1 + (l - c - r)\,\heartsuit, \tag{99}$$

then inductively we choose $\mu_{l,k}$ with:

$$\text{[see (68), (64)]} \qquad \mu_{l,k} \geqslant \sum_{j=0}^{k-1} l\,\mu_{l,j} + (l-k)\,\delta_l + l\,(\delta_{c+r+1}+1) \\ +1 + (l-c-r)\,\heartsuit \qquad (k = 1 \cdots l). \tag{100}$$

If $l < N$, we end this step by setting:

$$\delta_{l+1} := l\,\mu_{l,l} \tag{101}$$

as the starting point for the next step $l+1$. At the end $l = N$, we make the integer d large enough:

$$d \geqslant (N+1)\,\mu_{N,N}. \tag{102}$$

12.2 Effective lower degree bounds

Recalling Sect. 5.4, we first provide an effective

Theorem 12.1 *For all $N \geqslant 3$, for $\heartsuit = 1$, for any $\epsilon_1, \ldots, \epsilon_{c+r} \in \{1, 2\}$, Theorem 5.4 holds for all $d \geqslant \prod_{l=c+r}^{N} l^2\,(l+1)^l - 2$.*

Proof Set $\delta_{c+r+1} = 2$ in (98), and choose all $\mu_{\bullet,\bullet}$ in (99), (100) to be:

$$\mu_{l,0} = l\,\delta_l + 4\,l + 1,$$
$$\mu_{l,k} = \sum_{j=0}^{k-1} l\,\mu_{l,j} + (l-k)\,\delta_l + 4\,l + 1 \qquad {\scriptstyle (k=1\cdots l)}.$$

By induction on k we can show that:

$$\mu_{l,k} = \delta_l \Big(l\,(l+1)^k - \frac{(l+1)^k - 1}{l} \Big) + (4l+1)(l+1)^k \qquad {\scriptstyle (k=1\cdots l)}.$$

In particular, we have:

$$\delta_{l+1} = l\,\mu_{l,l} = \delta_l \Big(l^2\,(l+1)^l - (l+1)^l + 1 \Big) + (4\,l+1)\,l\,(l+1)^l.$$

Therefore $\delta_{l+1} \geqslant (4\,l+1)\,l\,(l+1)^l$ for $l = c+r+1 \ldots N-1$. Consequently, for $l = c+r+2 \cdots N-1$ we have $-\delta_l\big((l+1)^l - 1\big) + (4l+1)l(l+1)^l < 0$, thus we receive:

$$\delta_{l+1} \leqslant l^2\,(l+1)^l\,\delta_l, \tag{103}$$

a neat estimate suitable for the induction! For convenience, we define $\delta_{N+1} := N\mu_{N,N}$, so that the above inequality still holds true for $l = N$.

Now, when $N \geqslant c+r+2$, using the estimates (103) for $l = c+r+2 \cdots N$ iteratively, and noting that $\delta_{c+r+2} < 6l^2(l+1)^l\big|_{l=c+r+1}$, we receive the estimate:

$$\begin{aligned}(N+1)\,\mu_{N,N} &= \frac{N+1}{N}\,\delta_{N+1} < \frac{N+1}{N}\,\delta_{c+r+2} \prod_{l=c+r+2}^{N} l^2\,(l+1)^l \\ &< \frac{N+1}{N}\,6 \prod_{l=c+r+1}^{N} l^2\,(l+1)^l \\ &< 12 \prod_{l=c+r+1}^{N} l^2\,(l+1)^l < \prod_{l=c+r}^{N} l^2\,(l+1)^l - 2,\end{aligned}$$

whence we conclude the proof. □

Hence, the *product coup* in Sect. 5.4 yields

Theorem 12.2 *In Theorem 5.1, for* ♥ $= 1$*, the lower bound* $\eth_0(-1) = \prod_{l=c+r}^{N} l^4\,(l+1)^{2l}$ *works. In particular, Theorem 1.2 follows.* □

Now, using the following two estimates:

$$\sum_{l=\lceil N/2\rceil}^{N} \ln l \leqslant \int_{N/2}^{N+1} \ln x \, dx = (x \ln x - x)\Big|_{N/2}^{N+1},$$

$$\sum_{l=\lceil N/2\rceil}^{N} l \ln (l+1) < \sum_{l=\lceil N/2\rceil}^{N} (l+1) \ln (l+1)$$

$$\leqslant \int_{N/2}^{N+2} x \ln x \, dx = \Big(\frac{1}{2} x^2 \ln x - \frac{x^2}{4}\Big)\Big|_{N/2}^{N+2},$$

we can see that:

$$\ln \Big(\prod_{l=\lceil N/2\rceil}^{N} l^4 (l+1)^{2l}\Big) \leqslant \frac{3}{4} N^2 \ln N + o(N^2 \ln N).$$

This verifies the claimed estimate in Theorem 1.1.

13 Uniform very-ampleness of $\mathsf{Sym}^{\kappa}\, \Omega_X$

13.1 The canonical bundle of a projectivized vector bundle

Let X be an n-dimensional complex manifold, with canonical line bundle $\mathscr{K}_X$. Let E be a holomorphic vector bundle on X having rank e, with dual bundle $E^\vee$, and let $\pi : \mathbb{P}(E) \longrightarrow X$ be the projectivization of E. Then we have:

$$\mathscr{K}_{\mathbb{P}(E)} \cong \mathscr{O}_{\mathbb{P}(E)}(-e) \otimes \pi^* \wedge^e E^\vee \otimes \pi^* \mathscr{K}_X,$$

as follows by taking the determinants of the relative Euler exact sequence and the exact sequence of relative differentials.

In applications, we are interested in the case where $X \subset V$ for some complex manifold V of dimension $n+r$, and $E = \mathrm{T}_V\big|_X$, thus we receive:

$$\mathscr{K}_{\mathbb{P}(\mathrm{T}_V|_X)} \cong \mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(-n-r) \otimes \pi^* \mathscr{K}_{\mathscr{V}}\big|_X \otimes \pi^* \mathscr{K}_{\mathscr{X}}. \tag{104}$$

Moreover, assume that X, V are some smooth complete intersections in $\mathbb{P}^N_{\mathbb{C}}$, so their canonical line bundles $\mathscr{K}_X$, $\mathscr{K}_V$ have neat expressions by the following classical theorem.

Theorem 13.1 *For a smooth complete intersection:*

$$Y := D_1 \cap \cdots \cap D_k \subset \mathbb{P}^N_{\mathbb{C}}$$

with divisor degrees:

$$\deg\ D_i \;=\; d_i \qquad (i=1\cdots k),$$

the canonical line bundle $\mathscr{K}_Y$ of Y satisfies:

$$\mathscr{K}_Y \;\cong\; \mathscr{O}_Y\Big(-N-1+\sum_{i=1}^{k} d_i\Big).$$

□

13.2 Proof of the very-ampleness theorem 1.3

Now, assume $\mathbb{K} = \mathbb{C}$. Recall our Ampleness Theorem 1.2, $\mathsf{dim}_{\mathbb{C}}\, X = N - (c+r) = n$. Then the above formula (104) and Theorem 13.1 imply:

$$\mathscr{K}_{\mathbb{P}(\mathrm{T}_V|_X)} \cong \mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(-n-r) \,\otimes\, \pi_2^*\, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}} \Big(-2\,(N+1)+\sum_{i=1}^{c} d_i+\sum_{i=1}^{c+r} d_i\Big).$$

Also, recalling Theorem 5.1 and Proposition 4.3, for generic choices of $H_1,\dots,H_{c+r}$, for any positive integers $a > b \geqslant 1$ with $b/a < \heartsuit$, the negatively twisted line bundle below is ample:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(a) \,\otimes\, \pi_2^*\, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(-b).$$

It is now time to recall (c.f. [5,16])

Siu's result about the Fujita Conjecture *Let M be an n-dimensional complex manifold with canonical line bundle $\mathscr{K}_M$. If $\mathscr{L}$ is any positive holomorphic line bundle on M, then $\mathscr{L}^{\otimes m} \otimes \mathscr{K}_M^{\otimes 2}$ is very ample for all large $m \geqslant 2+\binom{3n+1}{n}$.*

Consequently, the line bundle below is very ample:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}\Big(m\,a-2n-2r\Big)\otimes\pi_2^*\,\mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}\Big(-m\,b-4\,(N+1)+2\sum_{i=1}^{c} d_i+2\sum_{i=1}^{c+r} d_i\Big). \tag{105}$$

Also recalling (11), for all $\ell \geqslant 3$, the line bundle below is very ample:

$$\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(1) \,\otimes\, \pi_2^*\, \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell). \tag{106}$$

Note the following two facts:

(A) if $\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(\kappa) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\star)$ is very ample, then so it is for every $\star' \geqslant \star$;
(B) the tensor product of any two very ample line bundles remains very ample.

We now consider the semigroup $\mathscr{G}$ of the usual Abelian group $\mathbb{Z} \oplus \mathbb{Z}$ generated by elements (ℓ_1, ℓ_2) such that $\mathscr{O}_{\mathbb{P}(\mathrm{T}_V|_X)}(\ell_1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^N_{\mathbb{K}}}(\ell_2)$ is very ample. Then, for all $m \geqslant 2 + \binom{3 \dim \mathbb{P}(\mathrm{T}_V|_X)+1}{\dim \mathbb{P}(\mathrm{T}_V|_X)}$, $\mathscr{G}$ contains:

$$\text{[see (105)]} \qquad \Big(m\,a - 2n - 2r,\ -m\,b - 4\,(N+1) + 2\sum_{i=1}^{c} d_i + 2\sum_{i=1}^{c+r} d_i\Big),$$

$$\text{[see (106)]} \qquad (1, \ell), \qquad \forall\, \ell \geqslant 3.$$

Paying no attention to optimality, taking:

$$b = 1,\ a = 2,\ m = -4\,(N+1) + 2\sum_{i=1}^{c} d_i + 2\sum_{i=1}^{c+r} d_i + 3,$$

we receive that $(m\,a - 2n - 2r, -3) \in \mathscr{G}$. Adding $(1, 3) \in \mathscr{G}$, we receive $(m\,a - 2n - 2r + 1, 0) \in \mathscr{G}$. Now, also using $(m\,a - 2n - 2r, 0) \in \mathscr{G}$, recalling Observation 5.6, we may take:

$$\kappa_0 \;=\; (m\,a - 2n - 2r - 1)\,(m\,a - 2n - 2r) \;\leqslant\; a^2\,m^2,$$

or the larger neater lower bound:

$$\kappa_0 \;=\; 16\Big(\sum_{i=1}^{c} d_i + \sum_{i=1}^{c+r} d_i\Big)^2.$$

Thus, we have proved the very-ampleness Theorem 1.3 for $\mathbb{K} = \mathbb{C}$. Remembering that *very-ampleness (or not) is preserved under any base change obtained by ambient field extension*, and noting the field extensions $\mathbb{Q} \hookrightarrow \mathbb{C}$ and $\mathbb{Q} \hookrightarrow \mathbb{K}$ for any field $\mathbb{K}$ with characteristic zero, by some standard arguments in algebraic geometry, we conclude the proof of the very-ampleness Theorem 1.3. □

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