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Article

Entire Curves Producing Distinct Nevanlinna Currents

Song-Yan Xie[1,2,*]

[1]Academy of Mathematics and System Science and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, China
[2]School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
[*]Corresponding author: Song-Yan Xie. Email: xiesongyan@amss.ac.cn

First, inspired by a question of Sibony, we show that in every compact complex manifold Y with certain Oka property, there exists some entire curve $f : \mathbb{C} \to Y$ generating all Nevanlinna/Ahlfors currents on Y, by holomorphic disks $\{f \upharpoonright_{\mathbb{D}(c,r)}\}_{c\in\mathbb{C},r>0}$. Next, we answer positively a question of Yau, by constructing some entire curve $g : \mathbb{C} \to X$ in the product $X := E_1 \times E_2$ of two elliptic curves E_1 and E_2, such that by using concentric holomorphic disks $\{g \upharpoonright_{\mathbb{D}_r}\}_{r>0}$ we can obtain infinitely many distinct Nevanlinna/Ahlfors currents proportional to the extremal currents of integration along curves $[\{e_1\} \times E_2]$, $[E_1 \times \{e_2\}]$ for all $e_1 \in E_1, e_2 \in E_2$ simultaneously. This phenomenon is new, and it shows tremendous holomorphic flexibility of entire curves in large scale geometry.

Dedicated to Julien Duval with admiration

1 Introduction

In 1967, Shoshichi Kobayashi introduced a pseudo-distance d_X intrinsically associated with any complex manifold X. When d_X is a true distance, X is called Kobayashi hyperbolic (cf. [32]). In the compact case, by Brody Lemma [5], X is Kobayashi hyperbolic if and only if it contains no entire curve, that is, nonconstant holomorphic map from the complex line $\mathbb{C}$ into X. Observing that most Riemann surfaces are hyperbolic, Kobayashi expected the same phenomenon for "most" higher dimensional complex manifolds [32]. In this direction, the leading problem is the famous Green–Griffiths conjecture [24], which stipulates that for every complex projective variety X of general type, all entire curves $f : \mathbb{C} \to X$ shall be factored through certain proper algebraic subvariety $Y \subsetneqq X$.

Partially advertised by the philosophical analogy between the value distribution of entire curves in Nevanlinna theory and the locations of rational points in Diophantine geometry (cf. Vojta's dictionary [47]), the Green–Griffiths conjecture has attracted much attention in the past decades. One key technique, which was introduced by A. Bloch [4] about a century ago, is by using negatively twisted k-jet differentials $\omega_1, \omega_2, \omega_3, \ldots$ on X as obstructions for the existence of entire curves $f : \mathbb{C} \to X$, since f must obey the ordinary differential equations $f^*\omega_1 \equiv 0, f^*\omega_2 \equiv 0, f^*\omega_3 \equiv 0, \ldots$. The reason is that the so-called fundamental vanishing theorem of entire curves (cf. [24, 40, 43]). The existence of such nontrivial negatively twisted k-jet differential forms was established by Merker [38] for general type smooth hypersurfaces in projective spaces, and then by Demailly [12] for general type compact complex manifolds, both requiring $k \gg 1$. One natural idea, which goes back at least to [24], is "controlling" the common zero loci defined by $\omega_1, \omega_2, \omega_3, \ldots$ in the k-jet space of X for concluding the desired hyperbolicity.

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However, in practice, such approach is very difficult. For instance, in the simplest case $k = 1$, there was a related conjecture of Debarre [11] that for general $c \geqslant N/2$ hypersurfaces $H_1, \ldots, H_c \subset \mathbb{CP}^N$ of large degrees $\gg 1$, the intersection $X := H_1 \cap \cdots \cap H_c$ shall have ample cotangent bundle T^*_X, in particular, the negatively twisted 1-jet differentials on X are abundant. The first proof of the Debarre ampleness conjecture was established in [48], using explicit 1-jet differentials obtained by Brotbek [6] (see also another proof [8] appeared shortly later). For Green–Griffiths conjecture on general hypersurfaces of large degrees in projective spaces, Siu introduced a strategy [44] of using slanted vector fields (cf. [37]), which explores certain symmetry of k-jet spaces of universal hypersurfaces, to facilitate controlling base loci of negatively twisted k-jet differentials. The existence of negatively twisted k-jet differentials for suitably small k can be established by certain Riemann–Roch calculations, see [2, 13] and references therein for the developments and state of the art. A breakthrough of Riedl and Yang [39] discovered that the Green–Griffiths conjecture for general hypersurfaces with smaller degrees can imply the Kobayashi hyperbolicity conjecture for general hypersurfaces with larger degrees (compare with [7, 46]).

While the Green–Griffiths conjecture for $\geqslant 3$ dimensional varieties remains dubious, it is likely to be true for compact projective surfaces of general type. One strong evidence is the celebrated work [35] of McQuillan, in which he verified the case of compact surfaces X with Chern number inequality $c_1^2 > c_2$ (see also [34]). This extra condition guarantees the existence of one negatively twisted 1-jet differential ω, thus inducing a multi-foliation $\mathcal{F}$ to which every entire curve $f : \mathbb{C} \to X$ must be tangent. For establishing the algebraic degeneracy of f, McQuillan introduced Nevanlinna currents for capturing asymptotic behavior of f (see Section 2), and the overall strategy of [35] is very deep and involved. Later, Brunella [9] provided a simplified proof of McQuillan's result, a key trick being to decompose every obtained Nevanlinna current $\mathbf{T}$ by Siu's theorem as $\mathbf{T} = \mathbf{T}_{\mathrm{alg}} + \mathbf{T}_{\mathrm{diff}}$ (see Section 2). In passing, Brunella raised the following two conjectures [9, page 200].

Conjecture 1.1. For every entire curve $f : \mathbb{C} \to X$, one can choose some increasing radii $\{r_i\}_{i\geqslant 1} \nearrow \infty$ such that the obtained Nevanlinna current $\mathbf{T} = \mathbf{T}_{\mathrm{alg}} + \mathbf{T}_{\mathrm{diff}}$ has either trivial singular part $\mathbf{T}_{\mathrm{alg}} = 0$ or trivial diffuse part $\mathbf{T}_{\mathrm{diff}} = 0$.

Conjecture 1.2. Some entire curve shall produce a Nevanlinna current $\mathbf{T} = \mathbf{T}_{\mathrm{alg}} + \mathbf{T}_{\mathrm{diff}}$ with both nontrivial singular part $\mathbf{T}_{\mathrm{alg}} \neq 0$ and nontrivial diffuse part $\mathbf{T}_{\mathrm{diff}} \neq 0$.

There are simplified analogues of Nevanlinna currents, called Ahlfors currents, by which Duval [19] obtained a deep, quantitative Brody Lemma and characterized complex hyperbolicity in terms of linear isoperimetric inequality for holomorphic disks (see also [31]). Using such currents, some geometric refinement [21] of the classical Cartan's Second Main Theorem and some higher dimensional analogue [29] of Weierstrass-Casorati Theorem were established. See also [14, 16] for recent key applications in complex dynamics. It is natural and fundamental to ask the following:

Question 1.3. Are all Nevanlinna/Ahlfors currents associated with the same entire curve cohomologically equivalent?

The Conjecture 1.1 (analogous to the "zero–one law" in probability) is still open, and the author believes that, by "Oka principle", certain counterexample shall exist. The Conjecture 1.2 (answer: yes) and Question 1.3 (answer: no) were solved by constructing exotic examples [30]. The large-scale behaviors of entire curves may have flexibility rather than rigidity.

Theorem 1.4 ([30]). There exists an entire curve $f : \mathbb{C} \to X$ such that, given any cardinality $|I| \in \mathbb{Z}_+ \cup \{\infty\}$ and any a priori requirement that $\mathbf{T}_{\mathrm{diff}}$ is trivial or not, by taking certain increasing radii $\{r_i\}_{i\geqslant 1}$ tending to infinity, the sequence of holomorphic disks $\{f \restriction_{\mathbb{D}_{r_i}}\}_{i\geqslant 1}$ yields a Nevanlinna/Ahlfors current $\mathbf{T} = \mathbf{T}_{\mathrm{alg}} + \mathbf{T}_{\mathrm{diff}}$ with the desired shape $\mathbf{T}_{\mathrm{alg}} = \sum_{i\in I} \alpha_i \cdot [C_i]$ in Siu's decomposition.

In hindsight, the above result reflects Oka property of X (cf. [22]).

Convention. We denote by $\mathbb{D}(a, r)$ (resp. $\overline{\mathbb{D}}(a, r)$) the open (resp. closed) disk centered at $a \in \mathbb{C}$ with radius r. When $a = 0$, we write $\mathbb{D}_r$ (resp. $\overline{\mathbb{D}}_r$) for short of $\mathbb{D}(0, r)$ (resp. $\overline{\mathbb{D}}(0, r)$).

After receiving an early manuscript of [30], Sibony [41] asked the following:

Question 1.5. Show that for $Y = \mathbb{CP}^n$, or complex torus, all Ahlfors currents can be obtained by single entire curve $f : \mathbb{C} \to Y$.

Here Sibony might allow us to use any holomorphic disk $\{f \upharpoonright_{\mathbb{D}(a,r)}\}_{a\in\mathbb{C},r>0}$ not necessarily centered at the origin ("(...) I suspect, that for $\mathbb{P}^n$, or torus, all the Ahlfors currents can be obtained just using one map, see the discussion on Birkhoff approach [3] in the enclosed paper [15] (...)" — Sibony [41]). He also suggested a hint [26]. Around the same time, Yau asked the following question during a seminar talk given by the author in Tsinghua.

Question 1.6. Can we construct an entire curve $g : \mathbb{C} \to E_1 \times E_2$ in the product of two elliptic curves E_1, E_2 such that by using concentric holomorphic disks $\{g \upharpoonright_{\mathbb{D}_r}\}_{r>0}$ we can produce two Nevanlinna/Ahlfors currents supported in distinct irreducible algebraic curves $\mathcal{C}_1, \mathcal{C}_2$, respectively?

The motivation of the above question is wondering whether there exists any entire curve $f : \mathbb{C} \to X$ producing two distinct Nevanlinna/Ahlfors currents (with different cohomology classes) supported on distinct curves $C_1, C_2 \subset X$, respectively. By a result [18] of Duval, C_1, C_2 must be either rational or elliptic. The simplest such X shall be the product of two curves with genus $\leqslant 1$. By using Weierstrass $\mathcal{P}$-functions mapping elliptic curves onto $\mathbb{CP}^1$, we only need to consider the case that $X = E_1 \times E_2$ is the product of two elliptic curves.

Definition 1.7. A connected complex manifold Y with a complete distance function $\mathbf{d}_Y$ is said to have the weak Oka-1 property, if for any two disjoint closed disks D_1, D_2 contained in a larger closed disk $D \subset \mathbb{C}$, for any holomorphic map $F : U \to Y$ from a neighborhood U of $D_1 \cup D_2$, for any error bound $\mathsf{e} > 0$, there exists some holomorphic map $\hat{F} : \hat{U} \to Y$ defined on some neighborhood $\hat{U}$ of D, which approximates F on $D_1 \cup D_2$ closely

$$\mathbf{d}_Y\,(\hat{F}(z), F(z)) \leqslant \mathsf{e} \qquad {\scriptstyle (\forall z \in D_1 \cup D_2)}.$$

Examples of such manifolds Y include all Oka manifolds, for example, $\mathbb{CP}^n$, complex tori, etc. [22]. Recently, Alarcón and Forstnerič [1] introduced a larger class of manifolds termed *Oka-1*, which clearly satisfy the above defined property. A remarkable result in [1] states that all rationally connected (e.g., Fano) manifolds are Oka-1.

Theorem A. Let Y be a compact complex manifold satisfying the weak Oka-1 property. Then there exists an entire curve $f : \mathbb{C} \to Y$ generating all Nevanlinna/Ahlfors currents on Y by holomorphic disks $\{f \upharpoonright_{\mathbb{D}(a,r)}\}_{a\in\mathbb{C},r>0}$.

Our Algorithm for producing such f also found applications in designing universal holomorphic/meromorphic functions in several variables with slow growth (cf. [10, 25]), hence solving an open problem [15, Problem 9.1] raised by Dinh and Sibony.

Theorem B. There exists some entire curve $g : \mathbb{C} \to X$ in the product $X := E_1 \times E_2$ of two elliptic curves E_1 and E_2, such that the concentric holomorphic disks $\{g \upharpoonright_{\mathbb{D}_r}\}_{r>0}$ can generate infinitely many distinct Nevanlinna/Ahlfors currents proportional to the extremal currents of integration along curves $[\{e_1\} \times E_2]$, $[E_1 \times \{e_2\}]$ for all $e_1 \in E_1, e_2 \in E_2$ simultaneously.

Conjecture 1.8. Let X be a compact Oka manifold. There should be some entire curve $g : \mathbb{C} \to X$ such that $\{g \upharpoonright_{\mathbb{D}_r}\}_{r>0}$ generate all Nevanlinna/Ahlfors currents on X.

Remark 1.9. Concerning the Green–Griffiths conjecture for low degree surfaces in $\mathbb{CP}^3$, one challenging problem is whether there exists any smooth hyperbolic surface with degree 5. For degree 6 and above, such examples exist [17, 27] by using Zaidenberg's deformation method [49]. Another key open problem in the subject is whether hyperbolicity property preserves along large deformations, for example, in the complement of certain pluripolar set in parameter space. The classical Brody Lemma only deals with small deformations. The two highlighted problems shall shed light for the Green–Griffiths conjecture for general surfaces in $\mathbb{CP}^3$ with degrees $\geqslant 5$.

Here is the structure of this paper. In Section 2, we present the definitions of Nevanlinna/Ahlfors currents. In Section 3, we introduce an Algorithm for Theorem A. In Section 4, we construct twisted entire curves to show Theorem B. The insight comes from Oka theory and complex dynamics. The construction is based on a sophisticated induction process (Ping Pong), using classical complex analysis in full strength, employing notably the Riemann mapping theorem, the little Picard theorem, and Runge's approximation theorem in every step.

2 Nevanlinna and Ahlfors currents

Let X be a compact complex manifold equipped with a Hermitian form ω. Let $f : \mathbb{D}_R \to X$ be a nonconstant holomorphic disk, smooth up to the boundary (i.e., f is a restriction of holomorphic map defined on a neighborhood of $\overline{\mathbb{D}}_R$). We can associate with f a positive current T_f of bidimension $(1,1)$, which evaluates every η in the set $\mathcal{A}^{(1,1)}(X)$ of smooth $(1,1)$–forms on X by

$$T_f(\eta) := \int_0^R \frac{\mathrm{d}t}{t} \int_{\mathbb{D}_t} f^*\eta.$$

The reason of such definition roots in Jensen's formula. Set

$$L_f(\omega) := \int_0^R \mathrm{Length}_\omega\big(f(\partial\mathbb{D}_t)\big)\, \frac{\mathrm{d}t}{t}.$$

Let $\{f_i : \mathbb{D}_{r_i} \to X\}_{i\geqslant 1}$ be a sequence of nonconstant holomorphic disks smooth up to the boundary, such that the length-area condition holds

$$\lim_{i\to\infty} \frac{L_{f_i}(\omega)}{T_{f_i}(\omega)} = 0. \tag{1}$$

Consider the family of positive currents of bounded mass $\{\Phi_{f_i}\}_{i\geqslant 1}$ defined as

$$\Phi_{f_i}(\eta) := \frac{T_{f_i}(\eta)}{T_{f_i}(\omega)} \qquad (\forall\, i \geqslant 1;\ \forall\, \eta \in \mathcal{A}^{(1,1)}(X)). \tag{2}$$

Then by Banach–Alaoglu's theorem, some subsequence $\{\Phi_{f_{k_\ell}}\}_{\ell\geqslant 1}$ converges in weak topology to a positive current Φ. The condition (1) guarantees that Φ is in fact closed (cf. [28, page 55]). Such a limit current is called Nevanlinna current.

In particular, given an entire curve $f : \mathbb{C} \to X$, by Ahlfors lemma (cf. [28, page 55]), one can find some increasing radii $\{r_i\}_{i\geqslant 1}$ tending to infinity, such that the sequence of restricted holomorphic disks $\{f \upharpoonright_{\mathbb{D}_{r_i}}\}_{i\geqslant 1}$ satisfies the length-area condition (1). Thus, we can receive some Nevanlinna current associated with the entire curve f.

The definition of Ahlfors currents is likewise and simpler. Again we start with a sequence of nonconstant holomorphic disks $f_i : \mathbb{D}_{r_i} \to X$ *smooth up to the boundary*, satisfying another length-area condition that

$$\lim_{i\to\infty} \frac{\mathrm{Length}_\omega\big(f_i(\partial\mathbb{D}_{r_i})\big)}{\mathrm{Area}_\omega\big(f_i(\mathbb{D}_{r_i})\big)} = 0. \tag{3}$$

From the sequence of normalized currents

$$\left\{\frac{[f_i(\mathbb{D}_{r_i})]}{\mathrm{Area}_\omega\big(f_i(\mathbb{D}_{r_i})\big)}\right\}_{i\geqslant 1} \tag{4}$$

of bounded mass, by compactness and diagonal argument, after passing to some subsequence, in the limit one receives some positive bidimension $(1,1)$ Ahlfors current, which is in fact closed because of the condition (3).

In particular, given an entire curve $f : \mathbb{C} \to X$, by Ahlfors lemma (cf. [20]), we can find some increasing radii $\{r_i\}_{i \geqslant 1} \nearrow \infty$, such that the sequence of restricted holomorphic disks $\{f \restriction_{\mathbb{D}_{r_i}}\}_{i \geqslant 1}$ satisfies the length-area condition (3). Thus, f produces some Ahlfors current.

One advantage of Nevanlinna currents over Ahlfors currents is that, the associated Nevanlinna currents with an entire curve $f : \mathbb{C} \to X$ are always nef provided that f is algebraically nondegenerate (cf. [36]).

Since every Nevanlinna/Ahlfors current $\mathbf{T}$ is positive and closed, by Siu's decomposition theorem [42], $\mathbf{T} = \mathbf{T}_{\text{alg}} + \mathbf{T}_{\text{diff}}$, where the singular part $\mathbf{T}_{\text{alg}} = \sum_{i \in I} \alpha_i \cdot [C_i]$ is some positive linear combination ($\alpha_i > 0$; $I \subset \mathbb{Z}_+$, could be $\emptyset$) of currents of integration on distinct irreducible algebraic curves C_i, and where the diffuse part $\mathbf{T}_{\text{diff}}$ is a positive closed $(1,1)$–current having zero Lelong number along any algebraic curve. A result [18] of Duval shows that, if the singular part $\mathbf{T}_{\text{alg}}$ is nontrivial, then every irreducible component C_i in the support of $\mathbf{T}_{\text{alg}}$ must be rational or elliptic.

3 Patching together infinite disjoint holomorphic disks

Throughout this section, we fix a Hermitian form ω on Y, which induces a distance $\mathbf{d}$. The proof of Theorem A relies on the following two observations.

Observation 3.1. We can select some countable holomorphic disks $\{f_i : \mathbb{D}_{r_i} \to Y\}_{i \geqslant 1}$ smooth up to the boundary, from which all Nevanlinna/Ahlfors currents on Y can be generated.

Proof. For every $r > 0$, denote by $\mathbf{Hol}(\mathbb{D}_r, Y)$ the set of holomorphic disks $f : \mathbb{D}_r \to Y$. The space $\mathbf{Hol}(\mathbb{D}_r, Y)$ has a natural distance $\mathbf{d}_r$, which associates $f, g : \mathbb{D}_r \to Y$ with the supremum deviation

$$\mathbf{d}_r(f, g) := \sup_{z \in \mathbb{D}_r} \mathbf{d}(f(z), g(z)).$$

For any $m > 0$, denote by $\mathbf{Hol}(\mathbb{D}_r, X)_m$ the subset of $\mathbf{Hol}(\mathbb{D}_r, Y)$ defined by the uniform derivative bound $||df||_\omega \leqslant m$. For any sequence of holomorphic disks $\{f_i\}_{i \geqslant 1}$ in $\mathbf{Hol}(\mathbb{D}_r, X)_m$, by mimicking the proof of Montel's theorem about normal families, that is, by using the Arzelà–Ascoli theorem and Cauchy's integral formula, we can find some subsequence $\{f_{n_i}\}_{i \geqslant 1}$ uniformly convergent to certain continuous map $f_\infty : \mathbb{D}_r \to Y$, which is in fact an element in $\mathbf{Hol}(\mathbb{D}_r, Y)_m$. Summarizing, any infinite sequence of elements in $\mathbf{Hol}(\mathbb{D}_r, Y)_m$ has an accumulation point. Therefore, $\mathbf{Hol}(\mathbb{D}_r, Y)_m$ is compact for every $r, m > 0$. Hence, we can select a dense countable subset $\mathcal{D}_{r,m}$ of $\mathbf{Hol}(\mathbb{D}_r, X)_m$ with respect to the distance $\mathbf{d}_r$.

We try to use the union

$$\mathcal{D} := \cup_{r \in \mathbb{Q}_+, m \in \mathbb{Z}_+} \mathcal{D}_{r,m}$$

as our desired family of countable holomorphic disks. However, there is one subtlety, that an element $f : \mathbb{D}_r \to Y$ in $\mathcal{D}$ might not be smooth up to the boundary (i.e., f is a restriction of holomorphic map defined on a neighborhood of $\overline{\mathbb{D}}_r$). The trick to remedy this is by replacing each $f \in \mathcal{D}$ by countably holomorphic disks $\{f_\ell\}_{\ell \geqslant 1}$ obtained by scaling

$$f_\ell(z) := f(\epsilon_\ell \cdot z) \qquad (\ell \geqslant 1; \forall z \in \mathbb{D}_r),$$

where $\{\epsilon_\ell\}_{\ell \geqslant 1} \nearrow 1$, say $\epsilon_\ell = 1 - 1/2^\ell$. The *new* $\mathcal{D}$, which remains countable, suffices for our purpose.

The remaining argument goes as follows. For any holomorphic disk $f : \mathbb{D}_r \to Y$ smooth up to the boundary, automatically $||df||_\omega$ is uniformly bounded from above. The reparametrization of f

$$f_1(\bullet) := f(r \cdot \bullet) : \mathbb{D}_1 \to Y$$

by scaling gives the same current $\Phi_{f_1} = \Phi_f$ by (2). Since $||df_1||_\omega < \infty$, f_1 can be approximated by a sequence $\{f_{1,j}\}_{j\geqslant 1}$ in $\mathcal{D}$ with respect to $\mathbf{d}_1$. Thus, $\Phi_{f_1} = \lim_{j\geqslant 1} \Phi_{f_{1,j}}$. This concludes the proof. ■

Observation 3.2. Given countable holomorphic disks $\{f_i : \mathbb{D}_{r_i} \to Y\}_{i\geqslant 1}$ smooth up to the boundary, we can construct an entire curve $f : \mathbb{C} \to Y$ such that every f_i can be approximated at any precision by holomorphic disks of f, that is, there exists some sequence $\{c_{i,\ell}\}_{\ell\geqslant 1}$ in $\mathbb{C}$ such that

$$\lim_{\ell\to\infty} f(z + c_{i,\ell}) = f_i(z) \qquad (\forall\, i \geqslant 1)$$

for all $z \in \mathbb{D}_{r_i}$ uniformly.

Proof. Fix a bijection $\varphi = (\varphi_1, \varphi_2) : \mathbb{Z}_+ \xrightarrow{\sim} \mathbb{Z}_+ \times \mathbb{Z}_+$. Take a sequence of shrinking positive numbers $\{\epsilon_i\}_{i\geqslant 1}$, say $\epsilon_i = 1/2^i$, such that $\lim_{j\to\infty} \sum_{i\geqslant j} \epsilon_i = 0$. We make another sequence of holomorphic disks $\{g_j : \mathbb{D}_{\hat{r}_j} \to Y\}_{j\geqslant 1}$ such that each f_i repeats infinitely many times in $\{g_j\}_{j\geqslant 1}$, for instance we can take $g_j := f_{\varphi_1(j)}$, $\hat{r}_j := r_{\varphi_1(j)}$. Now we construct a desired entire curve f by taking the limit of some convergent holomorphic disks $\{F_i : \mathbb{D}_{R_i} \to Y\}_{i\geqslant 1}$ as $R_i \nearrow \infty$.

In *Step* 1, we set three key data $F_1 := g_1$, $R_1 := \hat{r}_1$, $c_1 := 0$.

Subsequently, in *Step* $i+1$ for $i = 1, 2, 3, \ldots$, we choose a center $c_{i+1} \in \mathbb{C}$ far away from the origin, so that the two disks $\overline{\mathbb{D}}_{R_i}$ and $\overline{\mathbb{D}}(c_{i+1}, \hat{r}_{i+1})$ stay apart. We define $\hat{g}_{i+1}(\bullet) := g_{i+1}(\bullet - c_{i+1})$ on a neighborhood of $\overline{\mathbb{D}}(c_{i+1}, \hat{r}_{i+1})$ by translation of g_{i+1}. Now we take a large disk $\overline{\mathbb{D}}_{R_{i+1}}$ having radius $R_{i+1} > |c_{i+1}| + \hat{r}_{i+1}$, so that it contains $\overline{\mathbb{D}}_{R_i}$ and $\overline{\mathbb{D}}(c_{i+1}, \hat{r}_{i+1})$ in the interior. By the assumed weak Oka-1 property of Y, we can "extend" F_i and $\hat{g}_{i+1}$ to some holomorphic disk F_{i+1} defined in a neighborhood of $\overline{\mathbb{D}}_{R_{i+1}}$ such that

$$\sup_{z\in\overline{\mathbb{D}}_{R_i}} \mathbf{d}(F_{i+1}(z), F_i(z)) \leqslant \epsilon_{i+1}, \qquad \sup_{z\in\overline{\mathbb{D}}(c_{i+1},\hat{r}_{i+1})} \mathbf{d}(F_{i+1}(z), \hat{g}_{i+1}(z)) \leqslant \epsilon_{i+1}. \tag{5}$$

Since $\lim_{j\to\infty} \sum_{i\geqslant j} \epsilon_i = 0$, it is clear that $\{F_i\}_{i\geqslant 1}$ converges to an entire curve $f : \mathbb{C} \to Y$. Moreover, by using (5) repeatedly, we see that f approximates each $\hat{g}_{i+1}$ well

$$\begin{aligned} \sup_{z\in\overline{\mathbb{D}}(c_{i+1},\hat{r}_{i+1})} \mathbf{d}(f(z), \hat{g}_{i+1}(z)) &\leqslant \sup_{z\in\overline{\mathbb{D}}(c_{i+1},\hat{r}_{i+1})} \mathbf{d}(F_{i+1}(z), \hat{g}_{i+1}(z)) + \sum_{j\geqslant i+1} \sup_{z\in\overline{\mathbb{D}}_{R_j}} \mathbf{d}(F_{j+1}(z), F_j(z)) \\ &\leqslant \epsilon_{i+1} + \sum_{j\geqslant i+1} \epsilon_{j+1} \to 0 \qquad (\text{as } i \to \infty). \end{aligned}$$

In other words, $\sup_{z\in\overline{\mathbb{D}}_{\hat{r}_{i+1}}} \mathbf{d}\big(f(c_{i+1} + z), g_{i+1}(z)\big) \to 0$ as $i \to \infty$. This finishes the proof. ■

The above Algorithm for constructing f also plays a key rôle in the solutions [10, 25] to an open problem [15, Problem 9.1] raised by Dinh and Sibony.

Proof of Theorem A. Combining Observations 3.1 and 3.2, Theorem A follows directly. ■

Remark 3.3. In fact, a moment of reflection shows that, the obtained entire curve $f : \mathbb{C} \to Y$ in the above construction is actually universal in the sense of [25, 33].

Question 3.4. Is there a compact complex manifold with the weak Oka-1 property but is not Oka-1 (cf. [1])?

4 Twisted entire curves in the product of two elliptic curves

Given two elliptic curves $E_1 = \mathbb{C}/\Gamma_1$ and $E_2 = \mathbb{C}/\Gamma_2$ where Γ_1, Γ_2 are two lattices in $\mathbb{C}$. Let $\pi_1 : \mathbb{C} \to E_1$, $\pi_2 : \mathbb{C} \to E_2$ and $\pi : \mathbb{C} \times \mathbb{C} \to E_1 \times E_2 = X$ be the canonical projections. The standard Euclidean norm $||\bullet||_{\mathbb{C}^2}$ on $\mathbb{C}^2$ corresponds to the positive (1, 1)-form $\frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$, where (z_1, z_2) are the usual coordinate functions of $\mathbb{C}^2$. Descending via π, we receive a reference Hermitian form ω on X.

For every $i = 1, 2$, we fix some countable everywhere dense sequence $\{e_{i,\ell}\}_{\ell\geqslant 1} \subset E_i$. Our desired entire curve will write as $g := \pi \circ F$, where $F = (G_1, G_2) : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ will be obtained by taking the limit of some convergent holomorphic disks

$$F_\ell = (G_{1,\ell}, G_{2,\ell}) : \mathbb{D}_{R_\ell} \to \mathbb{C} \times \mathbb{C}, \quad R_\ell \nearrow \infty \qquad (\ell = 0, 1, 2, \dots). \tag{6}$$

In *Step* 0, we set $R_0 = 1$, and choose two nonconstant holomorphic functions $G_{1,0}$, $G_{2,0}$ defined on a neighborhood V_0 of the unit disk $\mathbb{D}_1$.

Subsequently, in *Step* $2k-1$ and *Step* $2k$ for $k = 1, 2, 3, \dots$, using Runge's approximation theorem (cf. e.g., [23, page 94]) upon the holomorphic functions $G_{1,2k-2}$ and $G_{2,2k-2}$ defined on $V_{2k-2} \supset \overline{\mathbb{D}}_{R_{2k-2}}$, for both $i = 1, 2$, we approximate $G_{i,2k-2}$ on a smaller neighborhood $V'_{2k-2} \subset\subset V_{2k-2}$ (i.e., relatively compact) of $\overline{\mathbb{D}}_{R_{2k-2}}$ by some nonconstant entire functions $\hat{G}_{i,2k-2}$ within small error $\epsilon_{2k-2}/2023$, so that $\hat{F}_{2k-2} := (\hat{G}_{1,2k-2}, \hat{G}_{2,2k-2})$ is very close to F_{2k-2} on V'_{2k-2}

$$||\hat{F}_{2k-2}(z) - F_{2k-2}(z)||_{\mathbb{C}^2} \leqslant 2\epsilon_{2k-2}/2023 \qquad (\forall z \in V'_{2k-2}).$$

Plan of Step $2k-1$. To choose some large radius $R_{2k-1} > R_{2k-2} + 1$ and then to modify $\hat{F}_{2k-2}$ as $F_{2k-1} := (\hat{G}_{1,2k-2}, \hat{G}_{2,2k-2} \cdot H_{2k-1})$ by some auxiliary holomorphic function H_{2k-1} defined on a neighborhood of $\overline{\mathbb{D}}_{R_{2k-1}}$, such that F_{2k-1} is very close to $\hat{F}_{2k-2}$ (hence F_{2k-2}) on V'_{2k-2}

$$||F_{2k-1}(z) - \hat{F}_{2k-2}(z)||_{\mathbb{C}^2} \leqslant \frac{\epsilon_{2k-2}}{2023}, \ ||F_{2k-1}(z) - F_{2k-2}(z)||_{\mathbb{C}^2} \leqslant \frac{3\epsilon_{2k-2}}{2023} \qquad (\forall z \in V'_{2k-2}), \tag{7}$$

while the induced current $\Phi_{\pi \circ F_{2k-1} \upharpoonright \mathbb{D}_{R_{2k-1}}}$ for constructing Nevanlinna currents (see (2)) and the current $\frac{[\pi \circ F_{2k-1}(\mathbb{D}_{R_{2k-1}})]}{\mathrm{Area}(\pi \circ F_{2k-1}(\mathbb{D}_{R_{2k-1}}))}$ for constructing Ahlfors currents (see (4)) both have mass concentrated near $\{e_{1,k}\} \times E_2$ in a quantitative sense to be specified later.

Observation 1. By the little Picard theorem, the range of the nonconstant entire function $\hat{G}_{1,2k-2}$ cannot avoid the set $\pi_1^{-1}(e_{1,k}) \setminus \hat{G}_{1,2k-2}(\overline{\mathbb{D}}_{R_{2k-2}+1})$, which contains infinitely many points! Hence, we can find some $z_{2k-1} \in \mathbb{C} \setminus \overline{\mathbb{D}}_{R_{2k-2}+1}$ with $\hat{G}_{1,2k-2}(z_{2k-1}) \in \pi_1^{-1}(e_{1,k})$. The upshot is that

$$\pi \circ F_{2k-1}(z_{2k-1}) = \big(\pi_1 \circ \hat{G}_{1,2k-2}(z_{2k-1}), \pi_2 \circ (\hat{G}_{2,2k-2} \cdot H_{2k-1})(z_{2k-1})\big) = (e_{1,k}, *)$$

lies in the curve $\{e_{1,k}\} \times E_2$ no matter how we choose H_{2k-1}.

Idea. Fix a very small neighborhood U_{2k-1} of z_{2k-1} such that $\pi_1 \circ \hat{G}_{1,2k-2}$ takes value in a small neighborhood of $\pi_1 \circ \hat{G}_{1,2k-2}(z_{2k-1}) = e_{1,k}$, say

$$\pi_1 \circ \hat{G}_{1,2k-2}(U_{2k-1}) \subset \mathbb{D}(e_{1,k}, 2^{-(2k-1)-2023}) := \pi_1\big(\mathbb{D}(\hat{G}_{1,2k-2}(z_{2k-1}), 2^{-(2k-1)-2023})\big).$$

Now we demand that H_{2k-1} is almost 1 on $\overline{\mathbb{D}}_{R_{2k-1}} \setminus U_{2k-1}$ whereas H_{2k-1} oscillates violently on U_{2k-1} so that the currents $\Phi_{\pi \circ F_{2k-1} \upharpoonright \mathbb{D}_{R_{2k-1}}}$ and $\frac{[\pi \circ F_{2k-1}(\mathbb{D}_{R_{2k-1}})]}{\mathrm{Area}(\pi \circ F_{2k-1}(\mathbb{D}_{R_{2k-1}}))}$ have most mass near $\{e_{1,k}\} \times E_2$ in a quantitative sense to be specified later.

Observation 2. Set $R_{2k-1} := |z_{2k-1}| > R_{2k-2} + 1$. If we can find some holomorphic function $\mathcal{X}_{2k-1}$ defined on a neighborhood V_{2k-1} of $\overline{\mathbb{D}}_{R_{2k-1}}$ such that $|\mathcal{X}_{2k-1}| < 1$ on $\overline{\mathbb{D}}_{R_{2k-1}} \setminus U_{2k-1}$ whereas $|\mathcal{X}_{2k-1}(z_{2k-1})| > 1$, then $H_{2k-1} := 1 + \mathcal{X}_{2k-1}^{M_{2k-1}}$ for some very large integer $M_{2k-1} \gg 1$ will suffice for the goal of *Step* $2k-1$.

The existence of such $\mathcal{X}_{2k-1}$ is guaranteed by the following

Key Lemma. Let $z_0 \in \partial\mathbb{D}_R$, and let U be a neighborhood of z_0. Then one can find some holomorphic function $\mathcal{X}$ defined in a neighborhood of $\overline{\mathbb{D}}_R$ such that $|\mathcal{X}| < 1$ on $\overline{\mathbb{D}}_R \setminus U$ while $|\mathcal{X}(z_0)| > 1$.

The proof will be postponed to the end of this section. Now we check the conditions (1) and (3).

Length-Area Estimate (I). Fix $k \in \mathbb{Z}_+$. For the family of holomorphic disks

$$\overline{F}_{2k-1} := \pi \circ F_{2k-1} : \mathbb{D}_{R_{2k-1}} \longrightarrow X$$

parameterized by $M_{2k-1} \in \mathbb{Z}_+$, one has

$$\frac{L_{\overline{F}_{2k-1}}(\omega)}{T_{\overline{F}_{2k-1}}(\omega)} \to 0 \qquad \text{as } M_{2k-1} \text{ tends to infinity.} \tag{8}$$

Proof. Denote by $m_{2k-1} > 1$ the maximum modulus of $|\mathcal{X}_{2k-1}|$ over $\overline{\mathbb{D}}_{R_{2k-1}}$. The set

$$\{z \in \mathbb{D}_{R_{2k-1}} \;:\; 1 < m_{2k-1}^{2/3} < |\mathcal{X}_{2k-1}(z)| < m_{2k-1},\; \hat{G}_{2,2k-2} \cdot \mathcal{X}'_{2k-1}(z) \neq 0\} \tag{9}$$

is clearly open and nonempty. Take a small closed disk $\overline{\mathbb{D}}(c_{2k-1}, r_{2k-1})$ in it. Let $\widetilde{m}_{2k-1} > 0$ be the minimum modulus of $|\hat{G}_{2,2k-2} \cdot \mathcal{X}'_{2k-1}|$ on $\overline{\mathbb{D}}(c_{2k-1}, r_{2k-1})$. Now we compute the derivative

$$\begin{aligned}(\hat{G}_{2,2k-2} \cdot H_{2k-1})' &= \big(\hat{G}_{2,2k-2} \cdot (1 + \mathcal{X}_{2k-1}^{M_{2k-1}})\big)' \\ &= M_{2k-1} \cdot \mathcal{X}_{2k-1}^{M_{2k-1}-1} \cdot \hat{G}_{2,2k-2} \cdot \mathcal{X}'_{2k-1} + \hat{G}'_{2,2k-2} \cdot (1 + \mathcal{X}_{2k-1}^{M_{2k-1}}).\end{aligned} \tag{10}$$

Using the elementary inequality $|A+B|^2 \geqslant \frac{1}{4} \cdot |A|^2 - |B|^2$ for all complex numbers A, B, we can estimate $\overline{F}^*_{2k-1}\,\omega \geqslant \overline{F}^*_{2k-1}(\frac{\sqrt{-1}}{2} dz_2 \wedge d\bar{z}_2)$ on $\overline{\mathbb{D}}(c_{2k-1}, r_{2k-1})$ from below by

$$\begin{aligned} &\Big(\frac{1}{4} \cdot |M_{2k-1} \cdot \mathcal{X}_{2k-1}^{M_{2k-1}-1} \cdot \hat{G}_{2,2k-2} \cdot \mathcal{X}'_{2k-1}|^2 - |\hat{G}'_{2,2k-2} \cdot (1 + \mathcal{X}_{2k-1}^{M_{2k-1}})|^2\Big) \cdot \frac{\sqrt{-1}}{2} dz \wedge d\bar{z} \\ &\geqslant \Big(O^+(1) \cdot M_{2k-1}^2 \cdot |\mathcal{X}_{2k-1}|^{2\cdot(M_{2k-1}-1)} - O^+(1) \cdot \big(1 + |\mathcal{X}_{2k-1}|^{M_{2k-1}}\big)^2\Big) \cdot \frac{\sqrt{-1}}{2} dz \wedge d\bar{z} \\ [\spadesuit]\quad &\geqslant \Big(O^+(1) \cdot M_{2k-1}^2 \cdot |\mathcal{X}_{2k-1}|^{2\cdot(M_{2k-1}-1)} - O^+(1) \cdot 4|\mathcal{X}_{2k-1}|^{2M_{2k-1}}\Big) \cdot \frac{\sqrt{-1}}{2} dz \wedge d\bar{z} \\ &= \big(O^+(1) \cdot M_{2k-1}^2 - O^+(1) \cdot 4|\mathcal{X}_{2k-1}|^2\big) \cdot |\mathcal{X}_{2k-1}|^{2\cdot(M_{2k-1}-1)} \cdot \frac{\sqrt{-1}}{2} dz \wedge d\bar{z} \\ [\text{see } (9)]\quad &\geqslant \big(O^+(1) \cdot M_{2k-1}^2 - O^+(1)\big) \cdot (m_{2k-1}^{2/3})^{2\cdot(M_{2k-1}-1)} \cdot \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}, \end{aligned} \tag{11}$$

where we always abuse the notation $O^+(1)$ for different positive bounded constants independent of M_{2k-1} (but can depend on m_{2k-1}, $\widetilde{m}_{2k-1}$, $\mathcal{X}_{2k-1}$, $\hat{G}_{2,2k-2}$, $\hat{G}_{1,2k-2}$, R_{2k-1}, c_{2k-1}, r_{2k-1}), and where [♠] uses the fact that $|\mathcal{X}_{2k-1}| > 1$ on $\overline{\mathbb{D}}(c_{2k-1}, r_{2k-1})$ due to (9). Thus, we can bound the area growth from below, without using Jensen's formula, by

$$\begin{aligned} T_{\overline{F}_{2k-1}}(\omega) &= \int_0^{R_{2k-1}} \frac{dt}{t} \int_{\mathbb{D}_t} f^*\omega \\ &\geqslant \int_{|c_{2k-1}|+r_{2k-1}}^{R_{2k-1}} \frac{dt}{t} \int_{\overline{\mathbb{D}}(c_{2k-1}, r_{2k-1})} f^*\omega \quad [\text{since } \overline{\mathbb{D}}(c_{2k-1}, r_{2k-1}) \subset \overline{\mathbb{D}}_t \text{ for } t \geqslant |c_{2k-1}| + r_{2k-1}] \\ &\geqslant \big(O^+(1) \cdot M_{2k-1}^2 - O^+(1)\big) \cdot (m_{2k-1}^{2/3})^{2(M_{2k-1}-1)} \cdot O^+(1) \qquad [\text{see } (11)]. \end{aligned} \tag{12}$$

We emphasize a key point here that the exponent of $m_{2k-1} > 1$ is

$$\frac{2}{3} \cdot 2(M_{2k-1} - 1) \gg M_{2k-1}$$

for $M_{2k-1} \gg 1$.

Next, on $\overline{\mathbb{D}}_{R_{2k-1}}$, we have $|(\hat{G}_{2,2k-2} \cdot H_{2k-1})'| \leqslant O^+(1) \cdot M_{2k-1} \cdot m_{2k-1}^{M_{2k-1}}$ by (10). Thus,

$$\overline{F}^*_{2k-1}\,\omega \leqslant \big(O^+(1) \cdot M_{2k-1}^2 \cdot m_{2k-1}^{2\cdot M_{2k-1}} + O^+(1)\big) \cdot \frac{\sqrt{-1}}{2} dz \wedge d\bar{z} \tag{13}$$

on $\overline{\mathbb{D}}_{R_{2k-1}}$. Therefore, we can estimate the length growth from above by

$$\begin{aligned} L_{\overline{F}_{2k-1}}(\omega) &\leqslant \left(O^+(1)\cdot M_{2k-1}^2\cdot m_{2k-1}^{2\cdot M_{2k-1}}+O^+(1)\right)^{1/2}\cdot \int_0^{R_{2k-1}} \mathrm{Length}_{\mathbb{C}}\,(\partial\mathbb{D}_t)\,\frac{dt}{t} \\ &= O^+(1)\cdot \left(O^+(1)\cdot M_{2k-1}^2\cdot m_{2k-1}^{2\cdot M_{2k-1}}+O^+(1)\right)^{1/2}\cdot 2\pi\, R_{2k-1} \\ &\leqslant O^+(1)\cdot \left(O^+(1)\cdot M_{2k-1}\cdot m_{2k-1}^{M_{2k-1}}+O^+(1)\right). \end{aligned} \tag{14}$$

Hence, the desired estimate (8) follows directly by comparing the asymptotic growth rates of (14) and (12) with respect to the parameter $M_{2k-1}\gg 1$. ■

Length-Area Estimate (II). Fix $k\in\mathbb{Z}_+$. For the family of holomorphic disks

$$\overline{F}_{2k-1} := \pi\circ F_{2k-1} : \mathbb{D}_{R_{2k-1}} \longrightarrow X$$

parameterized by $M_{2k-1}\in\mathbb{Z}_+$, one has

$$\frac{\mathrm{Length}_{\omega}\left(\overline{F}_{2k-1}(\partial\mathbb{D}_{R_{2k-1}})\right)}{\mathrm{Area}_{\omega}\left(\overline{F}_{2k-1}(\mathbb{D}_{R_{2k-1}})\right)} \to 0 \qquad \text{as } M_{2k-1} \text{ tends to infinity.} \tag{15}$$

Proof. Following the preceding argument, by (13), we receive

$$\mathrm{Length}_{\omega}\left(\overline{F}_{2k-1}(\partial\mathbb{D}_{R_{2k-1}})\right) \leqslant \left(O^+(1)\cdot M_{2k-1}^2\cdot m_{2k-1}^{2\cdot M_{2k-1}}+O^+(1)\right)^{1/2}\cdot 2\pi\, R_{2k-1}.$$

By (11), we have

$$\mathrm{Area}_{\omega}\left(\overline{F}_{2k-1}(\mathbb{D}_{R_{2k-1}})\right) \geqslant \left(O^+(1)\cdot M_{2k-1}^2 - O^+(1)\right)\cdot (m_{2k-1}^{2/3})^{2\cdot(M_{2k-1}-1)}\cdot \pi\, r_{2k-1}^2. \tag{16}$$

Thus, the desired estimate follows directly by comparing the asymptotic growth rates in the above two estimates with respect to the parameter $M_{2k-1}\nearrow\infty$. ■

Now we move on to *Step* $2k$. Copy

$$(G_{1,2k-1}, G_{2,2k-1}) := F_{2k-1} = (\hat{G}_{1,2k-2}, \hat{G}_{2,2k-2}\cdot H_{2k-1})$$

defined on a neighborhood V_{2k-1} of $\overline{\mathbb{D}}_{R_{2k-1}}$. By Runge's approximation theorem, for each $i=1,2$, we can approximate $G_{i,2k-1}$ on a neighborhood $V'_{2k-1}\subset\subset V_{2k-1}$ of $\overline{\mathbb{D}}_{R_{2k-1}}$ by nonconstant entire function $\hat{G}_{i,2k-1}$ within small error $\epsilon_{2k-1}/2023$, such that $\hat{F}_{2k-1} := (\hat{G}_{1,2k-1}, \hat{G}_{2,2k-1})$ is very close to F_{2k-1}

$$||\hat{F}_{2k-1}(z) - F_{2k-1}(z)||_{\mathbb{C}^2} \leqslant 2\epsilon_{2k-1}/2023 \qquad (\forall z\in V'_{2k-1}).$$

Similarly, we can find some $z_{2k}\in\mathbb{C}$ with $|z_{2k}| =: R_{2k} > R_{2k-1}+1$ such that $\pi_2\circ\hat{G}_{2,2k-1}(z_{2k}) = e_{2,k}$.

Plan of Step $2k$. To modify $\hat{F}_{2k-1}$ as $F_{2k} := (\hat{G}_{1,2k-1}\cdot H_{2k}, \hat{G}_{2,2k-1})$ by some holomorphic function H_{2k} defined in a neighborhood V_{2k} of $\overline{\mathbb{D}}_{R_{2k}}$, such that F_{2k} is very close to $\hat{F}_{2k-1}$ and F_{2k-1} on V'_{2k-1}

$$||F_{2k}(z) - \hat{F}_{2k-1}(z)||_{\mathbb{C}^2} \leqslant \frac{\epsilon_{2k-1}}{2023}, \quad ||F_{2k}(z) - F_{2k-1}(z)||_{\mathbb{C}^2} \leqslant \frac{3\,\epsilon_{2k-1}}{2023} \qquad (\forall z\in V'_{2k-1}), \tag{17}$$

whereas the induced currents $\Phi_{\pi\circ F_{2k}\restriction_{\mathbb{D}_{R_{2k}}}}$ and $\frac{[\pi\circ F_{2k}(\mathbb{D}_{R_{2k}})]}{\mathrm{Area}(\pi\circ F_{2k}(\mathbb{D}_{R_{2k}}))}$ both have mass concentrated near $E_1\times\{e_{2,k}\}$ in a quantitative sense to be specified later.

The method of *Step* $2k-1$ works as well in *Step* $2k$. Fix a neighborhood U_{2k} of z_{2k} such that

$$\pi\circ\hat{G}_{2,2k-1}(U_{2k}) \subset \mathbb{D}(e_{2,k}, 2^{-2k-2023}) := \pi_2\left(\mathbb{D}(\hat{G}_{2,2k-1}(z_{2k}), 2^{-2k-2023})\right).$$

Choose some holomorphic function $\mathcal{X}_{2k}$ defined on a neighborhood V_{2k} of $\overline{\mathbb{D}}_{R_{2k}}$ such that $|\mathcal{X}_{2k}| < 1$ on $\overline{\mathbb{D}}_{R_{2k}} \setminus U_{2k}$ whereas $|\mathcal{X}_{2k}(z_{2k})| > 1$. Take $H_{2k} := 1 + \mathcal{X}_{2k}^{M_{2k}}$ for some very large integer $M_{2k} \gg 1$ to be specialized later. Thus, H_{2k} is almost 1 on $\overline{\mathbb{D}}_{R_{2k}} \setminus U_{2k}$ whereas H_{2k} oscillates violently inside U_{2k} so that the goal of *Step* $2k$ can be reached.

Technical Details in the Proof of Theorem B. Subsequently, in each Step $\ell \geqslant 1$, we find out the aforementioned $z_\ell, R_\ell = |z_\ell| > R_{\ell-1}+1, U_\ell$ and an auxiliary holomorphic function $\mathcal{X}_\ell$ defined on $V_\ell \supset \overline{\mathbb{D}}_{R_\ell}$. Now we choose some sufficiently large $M_\ell \gg 1$ and determine a very small error bound $0 < \epsilon_\ell \ll 1$ such that all the following sophisticated considerations hold true simultaneously.

($\heartsuit$). The holomorphic disk $\overline{F}_\ell = \pi \circ F_\ell : \mathbb{D}_{R_\ell} \longrightarrow X$ satisfies length-area estimates

$$\frac{L_{\overline{F}_\ell}(\omega)}{T_{\overline{F}_\ell}(\omega)} \leqslant 2^{-\ell}, \qquad \frac{\mathrm{Length}_\omega\big(\overline{F}_\ell(\partial\mathbb{D}_{R_\ell})\big)}{\mathrm{Area}_\omega\big(\overline{F}_\ell(\mathbb{D}_{R_\ell})\big)} \leqslant 2^{-\ell}. \tag{18}$$

($\diamondsuit$). Most area of the holomorphic disk $\overline{F}_\ell$ is either concentrated near the curve $\{e_{1,k}\}\times E_2$ when $\ell = 2k-1$ is odd or concentrated near the curve $E_1 \times \{e_{2,k}\}$ when $\ell = 2k$ is even. Precisely,

$$\frac{T_{\overline{F}_\ell}(\mathbf{1}_{X\setminus\hat{U}_\ell}\cdot\omega)}{T_{\overline{F}_\ell}(\omega)} < 2^{-\ell}, \qquad \frac{[\overline{F}_\ell(\mathbb{D}_{R_\ell})]}{\mathrm{Area}_\omega\big(\overline{F}_\ell(\mathbb{D}_{R_\ell})\big)}(\mathbf{1}_{X\setminus\hat{U}_\ell}\cdot\omega) = \frac{\int_{\mathbb{D}_{R_\ell}\setminus\overline{F}_\ell^{-1}(\hat{U}_\ell)} \overline{F}_\ell^*\,\omega}{\int_{\mathbb{D}_{R_\ell}} \overline{F}_\ell^*\,\omega} < 2^{-\ell}, \tag{19}$$

where $\hat{U}_\ell$ is either $\pi_1\big(\mathbb{D}(z_\ell, 2^{-\ell})\big)\times E_2$ (when $\ell = 2k-1$) or $\hat{U}_\ell = E_1 \times \pi_2\big(\mathbb{D}(z_\ell, 2^{-\ell})\big)$ (when $\ell = 2k$), and where $\mathbf{1}_{X\setminus\hat{U}_\ell}$ is the characteristic function of the set $X \setminus \hat{U}_\ell$.

($\clubsuit$). By our construction, F_ℓ is well-defined on all three domains $\mathbb{D}_{R_\ell} \subset\subset V'_\ell \subset\subset V_\ell$. We choose $0 < \epsilon_\ell \ll 1$ shrinking to zero very fast (set $\epsilon_0 := 1$) such that

$$\epsilon_\ell < \epsilon_{\ell-1}/2 \tag{20}$$

and such that for any small holomorphic perturbation $F_{\ell,\mathbf{p}}$ of F_ℓ within difference ϵ_ℓ on V'_ℓ (i.e., $|F_{\ell,\mathbf{p}} - F_\ell| < \epsilon_\ell$ on V'_ℓ), the estimates (18) and (19) are stable in the sense:

$$\frac{L_{\overline{F}_{\ell,\mathbf{p}}}(\omega)}{T_{\overline{F}_{\ell,\mathbf{p}}}(\omega)} \leqslant 2^{-\ell+1}, \quad \frac{T_{\overline{F}_{\ell,\mathbf{p}}}(\mathbf{1}_{X\setminus 2\hat{U}_\ell}\cdot\omega)}{T_{\overline{F}_{\ell,\mathbf{p}}}(\omega)} < 2^{-\ell+1}, \tag{21}$$

$$\frac{\mathrm{Length}_\omega\big(\overline{F}_{\ell,\mathbf{p}}(\partial\mathbb{D}_{R_\ell})\big)}{\mathrm{Area}_\omega\big(\overline{F}_{\ell,\mathbf{p}}(\mathbb{D}_{R_\ell})\big)} \leqslant 2^{-\ell+1}, \qquad \frac{\int_{\mathbb{D}_{R_\ell}\setminus\overline{F}_{\ell,\mathbf{p}}^{-1}(2\hat{U}_\ell)} \overline{F}_{\ell,\mathbf{p}}^*\,\omega}{\int_{\mathbb{D}_{R_\ell}} \overline{F}_{\ell,\mathbf{p}}^*\,\omega} < 2^{-\ell+1}, \tag{22}$$

where $\overline{F}_{\ell,\mathbf{p}} : \mathbb{D}_{R_\ell} \to X$ is the restriction of $\pi\circ F_{\ell,\mathbf{p}}$ on $\mathbb{D}_{R_\ell}$, and where $2\hat{U}_\ell \supset \hat{U}_\ell$ is either $\pi_1\big(\mathbb{D}(z_\ell, 2^{-\ell+1})\big)\times E_2$ (when $\ell = 2k-1$) or $E_1 \times \pi_2\big(\mathbb{D}(z_\ell, 2^{-\ell+1})\big)$ (when $\ell = 2k$).

The requirement ($\heartsuit$) is guaranteed by the key estimates (8), (15) and by specializing a parameter $M_\ell \gg 1$ sufficiently large.

The demand ($\diamondsuit$) follows from (9), (12), and (16) by specializing $M_\ell \gg 1$, since both $T_{\overline{F}_\ell}(\mathbf{1}_{X\setminus\hat{U}_\ell}\cdot\omega)$ and $\int_{\mathbb{D}_{R_\ell}\setminus\overline{F}_\ell^{-1}(\hat{U}_\ell)} \overline{F}_\ell^*\,\omega$ are uniformly bounded from above by some constant $O^+(1)$ independent of the parameter $M_\ell \in \mathbb{Z}_+$. Indeed, by our construction, for every $M_\ell \geqslant 1$, we have $U_\ell \subset\subset \overline{F}_\ell^{-1}(\hat{U}_\ell)$, while the sequence of holomorphic maps $\overline{F}_\ell$, parameterized by $M_\ell \in \mathbb{Z}_+$, converges uniformly to $\pi \circ \hat{F}_{\ell-1}$ on $\overline{\mathbb{D}}_{R_\ell} \setminus U_\ell$ because $|\mathcal{X}_\ell| < 1$ on $\overline{\mathbb{D}}_{R_\ell} \setminus U_\ell$.

The existence of ϵ_ℓ in the consideration ($\clubsuit$) can be proved by a *reductio ad absurdum* argument, since the $\mathcal{C}^0$–convergence of a sequence of holomorphic functions on a larger open set $V'_\ell \supset \overline{\mathbb{D}}_{R_\ell}$ can guarantee the $\mathcal{C}^1$–convergence on the smaller compact set $\overline{\mathbb{D}}_{R_\ell}$ by complex analysis.

The estimates (7) and (17) guarantee that the holomorphic disks $\{F_\ell\}_{\ell\geqslant 1}$ in (6) can converge to an entire curve $F = (G_1, G_2) : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$.

For every $\ell \geqslant 1$, on V'_ℓ, we have

$$
\begin{aligned}
||F - F_\ell|| &\leqslant \sum_{j=\ell}^{\infty} ||F_{j+1} - F_j|| \\
\text{[use (7), (17)]} \qquad &\leqslant \sum_{j=\ell}^{\infty} 3 \cdot \epsilon_j/2023 \\
\text{[check (20)]} \qquad &\leqslant \sum_{j=\ell}^{\infty} 3/2023 \cdot \Big(\frac{1}{2}\Big)^{j-\ell} \cdot \epsilon_\ell \\
&< \epsilon_\ell .
\end{aligned}
$$

Thus, (21) guarantees that $g = \pi \circ F$ produces a holomorphic disk $g_\ell := g \upharpoonright_{\mathbb{D}_{R_\ell}}$ satisfying

$$\frac{L_{g_\ell}(\omega)}{T_{g_\ell}(\omega)} \leqslant 2^{-\ell+1}, \quad \frac{T_{g_\ell}(\mathbf{1}_{X\setminus 2\hat{U}_\ell} \cdot \omega)}{T_{g_\ell}(\omega)} < 2^{-\ell+1}. \tag{23}$$

Hence, the sequence of concentric holomorphic disks $\{g \upharpoonright_{\mathbb{D}_{R_\ell}}\}_{\ell \geqslant 1}$ satisfy the length-area conditions (1), (3) for producing Nevanlinna/Ahlfors currents.

For any $e_1 \in E_1$, since $\{e_{1,\ell}\}_{\ell \geqslant 1}$ is dense in E_1, we can find some index subsequence $\{j_\ell\}_{\ell \geqslant 1} \nearrow \infty$ such that the corresponding point sequence $\{e_{1,j_\ell}\}_{\ell \geqslant 1}$ converges to e_1. Note that the sequence of holomorphic disks $\{g \upharpoonright \mathbb{D}_{R_{2j_\ell - 1}}\}_{\ell \geqslant 1}$ satisfies the the length-area condition (1) by the first inequality of (23). After passing to some subsequence, we thus obtain some positive closed Nevanlinna current T. By the second inequality of (23), noting that $2\hat{U}_{2j_\ell - 1}$ converges to the curve $e_1 \times E_2$ as $\ell \nearrow \infty$, T must have zero mass outside $e_1 \times E_2$, that is, T is supported on $e_1 \times E_2$. By Siu's decomposition theorem, T is proportional to $[\{e_1\} \times E_2]$.

Similarly, for every $e_2 \in E_2$, we can also select some sequence of increasing radii to generate a Nevanlinna current proportional to $[E_1 \times \{e_2\}]$.

Likewise, we can prove the analogous result about Ahlfors currents by using (22) instead of (21). ∎

Proof of the Key Lemma. Fix a small disk $\overline{\mathbb{D}}(z_0, 2\delta)$ in U. Draw a simple closed curve passing through z_0, which bounds a simply connected domain D strictly larger than $\overline{\mathbb{D}}_R$ except at z_0. Using the Riemann mapping theorem, we receive a biholomorphic map ψ from D to the unit disk $\mathbb{D}_1$. By compactness, the maximum modulus $\mathbf{m}$ of $|\psi|$ on $\overline{\mathbb{D}}_R \setminus \mathbb{D}(z_0, \delta)$ is strictly less than 1. Now we claim that

$$\mathcal{X}(\bullet) := (1 + \delta_1) \cdot \psi\big((1 - \delta_2) \cdot \bullet\big)$$

satisfies our requirement for some small $\delta_1, \delta_2 > 0$.

Indeed, we first require that $(1 + \delta_1) \cdot \mathbf{m} < 1$. Next, we choose $\delta_2 > 0$ sufficiently small such that

$$\frac{1}{1-\delta_2} \cdot \mathbb{D}(z_0, \delta) := \mathbb{D}\Big(\frac{z_0}{1-\delta_2}, \frac{\delta}{1-\delta_2}\Big) \subset \mathbb{D}(z_0, 2\delta) \qquad \Big(\text{equivalently}, \delta_2 < \frac{\delta}{|z_0| + 2\delta}\Big),$$

and that $|\mathcal{X}(z_0)| = (1 + \delta_1) \cdot |\psi\big((1 - \delta_2) \cdot z_0\big)| > 1$. Then $\mathcal{X}$ is well defined on

$$\frac{1}{1-\delta_2} \cdot \mathrm{D} := \Big\{\frac{z}{1-\delta_2} \ : z \in \mathrm{D}\Big\} \supset \overline{\mathbb{D}}_R$$

and satisfies our requirements. ∎

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